Stud. Univ. Babeş-Bolyai Math. 65(2020), No. 4, 641–649
DOI: 10.24193/subbmath.2020.4.10

Gradient-type deformations of cycles in EPH geometries

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Abstract. The aim of this paper is to study the cycles of EPH geometries through their homogeneous gradient-type deformations recently introduced by the author. A special topic is the orthogonality between a given cycle C and its deformations as well as between C and its rotated version $R(C)$.

Mathematics Subject Classification (2010): 51N25, 51M09, 53A40.

Keywords: EPH geometries, cycle, deformation, orthogonality, rotation.

1. Introduction

It is well-known that up to isomorphisms there are three 2-dimensional real algebras: $\mathbb{C} = \mathbb{R}[X]/(x^2+1)$, $\mathbb{D} = \mathbb{R}[X]/(x^2)$ and $\mathbb{A} = \mathbb{R}[X]/(x^2-1)$. The theory of the first algebra is richer than the following two, a fact corresponding to the field property of $\mathbb{C}$. Inspired by the terminology of [6, p. 1458] or [7, p. 2] we call *EPH geometries* these spaces and a common image consists in $A(\sigma) := \mathbb{R}[X]/(x^2-\sigma)$ with $\sigma := i^2 \in \{-1, 0, 1\}$ respectively and i the corresponding imaginary unit.

The recent papers [2] and [5], devoted to Finsler geometry, start with a deformation of a conic Γ obtained by deforming the gradient vector field for the quadratic form defining Γ. These deformations are inspired by the scaling (linear) transformation of Computer Graphics: $(x, y) \in \mathbb{R}^2 \to (\lambda_x \cdot x, \lambda_y \cdot y) \in \mathbb{R}^2$, following [8, p. 136]. Using the well-known invariants from the Euclidean geometry of conics we obtained the classifications of the new conics which depends on two scalars denoted α and β, having the role of λ_x, λ_y. The new conic of [2], denoted $\tilde{\Gamma}$, is a degenerate one and we could interpret the map $\Gamma \to \tilde{\Gamma}$ as a "curve shortening" transformation. The same fact holds for the new conic of [5], denoted Γ^m, if the initial conic Γ does not have linear terms.

In this note we use these classes of gradient-type deformation to a main object of EPH geometries, called *cycle*, which is a particular case of conic sections, invariant under the action of the group $SL(2, \mathbb{R})$ through Mobiüs transformations. A detailed analysis of the deformed cycles depends on the vanishing or not of σ as well as the vanishing or not of a parameter k separating the circles to lines. Also, we discuss the transformation of a square matrix associated to any cycle C.

Moreover, we treat these deformations in terms of $A(\sigma)$-numbers. In the second section we study the orthogonality of a given cycle C with its deformations restricting to the $\sigma \neq 0$ case. In the last section we introduce a natural rotation R in $A(\sigma)$ and we study the relationships between a given C and its rotated cycle $R(C)$.

2. The cycles of EPH geometries and their gradient-type deformations

In the two-dimensional Euclidean space $\mathbb{R}^2$ let us consider the conic Γ implicitly defined by $f \in C^\infty\left(\mathbb{R}^2\right)$ as: $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ where f is a quadratic function of the form $f(x, y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$ for the non-degenerate conics.

It is well-known that the gradient vector field of f, namely

$$\nabla f = \left(f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}\right),$$

gives important properties of Γ; for example, the centers of Γ are exactly the critical points of ∇f. Inspired by this fact we introduced recently:

Definition 2.1. Fix the scalars α, β with $\alpha\beta \neq 0$.
i) ([2, p. 86-87], [3, p. 60]) The (α, β)-*deformation* of Γ is the conic:

$$\tilde{\Gamma} = \Gamma_{\alpha,\beta} : \alpha\left[\frac{1}{2}f_x\right]^2 + \beta\left[\frac{1}{2}f_y\right]^2 = 0. \tag{2.1}$$

ii) ([5, p. 102]) The (α, β)-*mixed deformation* of Γ is the conic:

$$\Gamma^m = \Gamma^m_{\alpha,\beta} : \alpha y\left[\frac{1}{2}f_x\right] + \beta x\left[\frac{1}{2}f_y\right] = 0. \tag{2.2}$$

A main object in EPH geometries is given in [6, p. 1459], [7, p. 4]:

Definition 2.2. The common name *cycle* will be used to denote circles, parabolas and hyperbolas (as well as straight lines as their limits) in the respective EPH geometry.

An analytical study of a cycle can be done via the general equation given in [6, p. 1460] or [7, p. 6]:

$$C : f(u, v) := k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0 \tag{2.3}$$

and hence C is a conic section completely defined by the data $(k, l, n, m) \in \mathbb{P}^3$. As usual, if $k = 0$ then C can be called *a degenerate cycle*. In fact, in the cited works C is identified with the matrix:

$$C^s_{\breve{\sigma}} := \begin{pmatrix} l + \breve{\imath}sn & -m \\ k & -l + \breve{\imath}sn \end{pmatrix} \tag{2.4}$$

where s is a new parameter, usually equal to ± 1, and a new imaginary unit $\breve{\imath}$. Its square $\breve{\sigma} := \breve{\imath}^2$ belongs again to $\{-1, 0, 1\}$ but independently of σ.

Since C is a conic section we can apply the ideas of Definition 2.1 to introduce the gradient-type deformations of a cycle:

$$\begin{cases} \tilde{C} = C_{\alpha,\beta} : \alpha(ku - l)^2 + \beta(k\sigma v + n)^2 = 0, \\ C^m : \alpha v(ku - l) - \beta u(k\sigma v + n) = 0 \end{cases} \tag{2.5}$$

which yields immediately:

Proposition 2.3. *Since $\alpha \neq 0$ we have:*
i) $\tilde{C}$ is a cycle if and only if $\sigma(\alpha + \sigma\beta) = 0$,
ii) C^m is a cycle if and only if $k(\alpha - \beta\sigma) = 0$. In this case C^m is the straight line:

$$(\beta n)u + (\alpha l)v = 0.$$

Example 2.4. In the following we discuss the remarkable particular cases of the result above.
i) Suppose $\sigma = 0$. Then $\tilde{C}$ is the cycle:

$$\tilde{C} : (ku - l)^2 + \frac{\beta}{\alpha} n^2 = 0 \tag{2.6}$$

with the matrix:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} kl & -(l^2 + \frac{\beta}{\alpha} n^2) \\ k^2 & -kl \end{pmatrix}. \tag{2.7}$$

The degenerate case of an initial line i.e. $k = 0$ is possible if and only if $\alpha l^2 + \beta n^2 = 0$ which is relation (2.19) below. If $k \neq 0$ then, due to the projective character of the coefficients of a cycle, we get the matrix:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} l & -\frac{1}{k}(l^2 + \frac{\beta}{\alpha} n^2) \\ k & -l \end{pmatrix}. \tag{2.8}$$

If $\frac{\beta}{\alpha} > 0$ then $\tilde{C}$ is a void set for $n \neq 0$ while $n = 0$ gives the deformation:

$$C : ku^2 - 2lu + m = 0 \to \tilde{C} : ku = l \ (line : k \neq 0). \tag{2.9}$$

If $\frac{\beta}{\alpha} < 0$ then we have the lines:

$$\tilde{C} : ku - l = \pm\sqrt{-\frac{\beta}{\alpha}} n. \tag{2.10}$$

C^m is a cycle if and only if $k = 0$ which means that we have the mixed deformation:

$$C : 2lu + 2nv - m = 0 \ (line) \to C^m : (\beta n)u + (\alpha l)v = 0 \ (line). \tag{2.11}$$

If $\beta = -\alpha$ then these two lines are Euclidean orthogonal. From the matrix point of view the deformation (2.11) means:

$$C^s_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\imath}sn & -m \\ 0 & -l + \breve{\imath}sn \end{pmatrix} \to C^{m,s}_{\breve{\sigma}} = \begin{pmatrix} -\beta n + \breve{\imath}s(-\alpha l) & 0 \\ 0 & \beta n + \breve{\imath}s(-\alpha l) \end{pmatrix}. \tag{2.12}$$

ii) For $\sigma \neq 0$ we have that $\tilde{C}$ is a cycle only for $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$ and then:

$$\tilde{C} : \left[k(u + iv) - l + \frac{n}{i}\right]\left[k(u - iv) - l - \frac{n}{i}\right] = 0. \tag{2.13}$$

Hence, if $k \neq 0$ then $\tilde{C}$ consists in a single point: $M = (\frac{l}{k}, -\frac{n}{k\sigma})$. Let us point out that for $\sigma \neq 0$ we have $\frac{1}{\sigma} = \sigma$ and hence $M = (\frac{l}{k}, -\sigma\frac{n}{k})$ which is exactly the *e/h-center* of the initial cycle C, as it is introduced in formula (7) of [6, p. 1460] or [7, p. 7]. In conclusion, for $\sigma \cdot k \neq 0$ we have the deformation:

$$C \to \tilde{C} = its\ center. \tag{2.14}$$

The matrix corresponding to $\tilde{C}$ is:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} k(l + \breve{\imath}sn) & n^2\sigma - l^2 \\ k^2 & k(-l + \breve{\imath}sn) \end{pmatrix} \tag{2.15}$$

which for $k = 0$ becomes:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} 0 & n^2\sigma - l^2 \\ 0 & 0 \end{pmatrix} \tag{2.16}$$

while for $k \neq 0$, due to the projective character of the parameters of a cycle:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\imath}sn & \frac{1}{k}(n^2\sigma - l^2) \\ k & -l + \breve{\imath}sn \end{pmatrix}. \tag{2.17}$$

The same case $\sigma \cdot k \neq 0$ for ii) of proposition above gives $\beta = \frac{\alpha}{\sigma} = \sigma\alpha$ and C^m is the line:

$$C^m : nu + (\sigma l)v = 0. \tag{2.18}$$

For elliptic geometry the condition $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$ becomes the equality $\alpha = \beta$ discussed in [2, p. 89] and [3, p. 62]; it can be called *the diagonal case*. Remark that the elliptic center $\tilde{C}$ of (2.14) is obtained in [6, p. 1461] or [7, p. 8] from the vanishing condition $detC^s_{-1} = 0$.

Remark 2.5. The cycle C^m contains the origin $(u, v) = (0, 0) = O$. This fact holds for $\tilde{C}$ if and only if:

$$\alpha l^2 + \beta n^2 = 0. \tag{2.19}$$

With the discussion of above particular cases it results:
i) for $\sigma = 0$ the only available case is $\frac{\beta}{\alpha} < 0$ yielding:

$$l_{\pm} = \pm\sqrt{-\frac{\beta}{\alpha}}n. \tag{2.20}$$

ii) for $\sigma \neq 0$ since $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$ we get that for the elliptic geometry the only possible case is $O = M$ the center of C while for the hyperbolic geometry:

$$l_{\pm} = \pm n. \tag{2.21}$$

The gradient-type deformation of a standard (i.e. Euclidean) ellipse is discussed in example 2.2i) of [2, p. 87]. Let us point out that (2.20) and (2.21) coincide for $\beta = -\alpha$ which for the case ii) correspond to the hyperbolic geometry. Hence the above cases i) and ii) are completely different, both from σ and the sign of $\frac{\beta}{\alpha}$ points of view.

Returning to the general case of α and β we treat the considered deformations within $A(\sigma)$ following the model of [3] and [5]. More precisely, with the usual notation $z = u + iv \in A(\sigma)$ we derive the expression of C:

$$C : F(z, \bar{z}) := kz\bar{z} + Bz + \bar{B}\bar{z} + m = 0, \quad B := -l - \frac{n}{\sigma}i \in A(\sigma)\ (\sigma \neq 0). \tag{2.22}$$

For $\sigma = 0$ we have: $B = -l - \frac{n}{i}$. The inverse relationship between f and F is:

$$l = -\Re B, \quad n = -\sigma \Im B \tag{2.23}$$

with $\Re$ and $\Im$ respectively the real and imaginary part. By replacing in (2.5) the usual relations:

$$u = \frac{1}{2}(z + \bar{z}), \quad v = \frac{1}{2i}(z - \bar{z}) \tag{2.24}$$

we get:

$$\begin{cases} \tilde{C} : \alpha[k(z+\bar{z}) - 2l]^2 + \beta[ki(z-\bar{z}) + 2n]^2 = 0, \\ C^m : \alpha(z-\bar{z})[k(z+\bar{z}) - 2l] - \beta(z+\bar{z})[k\sigma(z-\bar{z}) + 2ni] = 0. \end{cases} \tag{2.25}$$

For the case $\sigma \neq 0$ we follow the discussion of Example 2.4ii and then:

$$\begin{cases} \tilde{C} : [k(z+\bar{z}) - 2l]^2 - \sigma[ki(z-\bar{z}) + 2n]^2 = 0, \\ C^m : (z-\bar{z})[k(z+\bar{z}) - 2l] - \sigma(z+\bar{z})[k\sigma(z-\bar{z}) + 2ni] = 0. \end{cases} \tag{2.26}$$

The second equation (2.26) reduces to:

$$C^m : Bz - \bar{B}\bar{z} = 0 \leftrightarrow Bz \in \mathbb{R} \tag{2.27}$$

and hence, for $B \neq 0$ we have the line: $z = \bar{B} \cdot \mathbb{R}$.
We finish this section by applying to the cycle C (not containing the origin, hence $m \neq 0$) the *inversion* $J : z \in A(\sigma)^* \to \frac{1}{z} = w$. We get a new cycle, expressed in w:

$$J(C) : mw\bar{w} + \bar{B}w + B\bar{w} + k = 0 \tag{2.28}$$

which means $J : (k, l, n, m) \to (m, l, -n, k)$. With (2.26)-(2.27) its gradient deformations for $\sigma \neq 0$ are:

$$\begin{cases} \widetilde{J(C)} : [m(w+\bar{w}) - 2l]^2 - \sigma[mi(w-\bar{w}) - 2n]^2 = 0, \\ J(C)^m : B\bar{w} - \bar{B}w = 0 \leftrightarrow \bar{B}w \in \mathbb{R}. \end{cases} \tag{2.29}$$

Again, if $B \neq 0$ then the second cycle from from above is the line: $w = B \cdot \mathbb{R}$.

3. Orthogonality in the geometry of cycles

In [6, p. 1462] or [7, p. 2] a Möbius-invariant (indefinite) inner product (depending on $\breve{\sigma}$) is defined on the set of cycles through:

$$< C^s_{\breve{\sigma}}, \hat{C}^s_{\breve{\sigma}} >:= Tr(C^s_{\breve{\sigma}} \cdot \overline{\hat{C}^s_{\breve{\sigma}}}) \tag{3.1}$$

which yields an associated $\breve{\sigma}$-orthogonality. Here, the bar means the conjugation with respect to $\breve{\imath}$.

For our setting we derive firstly the norms of a cycle and its gradient-type deformations for $k\sigma \neq 0$:

$$\begin{cases} \|C^s_{\breve{\sigma}}\|^2 = 2(l^2 - km - \breve{\sigma}n^2) = \|J(C)^s_{\breve{\sigma}}\|^2, \\ \|\tilde{C}^s_{\breve{\sigma}}\|^2 = 2(\sigma - \breve{\sigma})n^2, \quad \|C^{m,s}_{\breve{\sigma}}\|^2 = \frac{1}{2}(n^2 - \breve{\sigma}l^2). \end{cases} \tag{3.2}$$

Let us remark that:

$$det\ C^s_{\breve{\sigma}} = km + \breve{\sigma}n^2 - l^2 \to \|C^s_{\breve{\sigma}}\|^2 = \|J(C)^s_{\breve{\sigma}}\|^2 = -2det\ C^s_{\breve{\sigma}}. \tag{3.3}$$

Secondly, we study all the possible cases of orthogonality for our setting:

Theorem 3.1. *Let $\sigma \neq 0$ and the cycle C with $k \neq 0$. Then:*
1) C is $\breve{\sigma}$-orthogonal to its gradient deformation $\tilde{C}$ if and only if:

$$l^2 - km + (\sigma - 2\breve{\sigma})n^2 = 0. \tag{3.4}$$

2) C is $\breve{\sigma}$-orthogonal to its mixed-gradient deformation C^m if and only if:

$$(1 - \sigma\breve{\sigma})nl = 0. \tag{3.5}$$

3) $\tilde{C}$ is $\breve{\sigma}$-orthogonal to C^m if and only if (3.4) holds.
4) Suppose also $m \neq 0$. Then C is $\breve{\sigma}$-orthogonal to $J(C)$ if and only if:

$$2(l^2 + \breve{\sigma}n^2) - k^2 - m^2 = 0. \tag{3.6}$$

Proof. 1) A straightforward computation gives:

$$< C^s_{\breve{\sigma}}, \tilde{C}^s_{\breve{\sigma}} >= l^2 - km + (\sigma - 2\breve{\sigma})n^2. \tag{3.7}$$

2) The matrix of C^m from (2.18) is:

$$C^{m,s}_{\breve{\sigma}} = \frac{1}{2}\begin{pmatrix} n + \breve{\imath}s\sigma l & 0 \\ 0 & -n + \breve{\imath}s\sigma l \end{pmatrix} \tag{3.8}$$

and then:

$$< C^s_{\breve{\sigma}}, C^{m,s}_{\breve{\sigma}} >= (1 - \sigma\breve{\sigma})nl. \tag{3.9}$$

3) The same computation as above.
4) The matrix of $J(C)$ is:

$$J(C)^s_{\breve{\sigma}} := \begin{pmatrix} l - \breve{\imath}sn & -k \\ m & -l - \breve{\imath}sn \end{pmatrix} \tag{3.10}$$

and:

$$< C^s_{\breve{\sigma}}, JC^s_{\breve{\sigma}} >= 2(l^2 + \breve{\sigma}n^2) - m^2 - k^2. \tag{3.11}$$

which gives the conclusion. □

Example 3.2. Suppose $\sigma = \breve{\sigma}$. Then $1 - \sigma\breve{\sigma} = 0$ since $\sigma^2 = 1$ and then C^m is both orthogonally on C and $\tilde{C}$. In this case C is orthogonally to $\tilde{C}$ if and only if $l^2 - km - \breve{\sigma}n^2 = 0$ but from the first equation (3.2) this means that $\|C\| = 0$ i.e. C is also self-orthogonal.

Returning to the Möbius-type study of cycles we continue this section considering some transformation of cycles. The first one is inspired by [1, p. 2706]. Let $\alpha \in A(\sigma)$ with module $|\alpha| \neq 1$ and consider the map $T_\alpha : A(\sigma) \to A(\sigma)$:

$$T_\alpha(z) = z + \alpha\bar{z} := w. \tag{3.12}$$

It follows directly that T_α is a bijective map with the inverse:

$$z := T^{-1}_\alpha(w) = \frac{1}{1 - |\alpha|^2}(w - \alpha\bar{w}). \tag{3.13}$$

Replacing this expression of z in (2.22) we find the image of cycle C through T_α:

$$T_\alpha(C) : k|w - \alpha\bar{w}|^2 + (1 - |\alpha|^2)[(B - \bar{\alpha}\bar{B})w + (\bar{B} - \alpha B)\bar{w} + (1 - |\alpha|^2)m] = 0 \tag{3.14}$$

but this curve is not a cycle for $\alpha \cdot k \neq 0$.

The second transformation is a Blaschke factor B_a defined by $a \in A(\sigma)$ with module $|a| < 1$:

$$w := B_a(z) = \frac{z-a}{1-\bar{a}z}, \tag{3.15}$$

having the inverse:

$$z = B_{-a}(w) = \frac{w+a}{1+\bar{a}w}. \tag{3.16}$$

The Blaschke transformation of the cycle (2.22) is again a cycle:

$$B_a(C) : b_a(k)w\bar{w} + b_a(B)w + \overline{b_a(B)}\bar{w} + b_a(m) = 0 \tag{3.17}$$

with:

$$\begin{cases} b_a(k) = k + m|a|^2 + 2\Re(Ba), \\ b_a(B) = (k+m)\bar{a} + B + \bar{B}\bar{a}^2, \\ b_a(m) = m + k|a|^2 + 2\Re(Ba). \end{cases} \tag{3.18}$$

Example 3.3. Suppose that $|B| < 1$ and let $a = \bar{B}$. Then the Blaschke transformation of the coefficients is:

$$\begin{cases} b_{\bar{B}}(k) = k + (m+2)|B|^2, \\ b_{\bar{B}}(B) = (k+m+1+|B|^2)B, \\ b_{\bar{B}}(m) = m + (k+2)|B|^2. \end{cases} \tag{3.19}$$

The last transformation is a similarity defined by $a, b \in A(\sigma)$ with $a \neq 0$:

$$w := S_{a,b}(z) = az + b, \tag{3.20}$$

having the inverse:

$$z = \frac{1}{a}(w-b) = S_{\frac{1}{a},\frac{-b}{a}}(w). \tag{3.21}$$

The similarity transformation of the cycle (2.22) is again a cycle:

$$S_{a,b}(C) : kw\bar{w} + (B\bar{a} - k\bar{b})w + (\bar{B}a - kb)\bar{w} + m|a|^2 + k|b|^2 - 2\Re(Bb\bar{a}) = 0. \tag{3.22}$$

If the initial cycle C is non-degenerate then we restrict to the case $k = 1$ due to the projective character of the coefficients of C. Then a non-degenerate C is called *decomposable* if it is a product of lines:

$$C : (z - B)(\bar{z} - \bar{B}) = 0 \tag{3.23}$$

which means that $m = |B|^2 = l^2 - \sigma n^2$. A similarity preserves the class of decomposable cycles since its image is:

$$S_{a,b}(C) : (w - b + a\bar{B})(\bar{w} - \bar{b} + \bar{a}B). \tag{3.24}$$

From (3.3) it follows that a decomposable cycle has:

$$det\ C^s_{\breve{\sigma}} = (\breve{\sigma} - \sigma)n^2. \tag{3.25}$$

4. The rotation of a cycle

In this section we suppose that $\sigma \neq 0$. In $A(\sigma)$ we introduce the rotation map $R : (u, v) \to i \cdot (u, v) = (\sigma v, u)$; then its square is: $R^2 = \sigma I$. It follows that a given cycle C has an associated rotation cycle $R(C)$ with equation:

$$R(C) : k(\sigma^2 v^2 - \sigma u^2) - 2l\sigma v - 2nu + m = 0. \tag{4.1}$$

A short computation gives a more simple form:

$$R(C) : k(u^2 - \sigma v^2) + 2(\sigma n)u + 2lv - \sigma m = 0 \tag{4.2}$$

and then we have the deformation:

$$C = (k, l, n, m) \to R(C) = (k, -\sigma n, -l, -\sigma m). \tag{4.3}$$

The general rotation of conics is treated in [4].

Remark 4.1. Concerning the compositions $J \circ R$ and $R \circ J$ we have:

$$J \circ R(C) = (-\sigma m, -\sigma n, l, k), \quad R \circ J(C) = (m, \sigma n, -l, -\sigma k) \tag{4.4}$$

and then J and R anti-commutes in the hyperbolic setting respectively J and R commutes if and only if $l = 0$ in the complex setting: $\sigma = -1$.

In terms of associated matrix we have:

$$R(C)^s_{\breve{\sigma}} = \begin{pmatrix} -\sigma n - \breve{i}sl & \sigma m \\ k & \sigma n - \breve{i}sl \end{pmatrix}, \|R(C)^s_{\breve{\sigma}}\|^2 = 2(n^2 + \breve{\sigma} l^2 + \sigma km). \tag{4.5}$$

Then R preserves the norm of C if and only if:

$$(\sigma + 1)km + (\breve{\sigma} - 1)l^2 + (1 - \breve{\sigma})n^2 = 0. \tag{4.6}$$

Also, recall from section 2 that the e/h-center of C is $M(\frac{l}{k}, -\sigma\frac{n}{k})$ and hence its rotation is $R(M) = (-\frac{n}{k}, \frac{l}{k})$. But the center of $R(C)$ is $\bar{M} = (-\frac{\sigma n}{k}, \frac{\sigma l}{k})$ and then $\bar{M} = \sigma R(M)$; these points coincide for $\sigma = 1$.

Concerning the orthogonality of this new cycle with the previous three cycles we have:

Proposition 4.2. *Let C be a cycle with $k \neq 0$. Then:*
i) C is $\breve{\sigma}$-orthogonal to its rotated cycle $R(C)$ if and only if:

$$(\breve{\sigma} - \sigma)nl + (\sigma - 1)km = 0. \tag{4.7}$$

ii) $\tilde{C}$ is $\breve{\sigma}$-orthogonal to $R(C)$ if and only if:

$$2(\breve{\sigma} - \sigma)nl + \sigma(n^2 + km) - l^2 = 0. \tag{4.8}$$

iii) C^m is $\breve{\sigma}$-orthogonal to $R(C)$ if and only if:

$$\breve{\sigma} l^2 = n^2. \tag{4.9}$$

Proof. A straightforward computation gives:

$$< C^s_{\breve{\sigma}}, R(C)^s_{\breve{\sigma}} >= 2[(\breve{\sigma} - \sigma)nl + (\sigma - 1)km], \tag{4.10}$$

$$< \tilde{C}^s_{\breve{\sigma}}, R(C)^s_{\breve{\sigma}} >= 2(\breve{\sigma} - \sigma)nl + \sigma(n^2 + km) - l^2, \tag{4.11}$$

$$< C^{m,s}_{\breve{\sigma}}, R(C)^s_{\breve{\sigma}} >= 2\sigma(\breve{\sigma} l^2 - n^2] \tag{4.12}$$

which yields the conclusion. □

Example 4.3. Suppose that $\sigma = \breve{\sigma} = 1$. Then $R(C)$ is orthogonal to C and:
a) is orthogonal to $\tilde{C}$ if and only if: $l^2 = n^2 + km$; for $k = 1$ this means that C is decomposable,
b) is orthogonal to C^m if and only if: $l_\pm = \pm n$, which is exactly the relation (2.21).

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