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Equipolar meromorphic functions sharing a set

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Abstract. Two meromorphic functions f and g having the same set of poles are known as equipolar. In this paper we study some uniqueness results of equi-polar meromorphic functions sharing a finite set and improve some recent results of Bhoosnurmath-Dyavanal [4] and Banerjee-Mallick [3] by removing some unnecessary conditions on ramification indices as well as relaxing the condition on the nature of sharing of the value ∞ by f and g from counting multiplicity to ignoring multiplicity.

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1. Introduction, definitions and results

Let f and g and be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (Counting Multiplicities) and if we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [7].

Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$, the counting function of the zeros of $f - a$ of multiplicity one. We also denote by $N(r, a; f \mid\geq l)$, the counting function of those a-points of f whose multiplicities are $\geq l$. Similarly we denote by $\overline{N}(r, a; f \mid\geq l)$ the reduced counting function of the a-points of f of multiplicity $\geq l$. We put $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$. We put

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)};$$

$$\delta_2(a; f) = 1 - \limsup_{r\to\infty} \frac{N_2(r, a; f)}{T(r, f)}$$

and

$$\delta_{(2}(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f \mid\geq 2)}{T(r,f)}.$$

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and

$$E_f(S) = \bigcup_{a\in S}\{(z,p)\in\mathbb{C}\times\mathbb{N} : z \text{ is an } a\text{-point of } f \text{ of multiplicity } p\},$$

and

$$\overline{E}_f(S) = \bigcup_{a\in S}\{(z,1)\in\mathbb{C}\times\mathbb{N} : z \text{ is an } a\text{-point of } f\}.$$

If $E_f(S) = E_g(S)$, we say that f and g share the set S CM (Counting Multiplicity). On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM (Ignoring Multiplicity).

It will be convenient to denote by E, any subset of nonnegative real numbers of finite measure not necessary the same in each of its occurrence.

In 1976, Gross [6] considered the uniqueness problem of meromorphic functions when the functions under consideration share sets instead of values. In this direction Gross raised the following question:
Can one find finite sets S_j, $j=1,2$ such that any two non-constant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical ?

To answer the Question of Gross [6], in 1995, Yi [13] obtained the following results.

Theorem A. [13] *Let $S=\{z : z^n+az^{n-m}+b=0\}$, where n and m are two positive integers such that $m\geq 2$, $n\geq 2m+7$, with m and n having no common factor, a and b be two nonzero constants such that $z^n+az^{n-m}+b=0$ has no multiple root. If f and g be two non-constant meromorphic functions having no simple poles such that $E_f(S)=E_g(S)$ and $E_f(\{\infty\})=E_g(\{\infty\})$, then $f\equiv g$.*

Theorem B. [13] *Let $S=\{z : z^n+az^{n-1}+b=0\}$, where $n(\geq 9)$ be an integer and a and b be two nonzero constants such that $z^n+az^{n-1}+b=0$ has no multiple root. If f and g be two non-constant meromorphic functions such that $E_f(S)=E_g(S)$ and $E_f(\{\infty\})=E_g(\{\infty\})$, then either $f\equiv g$ or*

$$f\equiv\frac{-ah(h^{n-1}-1)}{h^n-1} \quad \text{and} \quad g\equiv\frac{-a(h^{n-1}-1)}{h^n-1},$$

where h is a non-constant meromorphic function.

Lahiri [8], in an attempt to investigate under which situation, $f\equiv g$, proved the following result.

Theorem C. [8] *Let S be defined as in Theorem B and $n(\geq 8)$ be an integer. If f and g be two non-constant meromorphic functions having no simple poles such that $E_f(S)=E_g(S)$ and $E_f(\{\infty\})=E_g(\{\infty\})$, then $f\equiv g$.*

Fang and Lahiri [5], improved Theorem C by reducing the cardinality of the same range set in the following result.

Theorem D. [5] *Let* $S = \{z : z^n + az^{n-1} + b = 0\}$, *where* $n(\geq 7)$ *be an integer and* a *and* b *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root. If* f *and* g *be two non-constant meromorphic functions having no simple poles such that* $E_f(S) = E_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$, *then* $f \equiv g$.

Below we give the definition of weighted sharing which will be required in the sequel.

Definition 1.1. [9, 10] *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$, *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k+1$ *times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f *and* g *share the value* a *with weight* k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ of multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [10] *Let* $S \subset \mathbb{C} \cup \{\infty\}$ *and* k *be a positive integer or* ∞. *We denote by* $E_f(S, k)$ *the set* $\bigcup_{a \in S} E_k(a; f)$.

Recently Bhoosnurmath-Dyavanal [4] proved the following result as an improvement of the above results by reducing the cardinality of the shared set S as well as weakening the condition on ramification indices.

Theorem E. [4] *Let* $S = \{z : z^n + az^{n-1} + b = 0\}$, *where* $n(\geq 5)$ *be an integer and* a *and* b *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root. If* f *and* g *be two non-constant meromorphic functions such that* $E_f(S, \infty) = E_g(S, \infty)$ *and* $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$. *Also* $N(r, 0; f \mid= 1) = S(r, f)$ *and* $N(r, 0; g \mid= 1) = S(r, g)$ *and* $\Theta(\infty; f) > \frac{2}{n-1}$ *and* $\Theta(\infty; g) > \frac{2}{n-1}$, *then* $f \equiv g$.

With the aid of weighted sharing Banerjee-Mallick [3] improved Theorem E as follows.

Theorem F. [3] *Let* $S = \{z : z^n + az^{n-1} + b = 0\}$, *where* $n(\geq 5)$ *be an integer and* a *and* b *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root. If* f *and* g *be two non-constant meromorphic functions satisfying* $E_f(S, m) = E_g(S, m)$ *and* $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$. *Also* $N(r, 0; f \mid= 1) = S(r, f)$ *and* $N(r, 0; g \mid= 1) = S(r, g)$ *and* $\Theta_f + \Theta_g > \frac{4}{n-1}$. *If*

(i) $m \geq 2$ *and* $n \geq 5$ *;*

(ii) *or* $m = 1$ *and* $n \geq 6$ *;*

(iii) *or* $m = 0$ *and* $n \geq 10$,

then $f \equiv g$, *where* $\Theta_f = \delta_{(2}(0; f) + \Theta(\infty; f) + \Theta(-a\frac{n-1}{n}; f)$ *and* Θ_g *is defined similarly.*

In this paper we give two-fold improvements to Theorem F as follows. Firstly we show that we can reach the conclusion of Theorem F without assuming the condition

$$\Theta_f + \Theta_g > \frac{4}{n-1}.$$

Secondly, we prove our theorem merely assuming that f and g share the value ∞ with weight 0. That is we reduce the CM sharing of ∞ by f and g to IM sharing. We also show that the cardinality of the shared set S can be reduced to 9 from 10 when $m = 0$. We state below our theorem.

Theorem 1.1. *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n(\geq 5)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions satisfying $E_f(S, m) = E_g(S, m)$, $E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$ and $N(r, 0; f \mid= 1) = S(r, f)$ and $N(r, 0; g \mid= 1) = S(r, g)$. Then, $f \equiv g$, if any one of the following holds.*

(i) $m = 2$, $n \geq 5$;

(ii) $m = 1$, $n \geq 6$;

(iii) $m = 0$, $n \geq 9$.

Definition 1.3. [10] *Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, and an a-point of g of multiplicity q. We denote by $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g where $p > q(q > p)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$. We also denote by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$. similarly we denote by $\overline{N}_E^{(2}(r, a; f)$, the reduced counting function of those a-points of f such that $p = q \geq 2$.*

2. Lemmas

In this section we present some lemmas which will be required to establish our results. Let f and g be two nonconstant meromorphic functions and we define

$$F = \frac{f^{n-1}(f + a)}{-b}, \quad G = \frac{g^{n-1}(g + a)}{-b}. \tag{2.1}$$

In the lemmas several times we use the function H defined by

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}.$$

Lemma 2.1. [12] *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$, $b_m \neq 0$. Then $T(r, R(f)) = dT(r, f) + S(r, f)$, where $d = max\{m, n\}$.

Lemma 2.2. [14] *If F, G be two non-constant meromorphic functions such that they share $(1, 0)$ and $H \not\equiv 0$ then,*
$N_E^{1)}(r, 1; F \mid= 1) = N_E^{1)}(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G)$.

Lemma 2.3. [2] *Let f and g be two nonconstant meromorphic functions sharing $(1, m)$, $0 \leq m < \infty$. Then*
$\overline{N}(r, 1; f) + \overline{N}(r, 1; g) - N_E^{1)}(r, 1; f) + \left(m - \frac{1}{2}\right) \overline{N}_*(r, 1; f, g) \leq \frac{1}{2}[N(r, 1; f) + N(r, 1; g)]$.

Lemma 2.4. *Let $H \not\equiv 0$ and $E_f(S, 0) = E_g(S, 0)$ and $E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$. Then, if F and G be given by (2.1),*

$$\begin{aligned} & N(r, H) \\ \leq\ & \overline{N}(r, 0; F \mid\geq 2) + \overline{N}(r, 0; G \mid\geq 2) + \overline{N}(r, c; F \mid\geq 2) + \overline{N}(r, c; G \mid\geq 2) \\ +\ & \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, F) \\ +\ & S(r, G), \end{aligned}$$

for $c \in \mathbb{C}\backslash\{0, 1\}$. Here, $\overline{N}_0(r, 0; F')$, denotes the reduced counting function of the zeros of F', which are not the zeros of $F(F-1)(F-c)$. Similarly we define $\overline{N}_0(r, 0; G')$.

Proof. From the definition of H, it follows that that the poles of H occur at the
(i) multiple zeros of F and G;
(ii) poles of F and G of different multiplicities;
(iii) 1-points of F and G of different multiplicities;
(iv) multiple c-points of F and G;
(v) the zeros of F' which are not the zeros of $F(F-1)(F-c)$;
(vi) the zeros of G' which are not the zeros of $G(G-1)(G-c)$.
Since the poles of H are all simple, the lemma follows easily. □

Lemma 2.5. [11] *If two non-constant meromorphic functions f and g share $(\infty, 0)$. Then $f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2$, for $n \geq 2$.*

Lemma 2.6. *Let f and g be two non-constant meromorphic functions such that $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$, where $n \geq 5$ is an integer. If $N(r, 1; f \mid= 1) = S(r, f)$ and $N(r, 1; g \mid= 1) = S(r, g)$, then $f \equiv g$.*

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a). \tag{2.2}$$

Clearly (2.2) implies that f and g share (∞, ∞). Suppose $f \not\equiv g$. Let $y = \frac{g}{f}$. Then (2.2) implies that $y \not\equiv 1$, $y^{n-1} \not\equiv 1$, $y^n \not\equiv 1$ and

$$f \equiv -a\frac{1 - y^{n-1}}{1 - y^n} \tag{2.3}$$

$$\begin{aligned} \equiv\ & a\left(\frac{y^{n-1}}{1 + y + y^2 + \ldots + y^{n-1}} - 1\right) \\ =\ & -a\frac{1 + y + y^2 + \ldots + y^{n-2}}{1 + y + y^2 + \ldots + y^{n-1}}. \end{aligned}$$

Case 1. Let $y = \frac{g}{f}$ =constant, then it follows from (2.3) that f is constant, which is impossible.
Case 2. Let $y = \frac{g}{f}$ be non-constant.

Using Lemma 2.1, we note from (2.2), $T(r,f) = T(r,g) + O(1)$ and hence $S(r,f) = S(r,g) = S(r)$, say.

Let z_0 be a zero of $f + a$. Then in view of (2.2), z_0 must be a zero of either $g + a$ or g. If possible suppose that z_0 is a zero of $g + a$. Then $y(z_0) = 1$ and from (2.3) we obtain $f(z_0) = -a(\frac{n-1}{n}) \neq -a$, that is $f(z_0) + a = -a(\frac{n-1}{n}) \neq 0$ which is a contradiction to our assumption. Therefore z_0 must be a zero of g. Thus we have

$$\{z : f(z) + a = 0\} \subseteq \{z : g(z) = 0\}. \tag{2.4}$$

Suppose z_0 be a zero of $f+a$ of multiplicity p and a zero of g of multiplicity q. Then in view of (2.2), $p = (n-1)q$. Thus $p = n-1$, if $q = 1$ or $p \geq 2(n-1)$, when $q \geq 2$. Thus the least multiplicity of a zero of $f + a$ is $n - 1$ and $f + a$ has no zero of multiplicity m such that $n - 1 < m < 2(n - 1)$.

We agree to denote by $\overline{N}(r, 0; f + a \mid g_{=1} = 0)$, the reduced counting function of the zeros of $f + a$ which are the zeros of g of multiplicity =1 and by $\overline{N}(r, 0; f + a \mid g_{\geq 2} = 0)$, the reduced counting function of the zeros of $f + a$ which are the zeros of g of multiplicity ≥ 2. Also we denote by $\overline{N}(r, 0; f + a \mid g = 0)$ the reduced counting function of the zeros of $f + a$, which are the zeros of g.

Now since $N(r, 0; g \mid= 1) = S(r, g)$, we have from (2.4) and above analysis,

$$\begin{aligned}
& \overline{N}(r,0;f+a) \\
= \; & \overline{N}(r,0;f+a \mid g=0) \\
= \; & \overline{N}(r,0;f+a \mid g_{=1}=0) + \overline{N}(r,0;f+a \mid g_{\geq 2}=0) \\
= \; & S(r,g) + \overline{N}(r,0;f+a \mid \geq 2(n-1)) \\
= \; & S(r,f) + \overline{N}(r,0;f+a \mid \geq 2(n-1)).
\end{aligned}$$

Hence

$$(2n-2)\overline{N}(r,0;f+a) \leq T(r,f) + S(r,f).$$

From (2.3) we observe that $T(r, f) = (n - 1)T(r, y) + S(r, y)$. Also

$$\begin{aligned}
& f + a\frac{n-1}{n} \\
= \; & -a\frac{1-y^{n-1}}{1-y^n} + a\frac{n-1}{n} \\
= \; & -a\frac{(n-1)y^n - ny^{n-1} + 1}{n(1-y^n)}.
\end{aligned} \tag{2.5}$$

If we put $p(y) = (n-1)y^n - ny^{n-1} + 1$, then $p(0) \neq 0$ and
$p'(y) = n(n-1)y^{n-2}\{y-1\}$ and $p''(y) = n(n-1)y^{n-3}\{(n-3)y - n + 2\}$.
Thus $p(1) = p'(1) = 0$. Hence $p(y) = 0$ has only one repeated root at $y = 1$.

Thus from (2.5) we obtain

$$\sum_{i=1}^{n-1}\overline{N}(r,u_i;y)\le\overline{N}(r,-a\frac{n-1}{n};f),$$

where u_i, $i=1,\ldots,n-1$ are the distinct zeros of $p(y)$.
Also from (2.3) we have

$$\sum_{j=1}^{n-1}\overline{N}(r,v_j;y)\le\overline{N}(r,\infty;f)\le T(r,f).$$

Since by our assumption $N(r,0;f\mid=1)=S(r,f)$, we have

$$\overline{N}(r,0;f)=N(r,0;f\mid=1)+\overline{N}(r,0;f\mid\ge 2)\le S(r,f)+\frac{1}{2}T(r,f).$$

Thus we have

$$\begin{aligned}&\sum_{j=1}^{n-2}\overline{N}(r,w_j;y)+\overline{N}(r,\infty;y)\\ \le\ &\overline{N}(r,0;f)=N(r,0;f\mid=1)+\overline{N}(r,0;f\mid\ge 2)\le\frac{1}{2}T(r,f)+S(r,f),\end{aligned}$$

where v_js, $j=1,2,\ldots,n-1$ are the distinct roots of $1+y+y^2+\ldots+y^{n-1}=0$ and w_js, $j=1,2,\ldots,n-2$ are the distinct roots of $1+y+y^2+\ldots+y^{n-2}=0$.
From (2.2) and (2.3) we note that the zeros of y occur at those zeros of g which are the zeros of $f+a$. Hence $\overline{N}(r,0;y)\le\overline{N}(r,0;f+a)$.
Also we have obtained $(2n-2)\overline{N}(r,0;f+a)\le T(r,f)+S(r,f)$. Thus, we obtain by the second main theorem,

$$\begin{aligned}&(3n-4)T(r,y)\\ \le\ &\sum_{j=1}^{n-1}\overline{N}(r,v_j;y)+\sum_{j=1}^{n-2}\overline{N}(r,w_j;y)+\sum_{i=1}^{n-1}\overline{N}(r,u_i;y)+\overline{N}(r,0;y)\\ +\ &\overline{N}(r,\infty;y)+S(r,y)\\ \le\ &\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,0;f+a)\\ +\ &S(r,f)\\ \le\ &\left\{1+\frac{1}{2}+1+\frac{1}{2n-2}\right\}T(r,f)+S(r,f)\\ \le\ &\left(\frac{5}{2}+\frac{1}{2n-2}\right)(n-1)T(r,y)+S(r,y),\end{aligned}$$

which leads to a contradiction for $n\ge 5$. This completes the proof of the Lemma. □

Lemma 2.7. *Let $S=\{z:z^n+az^{n-1}+b=0\}$, where $n(\ge 4)$ be an integer and a and b be two nonzero constants such that $z^n+az^{n-1}+b=0$ has no multiple root. If F and G are given by (2.1), then there exists an $\alpha\in\mathbb{C}\setminus\{0,a,b\}$, satisfying $N_2(r,\alpha;F)\le(n-1)T(r,f)+S(r,f)$, $N_2(r,\alpha;G)\le(n-1)T(r,g)+S(r,g)$, where*

$|\alpha| = \frac{(n-1)^{n-1}}{n^n} . \frac{|a|^n}{|b|}$, $\arg\alpha = \arg(\frac{a^n}{b})$ *or* $\arg\alpha = \arg(-\frac{a^n}{b})$, *according as n is even or odd. Here* $\arg z$ *denotes the principal argument of z for any* $z \in \mathbb{C} \setminus \{0\}$.

Proof. Let $p(z) = z^n + az^{n-1} + b$. Then $p'(z) = z^{n-2}\{nz + a(n-1)\}$. Thus $p'(z) = 0$ has roots at $z = 0$ and at $z = -\frac{a(n-1)}{n}$. Thus $p(z) = 0$ will have a repeated root at $-\frac{a(n-1)}{n}$ provided $p\left(-\frac{a(n-1)}{n}\right) = 0$ and this yields $b = (-1)^n \left(\frac{a}{n}\right)^n (n-1)^{n-1}$. Note that $p''\left(-\frac{a(n-1)}{n}\right) \neq 0$.

Thus $p(z) = 0$ has a repeated root at $-\frac{a(n-1)}{n}$ and hence only $n-1$ distinct roots provided $b = (-1)^n(\frac{a}{n})^n(n-1)^{n-1}$.

Let α be a nonzero complex number. Then

$$F - \alpha = \frac{f^{n-1}(f+a)}{-b} - \alpha = \frac{f^n + af^{n-1} + \alpha b}{-b}.$$

We choose α in such a manner that the equation $z^n + az^{n-1} + \alpha b = 0$ has repeated roots. It is clear from the above discussion that in this case we must have

$$\alpha b = (-1)^n \left(\frac{a}{n}\right)^n (n-1)^{n-1}.$$

This implies $|\alpha| = \frac{(n-1)^{n-1}}{n^n} \cdot \frac{|a|^n}{|b|}$, $\arg\alpha = \arg(\frac{a^n}{b})$ or $\arg\alpha = \arg(-\frac{a^n}{b})$, according as n is even or odd. If $w_1, w_2, \ldots, w_{n-1}$, be the distinct roots of $z^n + az^{n-1} + \alpha b = 0$, then we have

$$\begin{aligned}
& N_2(r, \alpha; F) \\
= \; & \overline{N}(r, \alpha; F) + \overline{N}(r, \alpha; F \mid\geq 2) \\
\leq \; & \sum_{i=1}^{n-1} \overline{N}(r, w_i; f) + \sum_{i=1}^{n-1} \overline{N}(r, w_i; f \mid\geq 2) + S(r, f) \\
= \; & \sum_{i=1}^{n-1} \{\overline{N}(r, w_i; f) + \overline{N}(r, w_i; f \mid\geq 2)\} + S(r, f) \\
= \; & \sum_{i=1}^{n-1} N_2(r, w_i; f) + S(r, f) \\
\leq \; & (n-1)T(r, f) + S(r, f).
\end{aligned}$$

This completes the proof. □

Lemma 2.8. *Let F, G be given by (2.1) and* $V = (\frac{F'}{F-1} - \frac{F'}{F}) - (\frac{G'}{G-1} - \frac{G'}{G}) \not\equiv 0$. *If* $\overline{N}(r, 0; f \mid= 1) = S(r, f)$ *and* $\overline{N}(r, 0; g \mid= 1) = S(r, g)$ *and f, g share* $(\infty, 0)$*; F, G, share* $(1, 0)$*, then*

$$\{n-1\}\overline{N}(r, \infty; f) \leq \left\{\frac{1}{2} + 1\right\} \{T(r, f) + T(r, g)\} + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).$$

Proof. Let z_0 be a pole of f and g of respective multiplicities p and q. Then from (2.1), around z_0, we have

$$F = \frac{A(z)}{(z-z_0)^{np}}, \quad G = \frac{B(z)}{(z-z_0)^{nq}}. \tag{2.6}$$

Where $A(z)$ and $B(z)$ are analytic at z_0, and $A(z_0) \neq 0$, $B(z_0) \neq 0$.
Thus

$$\frac{F'}{F-1} = \frac{A'}{A-(z-z_0)^{np}} - \frac{npA}{(z-z_0)[A-(z-z_0)^{np}]}$$

and

$$\frac{F'}{F} = \frac{A'}{A} - \frac{np}{z-z_0}.$$

Therefore a simple calculation yields,

$$\begin{aligned} \frac{F'}{F-1} - \frac{F'}{F} &= (z-z_0)^{np-1}\left\{\frac{A'}{A}.\frac{z-z_0}{A-(z-z_0)^{np}} - \frac{np}{A-(z-z_0)^{np}}\right\} \\ &= (z-z_0)^{np-1}\phi(z), \end{aligned}$$

say, where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Similarly we obtain,

$$\begin{aligned} \frac{G'}{G-1} - \frac{G'}{G} &= (z-z_0)^{nq-1}\left\{\frac{B'}{B}.\frac{z-z_0}{B-(z-z_0)^{nq}} - \frac{nq}{B-(z-z_0)^{nq}}\right\} \\ &= (z-z_0)^{nq-1}\psi(z), \end{aligned}$$

say, where $\psi(z)$ is analytic at z_0 and $\psi(z_0) \neq 0$. Therefore, around z_0,

$$V = (z-z_0)^{np-1}\phi(z) - (z-z_0)^{nq-1}\psi(z).$$

Thus V has a zero at z_0, of order at least $n-1$.
We note by Millux's theorem

$$\begin{aligned} & m(r,V) \\ = \;& m\left(r, \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right)\right) \\ \leq \;& m\left(r, \frac{F'}{F-1}\right) + m\left(r, \frac{G'}{G-1}\right) + m\left(r, \frac{F'}{F}\right) + m\left(r, \frac{G'}{G}\right) \\ = \;& S(r,F) + S(r,G) = S(r,f) + S(r,g). \end{aligned}$$

Hence from above analysis and by the first fundamental theorem, we have

$$\begin{aligned} & \{n-1\}\overline{N}(r,\infty;f) \\ \leq \;& N(r,0;V) \\ \leq \;& T(r,V) + O(1) \\ \leq \;& N(r,\infty;V) + S(r,f) + S(r,g) \\ \leq \;& \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,0;f+a) + \overline{N}(r,0;g+a) \\ + \;& \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g). \end{aligned}$$

Now since $\overline{N}(r,0;f\mid=1)=S(r,f)$ and $\overline{N}(r,0;g\mid=1)=S(r,g)$, we have

$$\overline{N}(r,0;f)\leq\frac{1}{2}T(r,f)+S(r,f)$$

and

$$\overline{N}(r,0;g)\leq\frac{1}{2}T(r,g)+S(r,g).$$

Therefore from above, we have

$$\begin{aligned}&\{n-1\}\overline{N}(r,\infty;f)\\ \leq\ &\left\{\frac{1}{2}+1\right\}\{T(r,f)+T(r,g\}+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

This completes the proof. □

Lemma 2.9. [1] *Let F and G be defined by (2.1) and F and G share $(1,m)$, $0\leq m<\infty$. Also let $w_1,\ldots,w_n$ be the distinct roots of the equation $z^n+az^{n-1}+b=0$, where $b\neq(-1)^n(\frac{a}{n})^n(n-1)^{n-1}$, $n\geq 3$. Then*

$$\overline{N}_L(r,1;F)\leq\frac{1}{m+1}\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}-N_{\bigodot}(r,0;f')+S(r,f),$$

where $N_{\bigodot}(r,0;f')=N(r,0;f'\mid f\neq 0,w_1,\ldots,w_n)$. Similar inequality holds for $\overline{N}_L(r,1;G)$.

Lemma 2.10. *Let F and G be defined by (2.1) and F and G share $(1,m)$, $0\leq m<\infty$. Also let $\overline{N}(r,0;f\mid=1)=S(r,f)$ and $\overline{N}(r,0;g\mid=1)=S(r,g)$. Then*

$$\begin{aligned}&\overline{N}_*(r,1;F,G)\\ \leq\ &\frac{1}{m+1}\left\{\frac{1}{2}[T(r,f)+T(r,g)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+S(r,f)+S(r,g).\end{aligned}$$

Proof. Since $\overline{N}_*(r,1;F,G)=\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)$ and from the condition of the Lemma it follows that

$$\overline{N}(r,0;f)\leq\frac{1}{2}T(r,f)+S(r,f)$$

and

$$\overline{N}(r,0;g)\leq\frac{1}{2}T(r,g)+S(r,g),$$

the Lemma follows from Lemma 2.9. □

Lemma 2.11. Let F and G be defined by (2.1) and F and G share $(1,m)$, $0\leq m<\infty$. Also let $\overline{N}(r,0;f\mid=1)=S(r,f)$ and $\overline{N}(r,0;g\mid=1)=S(r,g)$ and f and g share $(\infty,0)$. Then

$$\begin{aligned}&\left[n-1-\frac{2}{m+1}\right]\overline{N}(r,\infty;f)\\ \leq\ &\left[\frac{3}{2}+\frac{1}{2(m+1)}\right]\{T(r,f)+T(r,g)\}\\ +\ &S(r,f)+S(r,g).\end{aligned}$$

Proof. From Lemmas 2.8 and 2.10, we have

$$\begin{aligned}&\{n-1\}\overline{N}(r,\infty;f)\\ \leq\ &\left\{\frac{1}{2}+1+\frac{1}{2(m+1)}\right\}\{T(r,f)+T(r,g\}+\frac{2}{m+1}\overline{N}(r,\infty;f)\\ +\ &S(r,f)+S(r,g).\end{aligned}$$

The lemma follows easily from above. □

3. Proof of theorem

Proof of Theorem 1.1. **Case 1.** $H \not\equiv 0$. By Lemma 2.1, we obtain from the definitions of F and G, $T(r,F) = nT(r,f) + S(r,f)$, $T(r,G) = nT(r,g) + S(r,g)$.

We denote by $N_0(r,0;F')$, the counting function of the zeros of F' which are not the zeros of $F(F-1)(F-c)$, for some $c \in \mathbb{C}\setminus\{0,1\}$. Similarly we define $N_0(r,0;G')$. Now applying the second main theorem to F and G, we obtain for some $c \in \mathbb{C}\setminus\{0,1\}$,

$$\begin{aligned}&2\{T(r,F)+T(r,G)\}\\ \leq\ &\overline{N}(r,0;F)+\overline{N}(r,c;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,c;G)\\ +\ &\overline{N}(r,1;G)+\overline{N}(r,\infty;G)-N_0(r,0;F')-N_0(r,0;G')+S(r,f)+S(r,g),\end{aligned}$$

and hence

$$\begin{aligned}&2n\{T(r,f)+T(r,g)\}\\ \leq\ &\overline{N}(r,0;F)+\overline{N}(r,c;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,c;G)\\ +\ &\overline{N}(r,1;G)+\overline{N}(r,\infty;G)-N_0(r,0;F')-N_0(r,0;G')+S(r,f)+S(r,g).\end{aligned}$$

Using Lemma 2.2, Lemma 2.3 and 2.4 and 2.9 we have from above,

$$\begin{aligned}&2n\{T(r,f)+T(r,g)\} \qquad (3.1)\\ \leq\ &N_2(r,0;F)+N_2(r,c;F)+3\overline{N}(r,\infty;f)+N_2(r,0;G)+N_2(r,c;G)\\ +\ &\frac{n}{2}\{T(r,f)+T(r,g)\}+\left(\frac{3}{2}-m\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ \leq\ &2\overline{N}(r,0;f)+N_2(r,0;f+a)+(n-1)T(r,f)+3\overline{N}(r,\infty;f)\\ +\ &2\overline{N}(r,0;g)+N_2(r,0;g+a)+(n-1)T(r,g)+\frac{n}{2}\{T(r,f)+T(r,g)\}\\ +\ &\left(\frac{3}{2}-m\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Subcase 1.1. $m = 2$. We obtain from (3.1) using Lemma 2.8,

$$\begin{aligned} & \left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g)\} \qquad (3.2)\\ \leq\ & \overline{N}(r,\infty;f)+\frac{2.3}{2(n-1)}\{T(r,f)+T(r,g)\}+\left(\frac{2}{n-1}-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\ +\ & S(r,f)+S(r,g).\\ \leq\ & \frac{1}{2}\{T(r,f)+T(r,g)\}+\frac{2.3}{2(n-1)}\{T(r,f)+T(r,g)\}\\ +\ & \left(\frac{2}{n-1}-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g). \end{aligned}$$

But this leads to a contradiction for $n \geq 5$.

Subcase 1.2. $m = 1$. Then proceeding as in Subcase 1.1, the Lemma 2.2 with $m = 1$ and Lemma 2.3, yield the following.

$$\begin{aligned} & 2n\{T(r,f)+T(r,g)\}\\ \leq\ & 2\{T(r,f)+T(r,g\}+(n-1)\{T(r,f)+T(r,g\}+\frac{n}{2}\{T(r,f)+T(r,g\}\\ +\ & 3\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g). \end{aligned}$$

Using the Lemma 2.10 we obtain from above,

$$\begin{aligned} & \left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g\}\\ \leq\ & 3\overline{N}(r,\infty;f)+\frac{1}{2}.\frac{1}{1+1}\left[\frac{1}{2}T(r,f)+\frac{1}{2}T(r,g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]\\ +\ & S(r,f)+S(r,g)\\ =\ & \left\{\frac{3}{2}+\frac{1}{4}\right\}[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]+\frac{1}{8}\{T(r,f)+T(r,g\}\\ +\ & S(r,f)+S(r,g)\\ \leq\ & \left\{\frac{3}{2}+\frac{1}{4}+\frac{1}{8}\right\}\{T(r,f)+T(r,g\}+S(r,f)+S(r,g). \end{aligned}$$

This leads to a contradiction for $n \geq 6$.

Subcase 1.3. $m = 0$. Proceeding as in Subcase 1.2., we obtain using Lemmas 2.10 and 2.11 with $m = 0$,

$$\begin{aligned} & \left\{\frac{n}{2} - 1\right\} \{T(r,f) + T(r,g\} \\ \leq\ & 3\overline{N}(r,\infty;f) + \frac{3}{2}\overline{N}_*(r,1;F,G) \\ \leq\ & 3\overline{N}(r,\infty;f) + \frac{3}{2}\left\{\frac{1}{2}T(r,f) + \frac{1}{2}T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\right\} \\ +\ & S(r,f) + S(r,g) \\ =\ & 6.\frac{2}{n-3}\{T(r,f) + T(r,g)\} + \frac{3}{4}\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g) \\ =\ & \left(\frac{12}{n-3} + \frac{3}{4}\right)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g), \end{aligned}$$

this leads to a contradiction for $n \geq 9$.

Case 2. $H \equiv 0$. We have

$$F \equiv \frac{AG + B}{CG + D}, \tag{3.3}$$

where $AD - BC \neq 0$. Clearly from above and the definitions of F and G we have $T(r,F) = T(r,G) + O(1)$ and $T(r,f) = T(r,g) + O(1)$.

Subcase 2.1. $AC \neq 0$. Since f and g share $\{\infty\}$, it follows from (3.2) that ∞ is an exceptional value of f and g. So by the second main theorem we get,

$$\begin{aligned} & nT(r,f) \\ \leq\ & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,\frac{A}{C};F) + S(r,f) \\ \leq\ & \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) \\ \leq\ & 2T(r,f) + S(r,f), \end{aligned}$$

which leads to a contradiction for $n \geq 5$.

Subcase 2.2. Let $A \neq 0$ and $C = 0$. Then $F = \gamma G + \beta$, where $\gamma = \frac{A}{D} \neq 0$ and $\beta = \frac{B}{D}$. It is obvious that F and G cannot omit the value 1. For if F omits the value 1, then f(and g as well) omits the distinct roots of the equation $z^n + az^{n-1} + b = 0$, which certainly leads to a contradiction for $n \geq 3$.

Thus F and G assume the value 1 and we have from above

$$F = \gamma G + (1 - \gamma). \tag{3.4}$$

If $\gamma = 1$ we have $F \equiv G$ and by Lemma 2.6, we have $f \equiv g$.

So let $\gamma \neq 1$. Since $N(r,0;f \mid= 1) = S(r,f)$ and $N(r,0;g \mid= 1) = S(r,g)$, we have from (3.4) using the second main theorem,

$$\begin{aligned} & nT(r,f) \\ \leq\ & \overline{N}(r,0;F) + \overline{N}(r,1-\gamma;F) + \overline{N}(r,\infty;F) + S(r,f) \\ \leq\ & \frac{1}{2}T(r,f) + \overline{N}(r,0;f+a) + \frac{1}{2}T(r,g) + \overline{N}(r,0;g+a) + \overline{N}(r,\infty;f) + S(r,f) \\ \leq\ & 4T(r,f) + S(r,f). \end{aligned}$$

This leads to a contradiction for $n \geq 5$.
Subcase 2.3. $A = 0$, $C \neq 0$. Then clearly $B \neq 0$. Hence, $F \equiv \frac{1}{\zeta G+\eta}$. We can show as before that F and G cannot omit the value 1 and hence $F \equiv \frac{1}{\zeta G+1-\zeta}$. Let $\zeta = 1$. Then $FG \equiv 1$. This is a contradiction by Lemma 2.5.

So $\zeta \neq 1$. Now since f and g share ∞, the relation $F \equiv \frac{1}{\zeta G+1-\zeta}$, at once implies F cannot assume the values ∞ and 0, and therefore f cannot assume the values ∞, 0 and $-a$. This is impossible. This completes the proof of the theorem. □

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