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Majorization for certain classes of analytic functions defined by convolution structure

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Abstract. In this paper, we investigate majorization properties for certain classes of analytic functions defined by convolution structure. Also we point out some new and known consequences of our main result.

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1. Introduction

Let $f(z)$ and $g(z)$ be analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

For analytic function $f(z)$ and $g(z)$ in U , we say that $f(z)$ is majorized by $g(z)$ in U (see [10]) and write

$$f(z) << g(z) \quad (z \in U), \tag{1.1}$$

if there exists a function $\varphi(z)$, analytic in U such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U). \tag{1.2}$$

It may be noted that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

If $f(z)$ and $g(z)$ are analytic functions in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, (see [11, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $A(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (\ p \in \mathbb{N} = \{1, 2, ...\}) \tag{1.3}$$

which are analytic and $p-$valent in the open unit disc. We note that $A(1)=A$. Let $g(z)\in A(p)$, be given by

$$g(z)=z^{p}+\sum_{k=p+1}^{\infty}b_{k}z^{k},$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z^{p}+\sum_{k=p+1}^{\infty}a_{k}b_{k}z^{k}=(g*f)(z). \tag{1.4}$$

For $\lambda,\ell\geqslant 0, p\in\mathbb{N}, m\in\mathbb{N}_{0}=\mathbb{N}\cup\{0\}$ and $f(z),\ g(z)\in A(p)$, A. O. Mostafa, [12] defined the linear operator $D^{m}_{\lambda,\ell,p}(f*g)$ as follows:

$$D^{m}_{p,\ell,\lambda}(f*g)=z^{p}+\sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{m}a_{k}b_{k}z^{k}. \tag{1.5}$$

From (1.5), it is easy to verify that (see [12]),

$$\lambda z\left(D^{m}_{\lambda,\ell,p}(f*g)(z)\right)'=(\ell+p)\,D^{m+1}_{\lambda,\ell,p}(f*g)(z)-\left[p(1-\lambda)+\ell\right]D^{m}_{\lambda,\ell,p}(f*g)(z). \tag{1.6}$$

We note that:

(i) For $b_{k}=1$ or $g(z)=\frac{z^{p}}{1-z}$ we have $D^{m}_{\lambda,\ell,p}f(z)=I^{m}_{p}(\lambda,\ell)f(z)$, where the operator $I^{m}_{p}(\lambda,\ell)$ was introduced and studied by Cătaş [4], which contains intern the operators D^{m}_{p}, (see [2] and [8]) and D^{m}_{λ} (see [1]).

(ii) For $b_{k}=\frac{(\alpha_{1})_{k-p}\ldots(\alpha_{q})_{k-p}}{(\beta_{1})_{k=p}\ldots(\beta_{s})_{k-p}(1)_{k-p}}$, the operator

$$D^{m}_{\lambda,\ell,p}(f*g)(z)=I^{m,\ell}_{p,q,r,\lambda}(\alpha_{1},\beta_{1})f(z),$$

where the operator $I^{m,\ell}_{p,q,r,\lambda}(\alpha_{1},\beta_{1})f(z)$ was introduced and studied by El-Ashwah and Aouf [6], $\alpha_{1},\alpha_{2},...,\alpha_{q}$ and $\beta_{1},\beta_{2},...,\beta_{s}$ are real or complex number ($\beta_{j}\in\mathbb{C}\backslash\mathbb{Z}_{0}^{-}=\{0,-1,-2,...\};j=1,...,s;$)($q\leq s+1;q,s\in\mathbb{N}_{0},p\in\mathbb{N};z\in U$) and

$$(\theta)_{\nu}=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1 & (\nu=0;\theta\in\mathbb{C}^{*}=\mathbb{C}\backslash\{0\}),\\ \theta(\theta-1)...(\theta+\nu-1) & (\nu\in\mathbb{N};\theta\in\mathbb{C}).\end{cases}$$

Also, for many special operators of the operator $I^{m,\ell}_{p,q,r,\lambda}(\alpha_{1},\beta_{1})f(z)$ (see [6]).

(iii) For $m=0$, $b_{k}=\frac{(\alpha_{1})_{k-p}\ldots(\alpha_{q})_{k-p}}{(\beta_{1})_{k=p}\ldots(\beta_{s})_{k-p}(1)_{k-p}}$, the operator

$$D^{m}_{\lambda,\ell,p}(f*g)(z)=S^{j}_{p,q,s}(\gamma;\alpha_{1})f(z),$$

where the operator $S^{j}_{p,q,s}(\gamma;\alpha_{1})f(z)$, was introduced and studied by El-Ashwah [5].

(iv) For $m=0$ and $b_{k}=\frac{\Gamma(p+\alpha+\beta)\Gamma(k+\beta)}{\Gamma(p+\beta)\Gamma(k+\alpha+\beta)}$, the operator $D^{m}_{p,\ell,\lambda}(f*g)(z)=Q^{\alpha}_{p,\beta}(f)$ ($\alpha\geq 0,\beta>-1,p\in\mathbb{N}$), where the operator $Q^{\alpha}_{p,\beta}$ was introduced by Liu and Owa [9].

For $h(z)$ given by

$$h(z)=z^{p}+\sum_{k=p+1}^{\infty}c_{k}z^{k}$$

A function $f(z) \in A(p)$ is said to be in the class $S^{m,j}_{\lambda,\ell,p}(\gamma)$ of $p-$valent functions of complex order $\gamma \neq 0$ in U, if and only if

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(D^m_{\lambda,\ell,p}(f*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}}-p+j\right)\right\}>0$$

$$(p \in \mathbb{N};\; j \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}; \ell,\lambda \geq 0; \gamma \in \mathbb{C}^*;\; z \in U). \tag{1.7}$$

Clearly, we have the following relationships:

(i) $S^{0,0}_{\lambda,\ell,1}(\gamma) = S(\gamma)(\gamma \in \mathbb{C}^*)$,

(ii) $S^{0,1}_{\lambda,\ell,1}(\gamma) = \kappa(\gamma)\ (\gamma \in \mathbb{C}^*)$,

(iii) $S^{0,0}_{\lambda,\ell,1}(1-\alpha) = S^*(\alpha)\ (0 \leq \alpha < 1)$.

The classes $S(\gamma)$ and $\kappa(\gamma)$ are classes of starlike and convex functions of complex order $\gamma \neq 0$ in U which were studied by Nasr and Aouf [13] and $S^*(\alpha)$ is the class of starlike functions of order α in U.

Also, for $m = 0$ the operator $S^j_p(h;\gamma)$ was introduced and studied by El-Ashwah and Aouf [7].

Definition 1.1. Let $-1 \leq B < A \leq 1, p \in \mathbb{N}; j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*$,

$$|\gamma(A-B)+(p-j)B| < (p-j),\ f \in A(p).$$

Then $f \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$, the class of $p-$valent functions of complex order γ in U if and only if

$$\left\{1+\frac{1}{\gamma}\left(\frac{z(D^m_{\lambda,\ell,p}(f*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}}-p+j\right)\right\} \prec \frac{1+Az}{1+Bz}. \tag{1.8}$$

A majorization problem for the subclasses of analytic function has recently been investigated by Altintas et al. [3] and MacGregor [11]. In this paper we investigate majorization problem for the class $S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$ and some related subclasses.

2. Main results

Unless otherwise mentioned we shall assume throughout the paper that, $-1 \leq B < A \leq 1$, $\gamma \in \mathbb{C}^*$, $\ell, \lambda \geq 0$, $p \in \mathbb{N}$ and $m, j \in \mathbb{N}_0$.

Theorem 2.1. *Let the function $f \in A(p)$ and suppose that $g \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$. If $(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}$ is majorized by $(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}$ in U, then*

$$\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right| \leq \left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right| \qquad (|z| < r_1), \tag{2.1}$$

where $r_1 = r_1(p, \gamma, \lambda, \ell, A, B)$ is the smallest positive root of the equation

$$|\gamma\lambda(A-B)+(p+\ell)B|\, r^3 - [2\lambda\,|B| + (p+\ell)]\, r^2 -$$

$$[|\gamma\lambda(A-B)+(p+\ell)B| + 2\lambda]\, r + (p+\ell) = 0. \tag{2.2}$$

Proof. Since $(g*h)(z) \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$, we find from (1.8) that

$$1+\frac{1}{\gamma}\left(\frac{z(D^m_{\lambda,\ell,p}(g*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}}-p+j\right) = \frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$. From (2.3), we have

$$\frac{z(D^m_{\lambda,\ell,p}(g*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}}=\frac{(p-j)+[\gamma(A-B)+(p-j)B]w(z)}{1+Bw(z)}. \tag{2.4}$$

In view of

$$\begin{aligned}\lambda z\left(D^m_{\lambda,\ell,p}(f*g)(z)\right)^{(j+1)}&=(p+\ell)\left(D^{m+1}_{\lambda,\ell,p}(f*g)(z)\right)^{(j)}\\ &-[p(1-\lambda)+\lambda j+\ell]\left(D^m_{\lambda,\ell,p}(f*g)(z)\right)^{(j)}\\ &0\le j\le p;\ p\in\mathbb{N},\lambda>0;\ z\in U,\end{aligned} \tag{2.5}$$

(2.4) immediately yields the following inequality:

$$\left|(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|\le\frac{(p+\ell)(1+|B|\,|z|)}{(p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,|z|}\left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|. \tag{2.6}$$

Next, since $(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}$ is majorized by $(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}$ in U, from (1.2), we have

$$(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}=\varphi(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}. \tag{2.7}$$

Differentiating (2.7) with respect to z, we have

$$z(D^m_{\lambda,\ell,p}(f*h)(z))^{(j+1)}=z\varphi^{'}(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}+z\varphi(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j+1)}. \tag{2.8}$$

From (2.5) and (2.8), we have

$$(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}=\frac{\lambda z}{p+\ell}\varphi^{'}(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}+\varphi(z)(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}. \tag{2.9}$$

Thus, by noting that $\varphi(z)$ satisfies the inequality (see [14]),

$$\left|\varphi^{'}(z)\right|\le\frac{1-|\varphi(z)|^2}{1-|z|^2}\quad(z\in U),$$

we see that

$$\begin{aligned}&\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right|\\ &\le\left(|\varphi(z)|+\frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{\lambda\,|z|\,(1+|B|\,|z|)}{(p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,|z|}\right)\left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|,\end{aligned} \tag{2.10}$$

which upon setting

$$|z|=r\quad\text{and}\quad|\varphi(z)|=\rho\quad(0\le\rho\le1),$$

leads us to the inequality

$$\begin{aligned}&\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right|\\ &\le\frac{\Theta(\rho)}{(1-r^2)((p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,r)}\left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|,\end{aligned}$$

where

$$\begin{aligned}\Theta(\rho)=-r\lambda\,(1+|B|\,r)\,\rho^2+(1-r^2)\,[(p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,r]\,\rho\\ +r\lambda\,(1+|B|\,r)\,,\end{aligned} \tag{2.11}$$

takes its maximum value at $\rho = 1$, with $r_1 = r_1(p,\gamma,\lambda,\ell,A,B)$, where $r_1(p,\gamma,\lambda,\ell,A,B)$ is the smallest positive root of (2.2). Therefore the function $\Phi(\rho)$ defined by

$$\Phi(\rho) = -\sigma\lambda\,(1+|B|\,\sigma)\,\rho^2 + (1-\sigma^2)\,[(p+\ell) - |\gamma\lambda(A-B) + (p+\ell)\,B|\,\sigma]\,\rho \\ +\sigma\lambda\,(1+|B|\,\sigma) \tag{2.12}$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\Phi(\rho) \le \Phi(1) = (1-\sigma^2)\,[(p+\ell) - |\gamma(A-B) + (p+\ell)B|\,\sigma] \tag{2.13}$$
$$(0 \le \rho \le 1;\ 0 \le \sigma \le r_0(p,\gamma,j,A,B))\,.$$

Hence upon setting $\rho = 1$ in (2.12), we conclude that (2.1) holds true for $|z| \le r_1 = r_1(p,\gamma,\lambda,\ell,A,B)$, where $r_1(p,\gamma,\lambda,\ell,A,B)$, is the smallest positive root of (2.2). This completes the proof of Theorem 1.

Putting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following result.

Corollary 2.2. *Let the function $f \in A(p)$ and suppose that $g \in S^{m,j}_{\lambda,\ell,p}(\gamma)$.*

If $(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}$ is majorized by $(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}$ in U, then

$$\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right| \le \left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right| \qquad (|z| < r_1)\,,$$

where $r_1 = r_1(p,\gamma,\lambda,\ell)$ is given by

$$r_1 = r_1(p,\gamma,\lambda,\ell) = \frac{k - \sqrt{k^2 - 4(p+\ell)\,|2\gamma\lambda - (p+\ell)|}}{2\,|2\gamma\lambda - (p+\ell)|}, \tag{2.14}$$

where $k = 2\lambda + (p+\ell)) + |2\gamma\lambda - (p+\ell))|\,.$

Putting $A = 1$, $B = -1$ and $p = j = 1$ in Theorem 1, we obtain the following result.

Corollary 2.3. *Let the function $f \in A$ and suppose that $g \in S^{m,0}_{\lambda,\ell}(\gamma)$.*

If $(D^m_{\lambda,\ell}(f*h)(z))$ is majorized by $(D^m_{\lambda,\ell}(g*h)(z))$ in U, then

$$\left|(D^{m+1}_{\lambda,\ell}(f*h)(z))\right| \le \left|(D^{m+1}_{\lambda,\ell}(g*h)(z))\right| \qquad (|z| < r_2)\,,$$

where $r_2 = r_2(\gamma,\lambda,\ell)$ is given by

$$r_2 = r_2(\gamma,\lambda,\ell) = \frac{k - \sqrt{k^2 - 4(1+\ell)\,|2\gamma\lambda - (1+\ell)|}}{2\,|2\gamma\lambda - (1+\ell)|}, \tag{2.15}$$

where $k = 2\lambda + (1+\ell)) + |2\gamma\lambda - (1+\ell))|\,.$

Putting $A = \lambda = 1$, $B = -1, m = \ell = 0$, and $h(z) = \frac{z^p}{1-z}$ (or $c_{k+p} = 1$) in Theorem 1, we obtain the following result.

Corollary 2.4. *Let the function $f \in A(p)$* and suppose that $g \in S_p$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \le |g'(z)| \quad (|z| < r_3)\,,$$

where $r_3 = r_3(p,\gamma)$ is given by

$$r_3 = r_3(p;\gamma) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma - p|}}{2\,|2\gamma - p|},$$

where $k = 2 + p + |2\gamma - p|\,.$

Putting $\gamma = 1$ in Corollary 3, we obtain the following result.

Corollary 2.5. *Let the function* $f \in A(p)$ and suppose that $g \in S_p(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_4),$$

where r_4 is given by

$$r_4 = r_4(p) = \frac{k - \sqrt{k^2 - 4p\,|2-p|}}{2\,|2-p|},$$

where $k = 2 + p + |2-p|$

Remarks 2.6. (i) Putting $p = 1$ in Corollary 3 we obtain the results obtained by Altintas et al. [3],

(ii) Putting $p = 1$ in Corollary 4 we obtain the results obtained by MacGregor [10].

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