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On the Rockafellar function associated to a non-cyclically monotone mapping

Teodor Precupanu

Abstract. In an earlier paper, we have given a definition of the Rockafellar integration function associated to a cyclically monotone mapping considering only systems of distinct elements in its domain. Thus, this function can be proper for certain non-cyclically monotone mappings. In this paper we establish general properties of Rockafellar function if the graph of mapping does not contain finite set of accumulation elements where the mapping is not cyclically monotone. Also, some dual properties are given.

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1. Introduction

Let X be a real linear normed space and let X^* be its dual. Given a function $f : X \to \overline{\mathbb{R}}$ its subdifferential is the (multivalued) mapping $\partial f : X \to X^*$ defined by

$$\partial f(x) = \{x^* \in X^*; x^*(u - x) \leq f(u) - f(x), \text{ for all } u \in X\}, x \in X, \tag{1.1}$$

where $\overline{\mathbb{R}} = [-\infty, \infty]$. We also suppose the usual extension in convex analysis of addition by condition $\infty - \infty = \infty$. Here, we consider only proper function f, that is, its domain

$$\mathrm{Dom}\, f = \{x \in X; f(x) < \infty\} \tag{1.2}$$

is a nonvoid set and $f(u) > -\infty$ for all $u \in X$.

It is well known that any proper convex lower-semicontinuous function is subdifferentiable, that is

$$\mathrm{Dom}\, \partial f = \{x \in X; \partial f(x) \neq \emptyset\} \neq \emptyset. \tag{1.3}$$

The problem of integration with respect to this subdifferential was studied by many authors. In this line, a remarkable result was established by Rockafellar [7],

[8]: any maximal cyclically monotone mapping can be represented as the subdifferential of a proper convex lower-semicontinuous function. This function is unique up to an additive constant function. Also, the subdifferential of a proper convex lower-semicontinuous function is a maximal cyclically monotone mapping. We recall that a mapping $T : X \to X^*$ is cyclically monotone if

$$\sum_{i=0}^{n} x_i^*(x_i - x_{i+1}) \geq 0 \quad \text{for} \quad \text{all } (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0, n}, \tag{1.4}$$

where $x_{n+1} = x_0$, $n = 1, 2, \ldots$

If (1.4) is fulfilled for $n = 1$, then the mapping T is called monotone.

In the proof of Rockafellar's result it is used the following function associated to a cyclically monotone mapping $T : X \to X^*$

$$f_{x_0;T}(x) = \sup \Big\{ \sum_{i=1}^{n} x_i^*(x_{i+1} - x_i); (x_i, x_i^*) \in \operatorname{Graph} T, \\ i = \overline{0, n}, n = 1, 2, \ldots, x_{n+1} = x \Big\}, \tag{1.5}$$

for any $x \in X$, where x_0 is a fixed element in $\operatorname{Dom} T$.

We mention that in the papers [1,3,4,5,9] are given special properties using different concepts of subdifferential.

Obviously, Rockafellar (integration) function $f_{x_0;T}$ is convex and lower-semicontinuous. In fact, this function can be defined for any mapping T, but it is a proper function only in the case of cyclically monotone mappings. In an earlier paper [6] we established the following result:

Theorem 1.1. *Let us consider a mapping $T : X \to X^*$. The following statements are equivalent:*

(i) *T is a cyclically monotone mapping;*

(ii) *$f_{x_0;T}(x_0) = 0$ for any (one) element $x_0 \in \operatorname{Dom} T$;*

(iii) *$\operatorname{Dom} f_{x_0;T} \neq \emptyset$ for any (one) element $x_0 \in \operatorname{Dom} T$.*

Indeed, if T is not cyclically monotone, there exist $x_0, x_1, \ldots, x_n \in X$ such that

$$\sum_{i=0}^{n} x_i^*(x_i - x_{i+1}) = a < 0, \quad x_{n+1} = x_0.$$

Now, if we consider the finite system of elements $x_0, x_1, x_2, \ldots x_{k(n+1)} \in X$, where $x_i = x_{i+j(n+1)}$, $i = \overline{0, n}$, $j = \overline{0, k-1}$, then

$$f_{x_0,T}(x) \geq -ka + x_{k(n+1)}^*(x - x_{k(n+1)}) - x_{k(n+1)}^*(x_0 - x_{k(n+1)}) \\ = -ka + x_{k(n+1)}^*(x - x_0) = -ka + x_n^*(x - x_0), \text{ for all } k = 1, 2, \ldots$$

and so $f_{x_0;T}(x) = \infty$, for all $x \in X$.

Taking into account the equivalence (i)$\Leftrightarrow$(iii) we can give an other equivalent definition for Roackafellar function $f_{T;x_0}$ such that its domain is also nonvoid for some non cyclically monotone mappings (see also [6]).

In this paper we give special cases when Rockafellar function is proper. Also, we establish a subdifferential property and some dual inequalities.

2. A new definition of Rockafellar function

Let $T : X \to X^*$ be a proper multivalued mapping. Considering only systems of distinct elements in $\operatorname{Dom} T$ we obtain the following slight modification of the Rockafellar function associated to T as follows:

$$g_{x_0;T}(x) = \sup\Big\{ \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i); (x_i, x_i^*) \in \operatorname{Graph} T, \tag{2.1}$$
$$x_i \neq x_j \text{ for any } i \neq j, i, j = \overline{0,n}, n = 1, 2, \ldots, x_{n+1} = x\Big\}, \quad x \in X,$$

where x_0 is an fixed element of $\operatorname{Dom} T$.

This function is also convex and lower-semicontinuous.

Proposition 2.1. *If T is a cyclically monotone mapping then*

$$g_{x_0;T} = f_{x_0;T} \text{ for all } x_0 \in \operatorname{Dom} T.$$

Proof. Obviously, $f_{x_0;T} \geq g_{x_0;T}$. On the other hand, if we consider a system $(x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n}$, which contains two equal elements $x_k = x_{k+l}$, $l > 0$, $k, k + l = \overline{0,n}$, then by (1.4) we have

$$\sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) = \sum_{i=0}^{k-1}(x_{i+1} - x_i) + \sum_{i=k}^{k+l-1} x_i^*(x_{i+1} - x_i) + \sum_{i=k+l}^{n} x_i^n(x_{i+1} - x_i)$$
$$\leq \sum_{i=0}^{k-1} x_i^*(x_{i+1} - x_1) + \sum_{i=k+l}^{n} x_i^*(x_{i+1} - x_i).$$

Therefore, in the definition (1.4) of Rockafellar function $f_{x_0;T}$ we can omit the systems $x_i \in \operatorname{Dom} T$, $i = \overline{1,n}$ which contain equal elements.

Example 2.2. Let $T : \mathbb{R} \to \mathbb{R}$ defineed by

$$T(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ \alpha, & \text{if } x = 2, \\ \phi, & \text{if } x \overline{\in} [0,1] \cup \{2\}, \end{cases} \tag{2.2}$$

where $\alpha \in \mathbb{R}$. If $\alpha \geq 1$ obviously T is a cyclically monotone mapping, and so, in this case $f_{x_0;T} = g_{x_0;T}$ for any $x_0 \in \operatorname{Dom} T$. But, if $\alpha < 1$, then T is not cyclically monotone. Thus, $f_{x_0;T}(x) = \infty$ for all $x \in \mathbb{R}$. On the other hand, we remark that T is cyclically monotone on $[0,1]$. We denote by T_0 this mapping (the identity mapping of $[0,1]$. Now, by a standard calculus we obtain

$$f_{x_0;T_0}(x) = \begin{cases} -\frac{x_0^2}{2}, & \text{for } x < 0, \\ -\frac{x_0^2}{2} + \dfrac{x^2}{2}, & \text{for } x \in [0,1], \\ -\frac{x_0^2}{2} + x - \frac{1}{2}, & \text{for } x > 1. \end{cases} \tag{2.3}$$

It is obvious that the sums in the definitions of $g_{x_0;T}$ and $f_{x_0;T_0}$ are distinct only for systems of elements which contain the element $x = 2$. Thus, $f_{x_0;T}(x) \neq f_{x_0;T_0}(x)$ if and only if for a system $(x_i, x_i^*) \in \operatorname{Graph} T$, $i = \overline{0,n}$, which contains the element $x = 2$, the corresponding sum in (2.1) has a greater value than the sum where the element $x = 2$ is omitted. For example if $\alpha \geq 1$ and $x_0 \in [0,1]$, then $g_{x_0;T}(x) = f_{x_0;T_0}(x)$ for all $x \in (-\infty, 2)$, while $g_{x_0;T} > f_{x_0;T_0}(x)$ for any $x > 2$.

Also, if $\alpha < 1$, then $g_{x_0;T} > f_{x_0;T_0}(x)$. On the other hand, concerning the integration property we get that on $[0,1]$ we have the following equality

$$\partial g_{x_0;T} = \partial f_{x_0;T_0} = T_0.$$

Generally, $\operatorname{Dom} g_{x_0;T} \neq \emptyset$ only if T is cyclically monotone excepting a "small" subsets of its domain. In this line we give the following two results.

Theorem 2.3. *If there exists a system of accumulation elements $(x_i, x_i^*) \in \operatorname{Graph} T$, $i = \overline{1,k}$, $x_{k+1} = x_1$, such that*

$$\sum_{i=1}^{k} x_i^*(x_i - x_{i+1}) < 0,$$

then $g_{x_0;T}(x) = \infty$ for any $x \in X$.

Proof. Let us denote $\sum_{i=1}^{k} x_i^*(x_{i+1} - x_i) = \alpha > 0$. Now, given $\varepsilon > 0$, $M > 0$ we inductively define the sequence $(x_{i,n}, x_{i,n}^*) \in \operatorname{Graph} T$, $i = \overline{1,k}$ and $n = 1, 2, \ldots$ such that $x_{i,n} \neq x_{j,m}$ for any $i \neq j$, $n = 2, 3 \ldots$

$$\|x_{i,n} - x_i\| < \frac{\varepsilon}{2^{n+2}kM}, \quad \|x_{i,n}^* - x_i^*\| < \frac{\varepsilon}{2^{n+2}kM},$$

where $\|x_{i,n}\| \leq M$, $\|x_{i,n}^*\| \leq M$. Then, we have

$$|x_{i,n}^*(x_{i+1,n} - x_{i,n}) - x_i^*(x_{i+1} - x_i)| \leq \|x_{i,n}^*\| \|(x_{i+1,n} - x_{i+1}) + (x_i - x_{i,n})\|$$
$$+ \|x_i - x_{i+1}\| \|x_{i,n}^* - x_i^*\| \leq \frac{\varepsilon}{2^n k}, \text{ for any } i = \overline{1,k}, n = 1, 2, \ldots$$

Let x_0 be a given element in $\operatorname{Dom} T$ and $x \in X$. By hypotheses we can suppose that $x_0 \neq x_{i;m}$, for any $i = \overline{1,k}$, $m = 1, 2, \ldots$

Now, we consider the system of distinct elements $\{x_{1,1}, x_{1,2}, \ldots, x_{1,k}, \ldots, x_{1,n}, x_{2,n}, \ldots, x_{k,n}\}$ and corresponding sum of right hand of formula (2.1). We have

$$x_0^*(x_{1,1} - x_0) + \sum_{i=1}^{k} \sum_{m=1}^{n} x_{i,m}^*(x_{i+1,m} - x_{i,m}) + x_{k,n}^*(x - x_{k,n})$$
$$= x_0^*(x_{1,1} - x_0) + \sum_{i=1}^{k} \sum_{m=1}^{n} x_{i,m}^*(x_{i+1,m} - x_{i,m}) + x_{k,n}^*(x - x_{k,n}) + n\alpha$$
$$- n\sum_{i=1}^{k} x_i^*(x_{i+1} - x_i) = x_0^*(x_{1,1} - x_0) + n\alpha + \sum_{i=1}^{k} \sum_{m=1}^{n} [x_{i,m}^*(x_{i+1,m} - x_{i,m})$$
$$- x_i^*(x_{i+1} - x_i)] + x_{k,n}^*(x - x_{k,n}) \geq x_0^*(x_{1,1} - x_0) - k\sum_{m=1}^{n} \frac{\varepsilon}{k2^m} + n\alpha$$
$$+ x_{k,n}^*(x - x_{k,n}) \geq -\varepsilon + x_0^*(x_{1,1} - x_0) + x_{k,n}^*(x - x_{k,n}) + n\alpha,$$

for any $n = 1, 2, \ldots$ Since the sequence $(x_{i,n}, x^*_{i,n}) \in \operatorname{Graph} T$, $i = 1, k$ and $n = 1, 2, \ldots$ is bounded, according to the definition (2.1) we get $g_{x_0,T}(x) = \infty$ for all $x \in X$, as claimed.

The following result establishes a sufficient condition such that $\operatorname{Dom} g_{x_0;T} \neq \emptyset$.

Theorem 2.4. *Let $T_0 : X \to X^*$ be a cyclically monotone mapping. If $T : X \to X^*$ such that $Graph\, T = Graph\, T_0 \cup \{(u_i, u_i^*) \in X \times X^*, i = \overline{1, k}\}$ and there exists $M > 0$ such that*

$$(u_i^* - u^*)(u - u_i) \leq M, \text{ for all } i = \overline{1, k}, u \in Dom\, T_0, u^* \in T_0(X), \tag{2.4}$$

then $Dom\, g_{x_0;T} \supset Dom\, f_{x_0;T_0}$ for any $x_0 \in Dom\, T_0$.

Proof. Let x_0 be an element in $\operatorname{Dom} T_0$ and let $\{x_1, x_2, \ldots, x_n\} \subset \operatorname{Dom} T_0$ be a system of distinct elements. If we add only an element $u_j, j = \overline{1, k}$, then

$$\begin{aligned}
&x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1) + \ldots + x_{i_0}^*(u_j - x_{i_0}) \\
&+ u_j^*(x_{i_0+1} - u_j) + x_{i_0+1}^*(x_{i_0+2} - x_{i_0+1}) \\
&+ \ldots + x_{n-1}^*(x_n - x_{n-1}) + x_n^*(x - x_n) \\
&= \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) + x_n^*(x - x_n) + x_{i_0}^*(u_j - x_{i_0}) \\
&+ u_j^*(x_{i_0+1} - u_j) - x_{i_0}^*(x_{i_0+1} - x_{i_0}) \leq f_{x_0;T_0}(x) + (u_j^* - x_{i_0}^*)(x_{i_0+1} - u_j) \\
&\leq f_{x_0;T_0}(x) + M, \text{ for any } x \in \operatorname{Dom} f_{x_0,T_0}.
\end{aligned}$$

Therefore, according to the definition (2.1) of $g_{x_0;T}$ it follows that

$$g_{x_0;T}(x) \leq f_{x_0;T_0}(x) + kM, \text{ for all } x \in \operatorname{Dom} f_{x_0;T_0}, \text{ as claimed.}$$

Remark 2.5. Obviously, if $M = 0$ the mapping T is cyclically monotone,. Also, the convex function $g_{x_0;T}$ can be proper in some special case when $(\operatorname{Dom} T)\backslash(\operatorname{Dom} T_0)$ is an infinite set. Generally, the following inequality

$$f_{x_0;T_0} \leq g_{x_0;T} \tag{2.5}$$

holds.

In the next result we prove that the integration property of a mapping $T : X \to X^*$ can be generated by the subdifferential of $g_{x_0;T}$.

Theorem 2.6. *Let $T : X \to X^*$ be an extension of a cyclically monotone mapping T_0. If there exist $x_0 \in Dom\, T_0$ and $\overline{x} \in Dom\, f_{x_0;T_0}$ such that $f_{x_0;T_0}(\overline{x}) = g_{x_0;T}(\overline{x})$, then $\partial f_{x_0;T_0}(\overline{x}) \subset \partial g_{x_0;T}(\overline{x})$.*

Proof. If $x^* \in \partial f_{x_0;T_0}(\overline{x})$, using the inequality (2.5) we have

$$x^*(x - \overline{x}) \leq f_{x_0;T_0}(x) - f_{x_0;T_0}(\overline{x}) = f_{x_0;T_0}(x) - g_{x_0;T}(\overline{x}) \leq g_{x_0;T}(x) - g_{x_0;T}(\overline{x}),$$

for all $x \in X$, and so $x^* \in \partial g_{x_0;T}(\overline{x})$.

Remark 2.7. Let T_0 be a maximal cyclically monotone mapping such that $\operatorname{Graph} T_0 \subset \operatorname{Graph} T$. Then on the set $\{x \in X; f_{x_0;T_0}(x) = g_{x_0;T}(x)\}$ the function $f_{x_0;T_0}$ can be regarded as an integral of mapping T.

3. Some dual properties

Firstly, we recall some fundamental dual concepts in convex analysis (see, for example [2]). Given a function $f : X \to (-\infty, \infty]$, its conjugate $f^* : X^* \to (-\infty, \infty]$ is defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x); x \in X\}, x^* \in X^*. \tag{3.1}$$

If $A \subset X$ we define the support function

$$s_A(x^*) = \sup\{x^*(x); x \in A\}. \tag{3.2}$$

Similarly, we define the support function associated of a subset of X^*.

Now, we give an other equivalent form for the Rockafellar function $g_{x_0;T}$ with respect to dual space X^* namely

$$g_{x_0;T}(x) = \sup\Big\{\sum_{i=1}^{n}(x^*_{i-1} - x^*_i)(x_i) + x^*_n(x) - x^*_0(x_0); \tag{3.3}$$
$$(x_i, x^*_i) \in \operatorname{Graph} T, x_i \neq x_j \text{ for } i \neq j, i, j = \overline{1, n}, n = 1, 2, \dots\Big\},\ x \in X,$$

where $x_0 \in \operatorname{Dom} T$.

This formula leads to consider the Rockafellar function associated to the mapping T^{-1}. Indeed, if $x^*_0 \in \operatorname{Dom} T^{-1}$ by (2.1) we have

$$g_{x^*_0;T^{-1}}(x^*) = \sup\Big\{\sum_{i=1}^{n+1}(x^*_i - x^*_{i-1})(x_{i-1}); (x^*_i, x_i) \in \operatorname{Graph} T^{-1}, \tag{3.4}$$
$$x^*_i \neq x^*_j, \text{ for } i \neq j, i, j = \overline{0, n}, x^*_{n+1} = x^*, n = 1, 2, \dots\Big\},\ x^* \in X^*.$$

But, if the system $\{x^*_1, x^*_2, \dots, x^*_n\}$ is replaced by the system $\{x^*_n, x^*_{n-1}, \dots, x^*_1\}$ we obtain the equivalent formula

$$g_{x^*_0;T^{-1}}(x^*) = \sup\Big\{\sum_{i=2}^{n}(x^*_{i-1} - x^*_i)(x_i) + (x^*_n - x^*_0)(x_0) + (x^* - x^*_1)(x_1); x^*_i \neq x$$
$$(x^*_i, x_i) \in \operatorname{Graph} T^{-1}, x^*_i \neq x^*_j \text{ for } i \neq j, i, j = \overline{1, n}, n = 1, 2 \dots\Big\},\ x^* \in X^*. \tag{3.5}$$

Generally, if $\{x_1, x_2, \dots, x_n\}$ is a system of distinct elements of $\operatorname{Dom} T$ then the corresponding system $\{x^*_1, x^*_2, \dots, x^*_n\}$, $(x_i, x^*_i) \in \operatorname{Graph} T$, can have equal elements. Thus, in the following results we need to suppose that T is injective, that is $u^* \neq v^*$ whenever $u \neq v$, $(u, u^*), (v, v^*) \in \operatorname{Graph} T$.

Theorem 3.1. *Let $T : X \to X^*$ be an injective mapping. If $(x_0, x^*_0) \in Graph T$, then*

$$g_{x_0;T}(x) \le g_{x^*_0;T^{-1}}(x^*) + s_{Dom T}(x^0_0 - x^*) + s_{Dom T^{-1}}(x - x_0), \tag{3.6}$$

for all $(x, x^*) \in X \times X^*$.

Proof. Let (x_0, x_0^*) be an element in $\operatorname{Graph} T$. By hypothesis, if $\{u_1, u_2, \ldots, u_m\}$ is a system of distinct elements in $\operatorname{Dom} T$, then a corresponding system $\{u_1^*, u_2^*, \ldots, u_m^*\}$, where $(u_j, u_j^*) \in \operatorname{Graph} T$, $j = \overline{1,m}$, also contains only distinct elements in $\operatorname{Dom} T^{-1}$. According to (3.5), taking $u_0^* = x_0^*$, we obtain

$$\sum_{i=1}^{m}(u_{j-1}^* - u_j^*)(u_j) + u_m^*(x) - x_0^*(x_0) \le \sup\Big\{ \sum_{i=2}^{n}[(x_{i-1}^* - x_i^*)(x_i) + (x_n^* - x_0^*)(x_0)$$
$$+ (x^* - x_1^*)(x_1)] + [(x_0^* - x_1^*)(x_1)]$$
$$+ x_n^*(x) - x_0^*(x_0) - (x_n^* - x_0^*)(x_0) - (x^* - x_1^*)(x_1);$$
$$(x_i, x_i^*) \in \operatorname{Graph} T, x_i \ne x_j, \text{ for } i \ne j, i, j = \overline{1,n}\Big\}$$
$$\le \sup\Big\{ \sum_{i=2}^{n}(x_{i-1}^* - x_i^*)(x_i) + (x_0^* - x^*)(x_1) + x_n^*(x - x_0); (x_i, x_i^*) \in \operatorname{Graph} T,$$
$$x_i \ne x_j \text{ for } i \ne j, i, j = \overline{1,n}\Big\} + \sup\{(x_0^* - x^*)(x_1); x_1 \in \operatorname{Dom} T\}$$
$$+ \sup\{x_n^*(x - x_0); x_n^* \in \operatorname{Dom} T^{-1}\} = g_{x_0^*;T^{-1}}(x^*) + s_{\operatorname{Dom} T}(x_0^* - x^*)$$
$$+ s_{\operatorname{Dom} T^{-1}}(x - x_0), \text{ for all } u_j, u_j^* \in \operatorname{Graph} T, u_i \ne u_j \text{ for } i \ne j,$$
$$i, j = \overline{1,n}, \ (x, x^*) \in X \times X^*, n = 1, 2, \ldots$$

Now, passing to the supremum with respect to systems $\{u_1, u_2, \ldots, u_m\}$, $m = 1, 2, \ldots$ we obtain the inequality (3.6).

Remark 3.2. If T^{-1} is injective we have a converse inequality

$$g_{x_0^*;T^{-1}}(x^*) \le g_{x_0;T}(x) + s_{\operatorname{Dom} T}(x^* - x_0^*) + s_{\operatorname{Dom} T^{-1}}(x - x_0), \tag{3.7}$$

for all $(x, x^*) \in X \times X^*$. Consequently, if T is an one to one mapping, then we can obtain an estimation for $|g_{x_0;T}(x) - g_{x_0^*,T^{-1}}(x^*)|$, $(x, x^*) \in X \times X^*$. If T is a cyclically monotone mapping the inequality (3.6) was given in [6].

Concerning the conjugate of the Rockafellar function associated to a mapping T and the Rockafellar function associated to the mapping T^{-1} we have the following result.

Theorem 3.2. *Let $T : X \to X^*$ be an injective mapping. If $Tx_0 = \{x_0^*\}$, $x_0 \in DomT$, then*

$$g_{x_0;T}^*(x^*) \le x_0^*(x_0) - g_{x^*;T^{-1}}(x_0^*), \text{ for all } x^* \in \operatorname{Dom} T^{-1}. \tag{3.8}$$

Proof. By formula (3.3) we obtain

$$g_{x^*;T^{-1}(x_0^*)} = \sup \sum_{i=1}^{n-1}(x_{i-1}^* - x_i^*)(x_i) + (x_{n-1}^* - x^*)(x), x \in T^{-1}(x^*), x_i \ne x_j, \tag{3.9}$$
$$(x_i, x_i^*) \in \operatorname{Graph} T, i, j \in \overline{1, n-1}, n = 1, 2, \ldots\}, \text{ for any } x^* \in \operatorname{Dom} T^{-1}.$$

On the other hand, by definition (3.1) of the conjugate we have

$$g^*_{x_0;T}(x^*) = \sup\{x^*(x) - g_{x_0;T}(x); x \in X\}$$

$$= \sup\{x^*(x) - \sup\Big\{ \sum_{i=1}^{n}(x^*_{i-1} - x^*_i)(x_i) + x^*_n(x) - x^*_0(x_0); (x_i, x^*_i) \in \operatorname{Graph} T,$$

$$x_i \neq x_j, \text{ for } i \neq j, i, j \in \overline{1,n}, n = 1, 2, \dots \Big\}; x \in X\} = x^*_0(x_0)$$

$$+ \sup\inf \Big\{(x^* - x^*_n)(x) - \sum_{i=1}^{n-1}(x^*_{i-1} - x^*_i)(x_i) + (x^*_{n-1} - x^*_n)(x_n); (x_i, x^*_i) \in \operatorname{Graph} T,$$

$$x_i \neq x_j, i \neq j, i, j = \overline{1,n}, n = 1, 2, \dots \Big\}.$$

Now, if we take $(x_n, x^*_n) = (x, x^*)$, according to (3.9) we obtain

$$g^*_{x_0;T}(x^*) \leq x^*_0(x_0) - \sup \sum_{i=1}^{n-1}\{(x^*_{i-1} - x^*_i)(x_i) + (x^*_{n-1} - x^*)(x); (x_i, x^*_i) \in \operatorname{Graph} T,$$

$$x_i \neq x_j, \text{ for } i \neq j, i, j = \overline{1, n-1}, n = 2, 3, \dots\} = x^*_0(x_0) - g_{x^*;T^{-1}}(x^*_0),$$

for all $x^* \in \operatorname{Dom} T^{-1}$, as claimed.

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Teodor Precupanu
Faculty of Mathematics, "Al.I. Cuza" University
Bd. Carol I, nr. 11, 700506, Iaşi, Romania
e-mail: tprecup@uaic.ro