Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 137–146

On a subclass of convex functions

Olga Engel and Róbert Szász

Abstract. In this paper we study a subclass of convex functions. Among others we prove an interesting property regarding the composition of functions from this class. The basic tool of the proof is the theory of differential subordination.

Mathematics Subject Classification (2010): 30C45.

Keywords: Starlikeness, univalent function, composition of functions.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{A}$ the class of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

defined in U. We say that f is starlike in U if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a starlike domain in $\mathbb{C}$ with respect to 0.

It is well-known that $f \in \mathcal{A}$ is starlike in U if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{for all } z \in U.$$

The function $f \in \mathcal{A}$ is convex in U if and only if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a convex domain in $\mathbb{C}$. The function $f \in \mathcal{A}$ is convex if and only if

$$\mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U.$$

The subclass of $\mathcal{A}$ which contain convex functions will be denoted by $\mathcal{K}$.
We define the class S^{***} by the equality

$$S^{***} = \left\{ f \in \mathcal{A} : \left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{1.1}$$

We will prove in the followings that $S^{***} \subset \mathcal{K}$, we will determine the order of starlikeness of the class S^{***} and we will show that if

$$f, g \in S^{***}, \ \text{then} \ f \circ g \text{ is starlike in the disk } U(r_0),$$

where $r_0 = \sup\{r > 0 | g(U(r)) \subset U\}$.

2. Preliminaries

In order to prove the Main Result, we need the following results. These lemmas can be found in [1], p.24-25, and [2], p. 201-203.
Let Q be the class of analytic functions q in U which has the property that are analytic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \longrightarrow \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

Lemma 2.1. [Miller-Mocanu] *Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If $f \not\prec q$, then there are two points $z_0 = r_0 e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \setminus E(q)$ and a real number $m \in [n, \infty)$ for which $p(U_{r_0}) \subset q(U)$,*
(i) $p(z_0) = q(\zeta_0)$
(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$
(iii) $\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re}\left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right).$

The following result is a particular case of Lemma 2.1.

Lemma 2.2. [Miller-Mocanu] *Let $p(z) = 1 + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv 1$ and $n \geq 1$.*
If $\operatorname{Re} p(z) \not> 0,\ z \in U$, then there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that
(i) $p(z_0) = ix$
(ii) $z_0 p'(z_0) = y \leq -\frac{n(x^2+1)}{2}$,
(iii) $\operatorname{Re} z_0^2 p''(z_0) + z_0 p'(z_0) \leq 0$.

We also need the following result, which is a particular case of the Theorem 3.2d. from [1]. The next result is Theorem 3.2i. from [1].

Lemma 2.3. *Let h be convex in U, with $h(0) = 1$ and let n be a positive integer. If q is the analytic solution of*

$$q(z) + \frac{zq'(z)}{q(z)} = h(z), \quad q(0) = 1,$$

and if $\operatorname{Re} q(z) > 0,\ \ z \in U$, then q is univalent. If $p(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \dots$ is an analytic function in U, and

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z),$$

then $p \prec q$, and q is the best dominant.

We also need the following result, which is a particular case of the Theorem 3.2d. from [1].

Lemma 2.4. *Let $\beta, \gamma \in \mathbb{C}$ and let n be a positive integer. Let $R_{\beta a+\gamma,n}$ be given by*

$$R_{c,n}(z) = 2C_n \frac{(z+b)(1+\overline{b}z)}{(1+\overline{b}z)^2 - (z+b)^2}, \quad C_n = \frac{n}{\operatorname{Re} c}[|c|\sqrt{1+2\operatorname{Re}(c/n)} + \operatorname{Im} c].$$

Let h be analytic in U, with $h(0) = a$, and let $\operatorname{Re}[\beta a + \gamma] > 0$. *If*

$$\beta h(z) + \gamma \prec R_{\beta a+\gamma,n(z)},$$

then the solution q of the equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z),$$

with $q(0) = a$ is analytic in U and satisfies $\operatorname{Re}[\beta q(z) + \gamma] > 0,\ z \in U$.
If $a \neq 0$, then the solution is given by

$$q(z) = z^{\gamma/n}[H(z)]^{\beta a/n} \left((\beta/n) \int_0^z [H(t)]^{\beta a/n} t^{(\gamma/n)-1} dt \right)^{-1} - \gamma/\beta,$$

where

$$H(z) = z \exp \int_0^z [(h(t) - a)/at] dt.$$

Lemma 2.5. *If $x > 0$, and $y \in \mathbb{R}$, then*

$$\operatorname{Re}(x+iy)^{\frac{1}{4}} \geq x^{\frac{1}{4}}.$$

Proof. We have

$$\operatorname{Re}(x+iy)^{\frac{1}{4}} = (x^2+y^2)^{\frac{1}{8}} \cos\left(\frac{1}{4}\arctan\frac{y}{x}\right)$$

and in order to prove the lemma we have to show that

$$(x^2+y^2)^{\frac{1}{2}} \cos^4\left(\frac{1}{4}\arctan\frac{y}{x}\right) \geq x. \tag{2.1}$$

Since

$$\cos^4\left(\frac{1}{4}\arctan\frac{y}{x}\right) = \frac{1}{4}\left(1 + \sqrt{\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{2}}\right)^2$$

the inequality (2.1) is equivalent to

$$\frac{1}{4}\left(1 + \sqrt{\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{2}}\right)^2 \geq \frac{x}{\sqrt{x^2+y^2}}.$$

Since $\frac{x}{\sqrt{x^2+y^2}} \in [0,1]$ it follows that there is a real number $\alpha \in [0, \frac{\pi}{2}]$ such that $\frac{x}{\sqrt{x^2+y^2}} = \cos\alpha$, and the previous inequality can be rewritten as follows

$$\cos^4 \frac{\alpha}{4} \geq \cos\alpha, \quad \alpha \in \left[0, \frac{\pi}{2}\right].$$

This inequality is equivalent to

$$\left(1 - \cos^2 \frac{\alpha}{4}\right)\left(7\cos^2 \frac{\alpha}{4} - 1\right) \geq 0, \ \alpha \in \left[0, \frac{\pi}{2}\right],$$

and the proof is done taking into account that $7\cos^2 \frac{\alpha}{4} \geq 7\cos^2 \frac{\pi}{8} > 1$. □

We need also the following lemma which can be found in [2], p.271.

Lemma 2.6. *Let* $g : [-\pi, \pi][0,1] \to \mathbb{C}$ *a function such that* $g(e^{i\theta}, .)$ *is integrable on* $[0,1]$, *for each* $\theta \in [-\pi, \pi]$. *If* $\alpha : [0,1] \to (0, \infty)$ *ia also integrable and*

$$\mathrm{Re}\, \frac{1}{g(e^{\theta}, t)} \geq \frac{1}{\alpha(t)}, \ \ \theta \in [-\pi, \pi], t \in [0,1],$$

then

$$\mathrm{Re}\, \frac{1}{\int_0^1 g(e^{i\theta}, t)dt} \geq \frac{1}{\int_0^1 \alpha(t)dt}, \ \ \theta \in [-\pi, \pi].$$

3. Main results

Theorem 3.1. *If* $f \in \mathcal{A}$ *and*

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{7}, \ \ z \in U,$$

then it follows that $f \in S^*$.

Proof. We will prove that $p(z) = \frac{zf'(z)}{f(z)} > 0, \quad z \in U.$
It is easily seen that

$$1 - \frac{zf''(z)}{f'(z)} = 2 - p(z) - \frac{zp'(z)}{p(z)},$$

and consequently the following equivalence holds

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{7}, \ \ z \in U \Leftrightarrow \left|2 - p(z) - \frac{zp'(z)}{p(z)}\right| < \sqrt{7}, \ \ z \in U. \tag{3.1}$$

If the condition $p(z) = \dfrac{zf'(z)}{f(z)} > 0, \quad z \in U$, does not hold, then according to the Miller-Mocanu lemma (Lemma 2.2) there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

$$p(z_0) = ix,$$

and

$$z_0 p'(z_0) = y \leq -\frac{1+x^2}{2}.$$

These equalities imply

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - i\left(x - \frac{y}{x}\right)\right| = \sqrt{4 + \frac{(x^2-y)^2}{x^2}}$$
$$\geq \sqrt{4 + \left(\frac{3x}{2} + \frac{1}{2x}\right)^2} \geq \sqrt{7}.$$

This inequality contradicts (3.1), and consequently $p(z) = \dfrac{zf'(z)}{f(z)} > 0,\ z \in U$ holds. □

Remark 3.2. The result of Theorem 3.1 shows that the following inclusion holds $S^{***} \subset S^*$.

We will determine the exact order of starlikeness of the class S^{***} in the followings.

Theorem 3.3. *If $f \in S^{***}$, then*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{\displaystyle\int_0^1 \left(\frac{\sqrt{\frac{5}{4}} - t}{\sqrt{\frac{5}{4}} - 1}\right)^{\frac{1}{4}} dt} = \frac{5}{4}\,\frac{\left(\sqrt{\frac{5}{4}} - 1\right)^{\frac{1}{4}}}{\left(\sqrt{\frac{5}{4}}\right)^{\frac{5}{4}} - \left(\sqrt{\frac{5}{4}} - 1\right)^{\frac{5}{4}}},\ z \in U.$$

The result is sharp.

Proof. The inequality $\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U$ is equivalent to the subordination

$$1 - \frac{zf''(z)}{f'(z)} \prec M\frac{Mz+1}{M+z},$$

where $M = \sqrt{\frac{5}{4}}$. Denoting $p(z) = \frac{zf'(z)}{f(z)}$ the subordination can be rewritten in the following form

$$2 - p(z) - \frac{zp'(z)}{p(z)} \prec M\frac{Mz+1}{M+z},$$

and this is equivalent to

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z) = 2 - M\frac{Mz+1}{M+z}. \tag{3.2}$$

If we denote by q the solution of the equation

$$q(z) + \frac{zq'(z)}{q(z)} = 2 - M\frac{Mz+1}{M+z} = h(z)$$

then $\operatorname{Re} q(z) > 0,\ \ z \in U$. Indeed if the inequality $\operatorname{Re} q(z) > 0,\ \ z \in U$ does not holds, then there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

$$q(z_0) = ix,$$

and

$$z_0 q'(z_0) = y \leq -\frac{1+x^2}{2}.$$

These equalities imply

$$2 < \left|2 - ix - \frac{y}{ix}\right| = \left|2 - q(z_0) - \frac{z_0 q'(z_0)}{q(z_0)}\right| = \left|M\frac{Mz_0+1}{M+z_0}\right| < \sqrt{\frac{5}{4}},$$

which is a contradiction. Thus $\operatorname{Re} q(z) > 0, \quad z \in U$, and h is a convex function, consequently Lemma 2.3 is applicable and we get $p(z) \prec q(z)$.
According to Lemma 2.4 we have

$$q(z) = \frac{H(z)}{\displaystyle\int_0^z H(t)t^{-1}dt},$$

where $H(z) = \displaystyle\int_0^z \frac{h(t)-1}{t}dt = z\left(1+\frac{z}{M}\right)^{M^2-1}$.
The subordination $p \prec q$ implies that

$$\operatorname{Re} p(z) > \inf_{|z|<1} q(z) = \inf_{|z|<1} \frac{H(z)}{\displaystyle\int_0^z H(t)t^{-1}dt} = \inf_{\theta\in[-\pi,\pi]} \frac{H(e^{i\theta})}{\displaystyle\int_0^{e^{i\theta}} H(s)s^{-1}ds}. \tag{3.3}$$

On the other hand we have

$$\inf_{\theta\in[-\pi,\pi]} \frac{H(e^{i\theta})}{\displaystyle\int_0^{e^{i\theta}} H(s)s^{-1}ds} = \inf_{\theta\in[-\pi,\pi]} \frac{1}{\displaystyle\int_0^1 \left(\frac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{M^2-1} dt}. \tag{3.4}$$

A simple calculation leads to

$$\operatorname{Re}\frac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}} = \operatorname{Re}\frac{M+e^{i\theta}}{M+te^{i\theta}} \geq \frac{M-1}{M-t},\ t\in[0,1],\ \theta\in[-\pi,\pi].$$

This inequality implies

$$\left(\operatorname{Re}\frac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}}\right)^{\frac{1}{4}} \geq \left(\frac{M-1}{M-t}\right)^{\frac{1}{4}},\ t\in[0,1],\ \theta\in[-\pi,\pi].$$

Putting $x+iy = \dfrac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}}$ in Lemma 2.5, we infer

$$\operatorname{Re}\frac{1}{\left(\dfrac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{\frac{1}{4}}} \geq \left(\operatorname{Re}\frac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}}\right)^{\frac{1}{4}} \geq \left(\frac{M-1}{M-t}\right)^{\frac{1}{4}},\ t\in[0,1],\ \theta\in[-\pi,\pi].$$

Since $M^2 - 1 = \frac{1}{4}$, we get

$$\mathrm{Re}\,\frac{1}{\left(\dfrac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{M^2-1}} \geq \left(\frac{M-1}{M-t}\right)^{M^2-1},\ t\in[0,1],\ \theta\in[-\pi,\pi].$$

Now we can apply Lemma 2.6 and it follows that

$$\mathrm{Re}\,\frac{1}{\displaystyle\int_0^1\left(\frac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{M^2-1}dt} \geq \frac{1}{\displaystyle\int_0^1\left(\frac{M-t}{M-1}\right)^{M^2-1}dt},\ \theta\in[-\pi,\pi]. \tag{3.5}$$

Finally (3.3), (3.4) and (3.5) imply

$$\mathrm{Re}\,p(z) \geq \frac{1}{\displaystyle\int_0^1\left(\frac{M-t}{M-1}\right)^{M^2-1}dt} = \frac{5}{4}\,\frac{\left(\sqrt{\frac{5}{4}}-1\right)^{\frac{1}{4}}}{\left(\sqrt{\frac{5}{4}}\right)^{\frac{5}{4}} - \left(\sqrt{\frac{5}{4}}-1\right)^{\frac{5}{4}}}. \qquad \square$$

Theorem 3.4. *We have $S^{***} \subset \mathcal{K}$.*

Proof. Let f be a function from the class S^{***}.
We will prove that $p(z) = 1 + \frac{zf''(z)}{f'(z)} > 0, \quad z \in U$.
It is easily seen that

$$1 - \frac{zf''(z)}{f'(z)} = 2 - p(z),$$

and consequently the following equivalence holds

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z\in U \Leftrightarrow |2-p(z)| < \sqrt{\frac{5}{4}},\ z\in U. \tag{3.6}$$

If the condition $p(z) = 1 + \frac{zf''(z)}{f'(z)} > 0, \quad z \in U$, does not hold, then according to the Miller-Mocanu lemma (Lemma 2.2) there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

$$p(z_0) = ix,$$

and

$$z_0 p'(z_0) = y \leq -\frac{1+x^2}{2}.$$

These equalities imply

$$|2 - p(z_0)| = |2 - ix| = \sqrt{4+x^2} > \sqrt{\frac{5}{4}}.$$

This inequality contradicts (3.6), and consequently

$$\mathrm{Re}\ p(z) = \mathrm{Re}\ \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U$$

holds. $\square$

Theorem 3.5. *If* $f \in S^{***}$, *then* $\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}$, $z \in U$.

Proof. The inequality $\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}$, $z \in U$ is equivalent to

$$p(z) = \frac{zf'(z)}{f(z)} \prec \sqrt{\frac{1+z}{1-z}} = q(z), \quad z \in U. \tag{3.7}$$

We will prove the subordination (3.7) using again the Miller-Mocanu lemma.
If the subordination (3.7) does not hold, then according to Lemma 2.1 there are two points $z_0 \in U$ and $\zeta_0 = e^{i\theta} \in \partial U$, and a real number $m \in [1, \infty)$, such that

$$p(z_0) = q(\zeta_0) = q(e^{i\theta}) = \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}} = \left(\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}\right)x,$$

where $x = \sqrt{|\cot\frac{\theta}{2}|}$, and

$$\frac{z_0 p'(z_0)}{p(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = m\frac{e^{i\theta}}{1-e^{2i\theta}}.$$

According to (3.6) the function f belongs to the class S^{***} if and only if

$$\left|2 - p(z) - \frac{zp'(z)}{p(z)}\right| < \sqrt{\frac{5}{4}}, \; z \in U. \tag{3.8}$$

On the other hand we have

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - q(\zeta_0) - m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right| = \left|2 - \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}} - m\frac{e^{i\theta}}{1-e^{2i\theta}}\right|$$
$$= \left|2 - \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}} - m\frac{i}{2\sin\theta}\right| = \left|2 - \sqrt{i\cot\frac{\theta}{2}} - m\frac{i}{2\sin\theta}\right|, \; \theta \in [-\pi, \pi].$$

Denoting $x = \sqrt{\left|\cot\frac{\theta}{2}\right|}$, it follows that $x \in (0, \infty)$, and in case $\theta \in [-\pi, 0]$, we have

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)x + im\frac{x^4+1}{4x^2}\right|$$
$$= \sqrt{\left(2 - \frac{x}{\sqrt{2}}\right)^2 + \left(\frac{x}{\sqrt{2}} + m\frac{x^4+1}{4x^2}\right)^2}. \tag{3.9}$$

If $\theta \in [0, \pi]$, then

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)x - im\frac{x^4+1}{4x^2}\right|$$
$$= \sqrt{\left(2 - \frac{x}{\sqrt{2}}\right)^2 + \left(\frac{x}{\sqrt{2}} + m\frac{x^4+1}{4x^2}\right)^2}. \tag{3.10}$$

Thus we get

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \sqrt{\left(2 - \tfrac{x}{\sqrt{2}}\right)^2 + \left(\tfrac{x}{\sqrt{2}} + m\tfrac{x^4+1}{4x^2}\right)^2}$$
$$\geq \sqrt{\left(2 - \tfrac{x}{\sqrt{2}}\right)^2 + \left(\tfrac{x}{\sqrt{2}} + \tfrac{x^4+1}{4x^2}\right)^2}. \qquad (3.11)$$

The inequality between the arithmetic and geometric means implies

$$\frac{x}{\sqrt{2}} + \frac{x^4+1}{4x^2} = \frac{x}{2\sqrt{2}} + \frac{x}{2\sqrt{2}} + \frac{x^2}{4} + \frac{1}{8x^2} + \frac{1}{8x^2} \geq \frac{5}{2^{\frac{6}{5}}} > 2. \qquad (3.12)$$

Finally (3.11) and (3.12) imply that

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| > \sqrt{\frac{5}{4}}.$$

This inequality contradicts (3.8) and consequently

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}, \; z \in U. \qquad \square$$

Theorem 3.6. *If $f \in S^{***}$, then $|\arg f'(z)| < \frac{\pi}{4}, \; z \in U$.*

Proof. The inequality $|\arg f'(z)| < \frac{\pi}{4}, \; z \in U$ is equivalent to

$$f'(z) \prec \sqrt{\frac{1+z}{1-z}} = q(z), \quad z \in U. \qquad (3.13)$$

If the subordination (3.13) does not hold, then according to Lemma 2.1 there are two points $z_0 \in U$ and $\zeta_0 = e^{i\theta} \in \partial U$, and a real number $m \in [1, \infty)$, such that

$$f'(z_0) = q(\zeta_0) = q(e^{i\theta}) = \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}},$$

$$\frac{z_0 f''(z_0)}{f'(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = m\frac{e^{i\theta}}{1-e^{2i\theta}} = \frac{im}{2\sin\theta}.$$

Thus we get

$$\left|1 - \frac{z_0 f''(z_0)}{f'(z_0)}\right| = \left|1 - \frac{im}{2\sin\theta}\right| = \sqrt{1 + \left(\frac{m}{2\sin\theta}\right)^2} \geq \sqrt{1 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}}.$$

This inequality contradicts $f \in S^{***}$. The contradiction implies that the subordination (3.13) holds, and the proof is done. $\square$

Now we are able to prove the result proposed in the Introduction regarding the composition of functions.

Theorem 3.7. *If $f, g \in S^{***}$, and $r_0 = \sup\{r \in (0,1] \big| f(U(r)) \subset U\}$, then $f \circ g$ will be starlike in $U(r_0)$.*

Proof. We have

$$\frac{z(f\circ g)'(z)}{(f\circ g)(z)} = \frac{zf'(g(z))}{f(g(z))}f'(z). \tag{3.14}$$

If $f, g \in S^{***}$, then Theorem 3.5 and Theorem 3.6 imply the inequalities

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4},\ z\in U,$$

and

$$|\arg f'(z)| < \frac{\pi}{4},\ z\in U.$$

The equality (3.14) implies that

$$\arg\frac{z(f\circ g)'(z)}{(f\circ g)(z)} = \arg\frac{zf'(g(z))}{f(g(z))} + \arg f'(z).$$

Thus we get

$$\left|\arg\frac{z(f\circ g)'(z)}{(f\circ g)(z)}\right| \le \left|\arg\frac{zf'(g(z))}{f(g(z))}\right| + |\arg f'(z)| \le \frac{\pi}{2},\quad z\in U(r_0).$$

This inequality means that

$$\mathrm{Re}\,\frac{z(f\circ g)'(z)}{(f\circ g)(z)} > 0,\ z\in U(r_0),$$

and consequently $f\circ g$ is starlike in $U(r_0)$. □

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Olga Engel
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca, Romania
e-mail: engel_olga@hotmail.com

Róbert Szász
Sapientia Hungarian University of Transilvania
Faculty of Tehnical and Human Sciences
1C, Corunca, 540485 Târgu-Mureş, Romania
e-mail: szasz_robert2001@yahoo.com