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A class of diffusion problem of Kirchhoff type with viscoelastic term involving the fractional Laplacian

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Abstract. This work is concerned with a class of diffusion problem of Kirchhoff type with viscoelastic term and nonlinear interior source in the setting of the fractional Laplacian. Under suitable conditions we prove the existence of global solutions and the exponential decay of the energy.

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1. Introduction

We consider the problem of finding $u = u(x,t)$ weak solutions to the following nonlinear heat equation of Kirchhoff type with variable exponent of nonlinearity, viscoelastic term and source term involving the fractional Laplacian

$$\begin{aligned} \left(1 + a|u|^{r(x)-2}\right) u_t \ +M(\|u\|^2_{w_0})(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau & \\ = |u|^{\rho-1} \quad \text{in } \Omega\times]0,\infty[, & \\ u = 0 \quad \text{in } \left(\mathbb{R}^N\setminus\Omega\right)\times[0,\infty[, & \\ u(x,0) = u^0(x) \quad \text{in } \Omega, & \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $M(t) = t^{\alpha-1} + 1$, $t \geq 0$, $s \in]0,1[$, $2 < \rho < 2^*_s = \frac{2N}{N-2s}$, $2 < \frac{N}{s}$, $\alpha > 1$; $g : [0,\infty[\to]0,\infty[$ belongs to $C^1([0,\infty[)$, $g(0) > 0$, $l = 1 - \displaystyle\int_0^\infty g(\tau)d\tau > 0$, $g'(t) \leq 0$ and r is a given continuous function.

This type of problems without viscoelastic term (that is $g = 0$), $r(x) = constant$ and $M(t) = 1$ have been considered by many authors with the standard Laplace operator

$(-\triangle)^s, s = 1$ and can be seen as special case of doubly nonlinear parabolic type equations

$$(\varphi(u))_t - \Delta u = f(u),$$

which appear in the mathematical modelling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, glaciology, see [3, 8, 7, 20, 33, 52] and the further references therein.
When $a = 0$, $M(t) = 1$ and $s = 1$, equation (1.1) is reduced to the following equation

$$u_t - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) d\tau = f(u). \tag{1.2}$$

This equation arises from the study of heat conduction in materials with memory. The questions of solvability and the long time behavior of solutions of the abstract evolutions equations of type

$$u_t - Bu + \int_0^t g(t-\tau)Au(\tau) d\tau = f(u),$$

where A and B are given operators, were studied in [12, 19, 36, 40].
Also, doubly nonlinear nonlocal parabolic equations

$$(\varphi(u))_t - div\sigma(\nabla u) = \int_0^t g(t-\tau) div\sigma(\nabla u(\tau)) d\tau + f(x,t,u),$$

were studied in [9, 30, 47, 48, 49, 50].
On the other hand, many fractional and nonlocal operators are actively studied in the recent years. This type of operators arises in a quite natural way in many interesting applications, such as, finance, physics, game theory, Lévy stable diffusion processes, crystal dislocation, one can see [10, 35, 51] and their references. Some general motivations regarding the fractional Laplacian can be explicitly found in the recent monograph [17]. Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} (u(y,t) - u(x,t))\, K(x-y)\, dy, \tag{1.3}$$

and variations of it, have been widely used to model diffusion processes, more precisely as stated in [26], if $u(x,t)$ is thought as a density of population at the point x at time t and $K(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} (u(y,t)K(x-y))\, dy$ is the rate at which individuals are arriving at position x from all other places and $\int_{\mathbb{R}^N} (u(x,t)K(x-y))\, dy$ is the rate which they are leaving location x to travel to all other sites. So the density u satisfies (1.3). For recent references on nonlocal diffusion problems, see [5, 1, 29]. If we consider the effects of total population, then equation (1.3) becomes

$$u_t = M\left(\int\int_{\mathbb{R}^N} |u(y,t) - u(x,t)|^2\, K(x-y)\, dx\, dy\right) \int_{\mathbb{R}^N} (u(y,t) - u(x,t))\, K(x-y)\, dy. \tag{1.4}$$

In particular, if $s \to 1^-$ and $K(x) = |x|^{-N-2s}$, then equation reduces to

$$u_t = -M \left(\int \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u,$$

which is equation (1.2), with $M(t) = 1$, $g(t) = 0$ and $f(t) = 0$. Thus it is natural to consider equation (1.1) as a generalization of the model (1.4). The main feature of the equation (1.1) is that contains an integrodifferential operator usually called memory term or viscoelastic term, which can be used to represent the damping or memory effect on the diffusion process.

The research on nonlinear problems with variable exponent growth conditions is an an attractive topic, and these problems have many applications in nonlinear elastic electrorheological fluids and image restoration, see [2, 16, 18, 53].

The study of Kirchhoff type problems has been receiving considerable attention in more recent years, see [31, 38, 42, 41]. The interest arises from their contribution to the modeling of many physical and biological phenomena. We refer for example the reader to the bibliography [4, 6, 11, 32, 37] and references therein. The first result concerning fractional Kirchhoff problems was obtained in Fiscella and Valdinoci [27]. In this paper, the fractional Kirchhoff equation was first introduced and motivated.

In [42], by using the sub-differential approach, Pucci et al obtained the well-posedness of solutions for problem (1.1) with $f(x,t)$ instead of $|u|^{p-2}u$. Moreover, the large-time behavior and extinction of solutions also are considered. With the help of potential well theory, Fu and Pucci [28] studied the existence of global weak solutions and established the vacuum isolating and blow-up of strong solutions, provided that $M \equiv 1$ and $2 < p \leq 2_s^* = 2N/(N-2s)$. However, the Kirchhoff function M is assumed to satisfy the non-degenerate condition in the above papers. In [41], Pan et al investigated for the first time the existence of global weak solutions for degenerate Kirchhoff-type diffusion problems involving fractional p-Laplacian, by combining the Galerkin method with potential well theory, for the special function $M(t) = t$; Mingqi et al. [38] proved the local existence and blow-up of solutions for the similar equation with more general conditions on M which cover the degenerate case.

In the works mentioned above, there are few about the global existence and exponential decay rate for doubly nonlinear parabolic equation, involving variable exponent conditions, with viscoelastic term in the fractional setting. Motivated by it, we intend to study global existence for the problem (1.1) by using Galerkin's method and also give the exponential decay rate of the energy via the energy perturbation method.

The plan of the paper is the following. In Section 2, we give the preliminaries for our research. In Section 3, by using the Galerkin approximation method we are able to prove global existence and finally, we obtain the exponential decay under certain class of initial data.

2. Preliminaries

In this section, we present some materials and assumptions needed in the rest of this paper.

We denote: $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$,

$$W = \left\{ u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^2(\Omega), \iint\limits_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy < \infty \right\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we define the following linear subspace of W,

$$W_0 = \{u \in W : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

The linear space W is endowed with the norm

$$\|u\|_W := \|u\|_{L^2(\Omega)} + \Big(\iint\limits_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \Big)^{1/2}.$$

It is easily seen that $\|\cdot\|_W$ is a norm on W and $C_0^\infty(\Omega) \subseteq W_0$.
The functional

$$\|u\|_{W_0} = \left(\iint\limits_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2},$$

is a equivalent norm on $W_0 = \{u \in W \;:\; u(x) = 0 \; a.e. \text{ in } \mathbb{R}^N \setminus \Omega\}$ which is a closed linear subspace of W. Furthermore $(W_0, \|\cdot\|_{W_0})$ is a Hilbert space with inner product

$$\langle u, v\rangle_{W_0} = \iint\limits_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy.$$

We review the main embedding results for the space W_0.

Lemma 2.1 ([44, 43, 46, 45])**.** *The embedding $W_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, 2_s^*]$, and compact for any $r \in [1, 2_s^*[$.*

Lemma 2.2 ([39, Lemma 2.1])**.** *Let $N \geq 1$, $0 < s < 1$, $p > 1$, $q \geq 1$, $\tau > 0$ and $0 < \theta < 1$ be such that $\frac{1}{\tau} = \theta\left(\frac{1}{p} - \frac{s}{N}\right) + \frac{1-\theta}{q}$ then*

$$\|u\|_{L^\tau(\mathbb{R}^n)} \leq \|u\|^\theta_{W^{s,p}(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}, \qquad \forall u \in C_0^1(\mathbb{R}^N).$$

Now, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [21, 22, 25, 23] for details.
Set

$$C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}),\ p(x) > 1, \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \max\{p(x) : x \in \overline{\Omega}\}, \quad p^- = \min\{p(x); x \in \overline{\Omega}\};$$

$$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

with the norm

$$\|u\|_{p(x)} \equiv \|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

becomes a Banach space [34]. We also define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u(x)\|_{p(x)} + \|\nabla u(x)\|_{p(x)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = \|\nabla u\|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Proposition 2.3 ([24, 25])**.** *(i) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\int_\Omega |uv|dx \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(x)}\|v\|_{p'(x)} \le 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ and $p_1(x) \le p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.4 ([25])**.** *Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)}\, dx$, then for $u \in W_0^{1,p(x)}(\Omega)$ and $(u_k) \subset W_0^{1,p(x)}(\Omega)$, we have*

1. $\|u\| < 1$ *(respectively $= 1; > 1$) if and only if $\rho(u) < 1$ (respectively $= 1; > 1$);*
2. *for $u \ne 0$, $\|u\| = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
3. *if $\|u\| > 1$, then $\|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$;*
4. *if $\|u\| < 1$, then $\|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$;*
5. $\|u_k\| \to 0$ *(respectively $\to \infty$) if and only if $\rho(u_k) \to 0$ (respectively $\to \infty$).*

For $x \in \Omega$, let us define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.5 ([23])**.** *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Lemma 2.6. *Let $2 < r < \rho < 2_s^*$.For each $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\|v\|_\rho^\rho \le \varepsilon\|v\|_{W_0}^2 + C_\varepsilon\|v\|_r^{kr},$$

for all $v \in W_0 \cap L^r(\Omega)$ where

$$k = \frac{2\rho(1-\theta)}{r(2-\rho\theta)} \quad , \quad \theta = \Big(\frac{1}{r} - \frac{1}{\rho}\Big)\Big(\frac{s}{N} - \frac{1}{2} + \frac{1}{r}\Big)^{-1}.$$

Proof. The conclusion of lemma immediately follows from Lemma 2.2 and Young's inequality. □

Lemma 2.7. [34, Theorem 1, pag 23] *Suppose that*

$$r \in L^{\infty}_{+}(\Omega),\ r^{-} \geq 2,\ w \in L^{r(x)}\left(\Omega\times]0,T[\right)\ \text{and}\ \frac{\partial}{\partial t}(|w|^{r(x)-2}w) \in L^{r'(x)}\left(\Omega\times]0,T[\right).$$

Then, for any $s, \tau \in [0,T]$, $s < \tau$ the following formula of integration by parts is correct:

$$\int_{s}^{\tau}\int_{\Omega} w\Big(\frac{1}{r(x)-1}|w|^{r(x)-2}w\Big)\,dx\,dt = \int_{\Omega}\frac{1}{r(x)}|w(\tau)|^{r(x)}\,dx - \int_{\Omega}\frac{1}{r(x)}|w(s)|^{r(x)}\,dx.$$

3. Global existence and exponential decay

In this section, we focus our attention on the global existence and exponential decay of the solution to problem (1.1).

Definition 3.1. Let $T > 0$. A weak solution of problem (1.1) is a function $u \in L^{\infty}(0,T;W_0)$, with $u_t \in L^2(0,T;L^2(\Omega))$ and $\left(|u|^{r(x)/2}\right)_t \in L^2\left(\Omega\times]0,T[\right)$ such that

$$\int_0^T\int_{\Omega}\Big(1+a|u|^{r(x)-2}\Big)u_t w\,dxdt + M(\|u\|^2_{w_0})\int_0^T\langle u,w\rangle_{W_0}\,dt$$
$$-\int_0^T\int_0^t g(t-\tau)\langle u(\tau),w\rangle_{W_0}d\tau dt = \int_0^T\int_{\Omega}|u|^{\rho-1}w\,dxdt,$$

for all $w \in L^2(0,T;W_0)$, and $u(x,0) = u^0(x) \in W_0$.

Theorem 3.2 (Local Solution). *Assume $u^0 \in W_0$, $2 < r^- < \rho < 2^*_s$, $\rho < 2 + \frac{2rs}{N}$, $r^+ \in]2, 2^*_s[$, then problem (1.1) has a unique weak solution u for T small enough.*

Proof. We prove the local existence of weak solutions by using the Faedo-Galerkin method benefited from the ideas of [14]. We choose a sequence $\{w_\nu\}_{\nu\in\mathbb{N}} \subseteq C_0^{\infty}(\Omega)$ such that $C_0^{\infty}(\Omega) \subseteq \overline{\bigcup_{\nu=1}^{\infty} V_m}^{C^1(\overline{\Omega})}$ and $\{w_\nu\}$ is a standard orthonormal basis with respect to the Hilbert space $L^2(\Omega)$ and an orthogonal basis in W_0, where

$$V_m = spam\{w_1, w_2, ...w_m\}.$$

Now, we construct approximate solutions u_m $(m = 1, 2, \cdots)$, of the problem (1.1), in the form

$$u_m(x,t) = \sum_{i=1}^{m} g_{jm}(t)w_j(x),$$

where the coefficient functions g_{jm} satisfy the system of ordinary differential equations

$$\int_\Omega \left(1+a|u_m(t)|^{r(x)-2}\right) u_{mt}(t)w_j\,dx + M(\|u_m(t)\|_{w_0}^2)\langle u_m(t), w_j\rangle_{W_0}$$
$$-\int_0^t g(t-\tau)\langle u_m(\tau), w_j\rangle_{W_0} d\tau dt = \int_\Omega |u_m(t)|^{\rho-1} w_j\,dx \tag{3.1}$$
$$j=1,2,\cdots m.$$
$$u_m(x,0)=u_m^0(x)\to u^0(x) \qquad \text{in} \quad W_0.$$

Let us show that the system (3.1) is locally solvable.
It is clear that (3.1) can be rewritten in the form

$$\frac{d}{dt}\Phi(g_m(t)) = -M\Big(\Big\|\sum_{i=1}^m g_{jm}(t)w_j(x)\Big\|_{W_0}^2\Big)Bg_m(t)+\int_0^t g(t-\tau)Bg_m(\tau)d\tau+F(g_m(t)), \tag{3.2}$$

where

$$g_m(t)=(g_{m1}(t), g_{m2}(t),\cdots,g_{mm}(t))^t, \qquad B=[\langle w_i, w_j\rangle]_{1\le i,j\le m},$$
$$\Phi(\eta)=(\Phi_1(\eta),\Phi_2(\eta),\cdots,\Phi_m(\eta))^t \qquad \text{with } \eta=(\eta_1,\eta_2,\cdots,\eta_m)\in\mathbb{R}^m,$$
$$\Phi_i(\eta)=\int_\Omega \Big\{\sum_{j=1}^m \eta_j w_j + \frac{a}{r(x)-1}\Big|\sum_{k=1}^m \eta_k w_k\Big|^{r(x)-2}\sum_{k=1}^m \eta_k w_k\Big\} w_i\,dx \qquad i=1,2,\cdots,m$$

and

$$F(\eta)=\left(\int_\Omega\Big|\sum_{k=1}^m \eta_j w_j\Big|^{\rho-1} w_1\,dx, \int_\Omega\Big|\sum_{k=1}^m \eta_j w_j\Big|^{\rho-1} w_2\,dx,\cdots,\int_\Omega\Big|\sum_{k=1}^m \eta_j w_j\Big|^{\rho-1} w_m dx\right)^t.$$

This system is equivalent to

$$\Phi(g_m(t))=\Phi(g_m(0))$$
$$+\int_0^t\left[-M\left(\Big\|\sum_{i=1}^m g_{jm}(t)w_j(x)\Big\|_{W_0}^2\right)Bg_m(t)+\int_0^\xi g(\xi-\tau)Bg_m(\tau)d\tau+F(g_m(\xi))\right]d\xi.$$

If ζ,η are to arbitrary elements of $\mathbb{R}^m$, we get

$$(\Phi(\zeta)-\Phi(\eta),\zeta-\eta)_{\mathbb{R}^m}\ge C_m|\zeta-\eta|_{\mathbb{R}^m}^2 \tag{3.3}$$

here C_m is a constant such that, for any g_m in $\mathbb{R}^m$

$$\int_\Omega |u_m|^2\,dx \ge C_m|g_m|_{\mathbb{R}^m}^2.$$

Then Φ is monotone coercive. Also it is obviously continuous. So, by the Brouwer theorem Φ is onto. In view of (3.3), Φ^{-1} is locally Lipchitz continuous.

Consider the map $L : C(0,T,\mathbb{R}^m) \to C(0,T,\mathbb{R}^m)$, defined by

$$L(g_m)(t) = \Phi^{-1}\Bigg(\Phi(g_m(0)) + \int_0^t \Bigg[-M\Bigg(\Bigg\| \sum_{i=1}^m g_{jm}(t) w_j(x) \Bigg\|_{W_0}^2 \Bigg) Bg_m(t) + \int_0^\xi g(\xi-\tau) Bg_m(\tau) d\tau + F(g_m(\xi)) \Bigg] d\xi \Bigg), \quad t \in [0,T].$$

It is not hard to prove that L is completely continuous and also, there exist (sufficient small) $T_m > 0$ and (sufficient large) $R > 0$ such that $L(\overline{B_R}) \subseteq \overline{B_R}$, where $\overline{B_R}$ is the ball in $C(0,T_m,\mathbb{R}^m)$ with center the origin and radius R. Consequently, by Schauder's theorem, the operator L has a fixed point in $C(0,T_m,\mathbb{R}^m)$. This fixed point is a solution of (3.2).

So, we can obtain an approximate solution $u_m(t)$ of (3.1) in V_m over $[0,T_m[$ and it can be extended to the whole interval $[0,T]$, for all $T > 0$, as a consequence of the a priori estimates that shall be proven in the next step.

The First Estimate

Multiplying (3.1) by $g_{jm}(t)$ and adding in $j = 1;\cdots;m$, we have

$$\int_\Omega \left(1 + a|u_m(t)|^{r(x)-2}\right) u_{mt}(t) u_m(t)\, dx + M(\|u_m(t)\|_{w_0}^2) \langle u_m(t), u_m(t)\rangle_{W_0} - \int_0^t g(t-\tau)\langle u_m(\tau), u_m(t)\rangle_{W_0} d\tau dt = \int_\Omega |u_m(t)|^{\rho-1} u_m(t)\, dx \tag{3.4}$$

which implies, integrating with respect to the time variable from 0 to t on both sides, using Lemma 2.7 that

$$S_m(t) = S_m(0) + \int_0^t d\lambda \int_0^\lambda g(\lambda-\tau)\langle u_m(\tau), u_m(\lambda)\rangle_{W_0}\, d\tau + \int_0^t \int_\Omega |u_m(t)|^{\rho-1} u_m(\tau)\, dx\, d\tau, \tag{3.5}$$

where

$$S_m(t) = \int_\Omega |u_m(t)|^2\, dx + a\int_\Omega \frac{1}{r(x)} |u_m(t)|^{r(x)}\, dx + \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2 \right) d\tau.$$

Let us introduce the function $\Theta(\lambda) = \int_0^\lambda g(\lambda-\tau)\|u_m(\tau)\|_{W_0}$. Estimating the second term on right-hand side of (3.5) we have

$$\int_0^t d\lambda \int_0^\lambda g(\lambda-\tau)\langle u_m(\tau), u_m(\lambda)\rangle_{W_0}\, d\tau \le \frac{1}{2}\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2 \right) d\tau + \frac{1}{2}\int_0^t \Theta^2(\lambda)\, d\lambda. \tag{3.6}$$

But, using Young Inequality and noting that $\int_0^\infty g(\tau)d\tau < 1$, we get

$$\int_0^t \Theta^2(\lambda)d\lambda \leq \int_0^\infty g(\tau)d\tau \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau. \tag{3.7}$$

Plugging (3.6)- (3.7)into (3.5), it follows that

$$S_m(t) \leq S_m(0) + \frac{1}{2}\left(1 + \int_0^\infty g(\tau)d\tau\right)\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau$$
$$+ \int_0^t \|u_m(t)\|_\rho^\rho \, d\tau. \tag{3.8}$$

To estimate the last term in (3.8) we use Lemma 2.6,

$$\int_0^t \|u_m(t)\|_\rho^\rho \, d\tau \leq \varepsilon \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau + c_0 \int_0^t S_m^k(\lambda)\, d\lambda, \tag{3.9}$$

where $k = \frac{2\rho(1-\theta)}{r^-(2-\rho\theta)} > 1$. Taking ε suitably small in (3.9), it follows from (3.5)-(3.9) that

$$S_m(t) \leq \hat{C}_0 + \hat{C}_1 \int_0^t S_m^k(\lambda)\, d\lambda. \tag{3.10}$$

Hence, by employing Bihari-Langenhop's inequality (cf. [13]), there exists a constant T_0 such that

$$S_m(t) \leq C_{T_0}, \qquad \forall t \in [0, T_0]. \tag{3.11}$$

The Second Estimate

Multiplying (3.1) by $g'_{jm}(t)$ and adding in $j = 1; \cdots ; m$, it holds that

$$\frac{d}{dt}\Big\{\frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u_m(t)\|_{W_0}^2 + \frac{1}{2}(g \diamond u)(t)$$
$$-\frac{1}{\rho}\int_\Omega |u_m(t)|^{\rho-1}u_m(t)\, dx\Big\} + \|u_{mt}(t)\|_2^2 + a\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx$$
$$= \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u_m(t)\|_{W_0}^2. \tag{3.12}$$

where $\quad (g \diamond u)(t) = \int_0^t g(t-\tau)\|u(t) - u(\tau)\|_{W_0}^2\, d\tau.$

Integrating (3.12) on $[0, t]$, $t \leq T_0$ we get

$$\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx + \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2$$
$$\leq \frac{1}{2\alpha}\|u_m(0)\|_{W_0}^{2\alpha} + \frac{1}{2}\|u_m(0)\||_{W_0}^2 - \frac{1}{\rho}\int_\Omega |u_m(0)|^{\rho-1}u_m(0)\, dx$$
$$+ \frac{1}{\rho}\int_\Omega |u_m(t)|^{\rho-1}u_m(t)\, dx.$$

From the assumptions on ρ and u^0, Lemma 2.6 and the estimate (3.11), it follows that

$$\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx + \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2 \leq M_1, \tag{3.13}$$

for some constant $M_1 > 0$.

By the above estimates (3.11) and (3.13), $\{u_m\}$ have subsequences still denoted by $\{u_m\}$ such that

$$u_m \to u \quad \text{weakly* in} \quad L^\infty(0,T_0;W_0), \tag{3.14}$$

$$u_{mt} \to u_t \quad \text{weakly in} \quad L^2(0,T_0;L^2(\Omega)), \tag{3.15}$$

$$\left(|u_m|^{r(x)/2}\right)_t \to \chi \quad \text{weakly in} \quad L^2(0,T_0;L^2(\Omega)). \tag{3.16}$$

Employing the same arguments as in [16] we can prove that

$$\chi = \left(|u|^{r(x)/2}\right)_t \quad |u_m|^{r(x)/2}u_{mt} \to |u|^{r(x)/2}u_t \text{ weakly in } L^2(\Omega\times]0,T_0[), \tag{3.17}$$

$$|u_m|^{\rho-1} \to |u|^{\rho-1} \quad \text{weakly in} \quad L^{\frac{\rho}{\rho-1}}(\Omega\times]0,T_0[). \tag{3.18}$$

Therefore, passing to the limit in (3.1) as $m \to +\infty$, by (3.14)– (3.18), we can show that u satisfies the initial condition $u(0) = u^0$ and

$$\int_0^T \int_\Omega \left(1 + a|u|^{r(x)-2}\right) u_t w\,dxdt + M(\|u\|_{w_0}^2)\int_0^T \langle u, w\rangle_{W_0}\,dt - \int_0^T \int_0^t g(t-\tau)\langle u(\tau), w\rangle_{W_0} d\tau dt = \int_0^T \int_\Omega |u|^{\rho-1}w\,dxdt,$$

for all $w \in L^2(0,T_0;W_0)$.
The uniqueness property of a solutions can be derived from [20, Theorem 3, p. 1095], observing that $\left(u + \frac{a}{r(x)-1}|u|^{r(x)-2}u\right) \in L^2(\Omega\times]0,T_0[)$ and $Au = M(\|u\|_{w_0}^2)(-\Delta)^s u$ is a monotone operator. We omit the details. □

Next, we consider the global existence and energy decay of solutions for problem (1.1). For this purpose we define the energy associated with problem (1.1) by

$$E(t) = \frac{1}{2\alpha}\|u(t)\|_{W_0}^{2\alpha} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 + \frac{1}{2}(g\diamond u)(t) - \frac{1}{\rho}\int_\Omega |u(t)|^{\rho-1}u(t)\,dx. \tag{3.19}$$

Then, we easily can check that

$$\frac{d}{dt}E(t) = \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u(t)\|_{W_0}^2 - \|u_t(t)\|_2^2 - a\int_\Omega |u(t)|^{r(x)-2}u_t^2(t)\,dx \leq 0$$

for any regular solution. This remains valid for weak solutions by simple density argument. This shows that $E(t)$ is a nonincreasing function.
Let C_* be the optimal constant satisfying the Sobolev inequality $\|u\|_\rho \leq C_*\|u\|_{W_0}$,

and $B_1 = \frac{C_*}{\sqrt{l}}$. We define the function $h(\lambda) = \frac{1}{2}\lambda^2 - \frac{B_1^\rho}{\rho}\lambda^\rho$. Then, we can verify that the function h is increasing in $]0, \lambda_1[$, decreasing in $]\lambda_1, \infty[$, $h(\lambda) \to -\infty$, as $\lambda \to \infty$ and h has a maximum at λ_1 with the maximum value

$$h(\lambda_1) = E_1 = \left(\frac{1}{2} - \frac{1}{\rho}\right) B_1^{-\frac{2\rho}{\rho-2}} = \frac{\rho-2}{2\rho} B_1^{-\frac{2\rho}{\rho-2}}.$$

where λ_1 is the first positive zero of the derivative function $h'(\lambda)$.
Here, note that

$$\begin{aligned} E(t) &\geq \frac{l}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(g \diamond u)(t) - \frac{1}{\rho}\|u(t)\|_\rho^\rho \\ &\geq \frac{1}{2}\left(l\|u(t)\|_{W_0}^2 + (g \diamond u)(t)\right) - \frac{B_1^\rho l^{\rho/2}}{\rho}\|u(t)\|_{W_0}^\rho \\ &\geq h\left(\sqrt{l\|u(t)\|_{W_0}^2 + (g \diamond u)(t)}\right), \ \forall t \geq 0. \end{aligned} \tag{3.20}$$

Now, we are ready to state our result.

Theorem 3.3. *Assume that hypotheses of Theorem* 3.2 *are satisfied. Consider* $u_0 \in W_0$*, satisfying*

$$0 < l^{1/2}\|u_0\|_{W_0} < \lambda_1, \tag{3.21}$$

$$\frac{1}{2\alpha}\|u^0\|_{W_0}^{2\alpha} + \frac{1}{2}\|u^0\|_{W_0}^2 - \frac{1}{\rho}\int_\Omega |u^0|^{\rho-1}u^0\, dx < \left(\frac{\rho-2}{2\rho}\right) B_1^{-\frac{2\rho}{\rho-2}}. \tag{3.22}$$

Then problem admits a global weak solution in time. In addition, if there exists a constant $\xi_0 > 0$ *such that* $g'(t) \leq -\xi_0 g(t)$*, then this solution satisfies*

$$E(t) \leq L_0 e^{-\gamma t}, \forall t \geq 0, \tag{3.23}$$

where L_0 *and* γ *are positive constants.*

Proof. We will get global estimates for $u_m(t)$ solution of the approximate system (3.1) under the conditions (3.21)–(3.22) for u^0. For this, it suffices to show that

$$E_m(t) + \int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx,$$

where $E_m(t)$ is defined in (3.19) with $u(t)$ replaced by $u_m(t)$, is bounded and independently of t. From (3.12) and the definition of energy, we have

$$E_m(t) + \int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx \leq E_m(0). \tag{3.24}$$

Due to convergence $u_{0m} \to u^0$ in W_0 we see that $E_m(0) < \left(\frac{\rho-2}{2\rho}\right) B_1^{-\frac{2\rho}{\rho-2}}$ for sufficiently large m. We claim that there exists an integer ν_0 such that

$$\sqrt{l\|u_m(t)\|_{W_0}^2 + (g \diamond u_m)(t)} < \lambda_1 \qquad \forall t \in [0, T_m[, m \geq \nu_0. \tag{3.25}$$

Suppose the claim is proved. Then $h\left(\sqrt{l\|u_m(t)\|_{W_0}^2 + (g \diamond u_m)(t)}\right) \geq 0$ and from (3.20), (3.24)–(3.25) we get

$$\|u_m(t)\|_{W_0}^{2\alpha} + \|u_m(t)\|_{W_0}^2 + \int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx \leq C. \tag{3.26}$$

where C is a constant independent of m.Thus, we obtain the global existence.
Proof of Claim: Suppose (3.25) is not true. Thus, for each $m > \nu_0$,there exists $t_1 \in [0, T_m[$ such that

$$\sqrt{l\|u_m(t_1)\|_{W_0}^2 + (g \diamond u_m)(t_1)} \geq \lambda_1. \tag{3.27}$$

Here, we observe that, from (3.21) and the convergence $u_{0m} \to u^0$ in W_0 there exists ν_1 such that

$$l^{1/2}\|u_m(0)\|_{W_0} < \lambda_1 \qquad \forall m > \nu_1.$$

Hence, by continuity there exists

$$t^* = \inf\{t \in [0, T_m[: \sqrt{l\|u_m(t)\|_{W_0}^2 + (g \diamond u_m)(t)} \geq \lambda_1\},$$

such that

$$\sqrt{l\|u_m(t^*)\|_{W_0}^2 + (g \diamond u_m)(t^*)} = \lambda_1. \tag{3.28}$$

By (3.20), we see that

$$E_m(t^*) \geq h\left(\sqrt{l\|u_m(t^*)\|_{W_0}^2 + (g \diamond u_m)(t^*)}\right) = h(\lambda_1) = E_1 \tag{3.29}$$

which contradicts $E_m(t) \leq E_m(0) < E_1,\ \forall t \geq 0$. Therefore our claim is true.
The above estimates permit us to pass to the limit in the approximate equation.
To show the uniform decay of the solution we introduce the perturbed energy functional

$$F(t) = E(t) + \varepsilon\Phi(t), \tag{3.30}$$

where ε is a positive constant which shall be determined later, and

$$\Phi(t) = \int_\Omega (|u|^2 + \frac{a}{r(x)}|u|^{r(x)})\,dx. \tag{3.31}$$

It is straightforward to see that $F(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 depending on ε such that for $t \geq 0$

$$\beta_1 E(t) \leq F(t) \leq \beta_2 E(t). \tag{3.32}$$

By taking the time derivative of the function F defined above in (3.30), using (3.20), and performing several integration by parts, we get

$$\begin{aligned}\frac{d}{dt}F(t) &= \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u(t)\|_{W_0}^2 - \|u_t(t)\|_2^2 - a\int_\Omega |u(t)|^{r(x)-2}u_t^2(t)\,dx \\ &-\varepsilon\|u(t)\|_{W_0}^{2\alpha} - \varepsilon\|u(t)\|_{W_0}^2 + \varepsilon\int_\Omega |u(t)|^{\rho-1}u(t)\,dx + \varepsilon\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\,d\tau.\end{aligned} \tag{3.33}$$

On the other hand, we can easily see that the condition $E(0) < E_1$ is equivalent to the inequality:

$$B_1^\rho \left(\frac{2\rho}{\rho - 2} E(0) \right)^{\frac{\rho-2}{2}} < 1. \tag{3.34}$$

From the assumption (3.21)–(3.22) and (3.24) we have

$$\begin{aligned} l\|u(t)\|_{W_0}^2 &\leq \left(1 - \int_0^t g(\tau)d\tau\right) \|u(t)\|_{W_0}^2 + (g \diamond u)(t) \\ &< \lambda_1^2 = B_1^{-\frac{2\rho}{\rho-2}}, \end{aligned}$$

which implies that

$$\begin{aligned} I(t) &= \left(1 - \int_0^t g(\tau)d\tau\right) \|u(t)\|_{W_0}^2 + (g \diamond u)(t) - \int_\Omega |u(t)|^{\rho-1} u(t)\, dx \\ &\geq l\|u(t)\|_{W_0}^2 + (g \diamond u)(t) - \|u(t)\|_\rho^\rho \\ &\geq l\|u(t)\|_{W_0}^2 - C_*^\rho \|u(t)\|_{W_0}^\rho \geq 0. \end{aligned}$$

So, we have

$$\begin{aligned} \left(\frac{\rho-2}{2\rho}\right) l\|u(t)\|_{W_0}^2 &\leq \frac{\rho-2}{2\rho} \left(1 - \int_0^t g(\tau)d\tau\right) \|u(t)\|_{W_0}^2 \\ &\leq \frac{\rho-2}{2\rho} \Big[\left(1 - \int_0^t g(\tau)d\tau\right) \|u(t)\|_{W_0}^2 + (g \diamond u)(t) \Big] + \frac{1}{\rho} I(t) \\ &\leq E(t) \leq E(0), \end{aligned}$$

then

$$l\|u(t)\|_{W_0}^2 \leq \frac{2\rho}{\rho - 2} E(0). \tag{3.35}$$

Using the above inequality, we can deduce that

$$\begin{aligned} \left| \int_\Omega |u|^{\rho-1} u \right| &\leq \|u(t)\|_\rho^\rho \\ &\leq C_*^\rho \|u(t)\|_{W_0}^\rho \frac{C_*^\rho}{l} \left(\frac{2\rho}{l(\rho-2)} E(0) \right)^{\frac{\rho-2}{2}} l\|u(t)\|_{W_0}^2 \\ &\equiv \theta l \|u(t)\|_{W_0}^2. \end{aligned} \tag{3.36}$$

From the Young inequality and the fact that

$$\int_0^t g(\tau)\, d\tau \leq \int_0^\infty g(\tau)\, d\tau = 1 - l,$$

it follows that

$$\begin{aligned}
&\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\, d\tau \\
&\leq \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}\Big\{\int_0^t g(t-\tau)\left(\|u(\tau)-u(t)\|_{W_0} + \|u(t)\|_{W_0}\right) d\tau\Big\}^2 \\
&\leq \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\eta)\left(\int_0^t g(t-\tau)\|u(t)\|_{W_0}\, d\tau\right)^2 \\
&+ \frac{1}{2}(1+\frac{1}{\eta})\left(\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{W_0}\, d\tau\right)^2 \\
&\leq \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\eta)(1-l)^2\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\frac{1}{\eta})(1-l)(g\diamond u)(t). \qquad (3.37)
\end{aligned}$$

for any $\eta > 0$. Now, letting $\eta = \frac{l}{1-l} > 0$ then (3.37) yields

$$\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\, d\tau \leq \frac{2-l}{2}\|u(t)\|_{W_0}^2 + \frac{1-l}{2l}(g\diamond u)(t). \qquad (3.38)$$

Substituting (3.38) into (3.33), we obtain

$$\frac{d}{dt}F(t) \leq -\frac{1}{2}\left(\xi_0 - \varepsilon\frac{1-l}{l}\right)(g\diamond u)(t) - \varepsilon\|u(t)\|_{W_0}^{2\alpha} - \frac{\varepsilon l}{2}\|u(t)\|_{W_0}^2 + \varepsilon\int_\Omega |u(t)|^{\rho-1}u(t)\, dx. \qquad (3.39)$$

Using the definition of $E(t)$ and (3.36) we have, for any positive constant M

$$\begin{aligned}
\frac{d}{dt}F(t) \leq -M\varepsilon E(t) + \varepsilon\left(\frac{M}{2\alpha} - 1\right)\|u(t)\|_{W_0}^{2\alpha} + \frac{\varepsilon}{2}\left[M + 2\theta l(1-\frac{M}{\rho}) - l\right]\|u(t)\|_{W_0}^2 \\
+\frac{1}{2}\left[\varepsilon(\frac{1-l}{l} + \frac{M}{2}) - \xi_0\right](g\diamond u)(t). \qquad (3.40)
\end{aligned}$$

At this point, we choose $1 > M > 0$ and $E(0)$ small sufficiently such that

$$\frac{M}{2\alpha} - 1 < 0 \qquad \text{and} \qquad M + 2\theta l\left(1 - \frac{M}{\rho}\right) - l < 0.$$

After M is fixed, we choose ε small enough such that

$$\varepsilon\left(\frac{1-l}{l} + \frac{M}{2}\right) - \xi_0 < 0.$$

Inequality (3.40) becomes

$$\frac{d}{dt}F(t) \leq -M\varepsilon E(t).$$

By (3.32), we have

$$\frac{d}{dt}F(t) \leq -M\beta_2\varepsilon F(t).$$

So $F(t) \leq Ce^{-Kt}$ where $K = M\beta_2\varepsilon > 0$. Consequently, by using (3.32) once again, we conclude the result.

Thus, the proof of Theorem 3.3 is achieved. □

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