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King-type operators related to squared Szász-Mirakyan basis

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Abstract. In this paper we study some approximation properties of a sequence of positive linear operators defined by means of the squared Szász-Mirakyan basis and prove that these operators behave better than the classical Szász-Mirakyan operators.

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1. Introduction

The operators defined by

$$S_n(f,x)=\sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad x\in[0,\infty),\ n=1,2,\dots,$$

where $s_{n,k}$ are

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},$$

were introduced and studied independently by Mirakyan [14], Favard [3] and Szász [17]. They usually are referred to as Szász-Mirakyan operators and the functions $s_{n,k}$ form the Szász-Mirakyan basis or the Poisson distribution.

Motivated by the article of Gavrea and Ivan [4] we study the following operators

$$A_n(f,x)=\frac{\displaystyle\sum_{k=0}^{\infty} s_{n,k}^2(x) f\left(\frac{k}{n}\right)}{\displaystyle\sum_{k=0}^{\infty} s_{n,k}^2(x)}, \quad x\ge 0,\ n=1,2,\dots \tag{1.1}$$

Herzog [5] introduced and studied the following sequence of positive linear operators

$$A_n^\nu(f,x) = \begin{cases} \dfrac{1}{I_\nu(nx)} \displaystyle\sum_{k=0}^{\infty} \dfrac{\left(\dfrac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+1+\nu)} \cdot f\left(\dfrac{2k}{n}\right), & x > 0 \\ f(0), & x = 0 \end{cases}$$

where I_ν is the modified Bessel function defined by

$$I_\nu(t) = \sum_{k=0}^{\infty} \frac{\left(\dfrac{nt}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+1+\nu)}.$$

For $\nu = 0$ the operators A_n^ν can be written in terms of the operators (1.1) by

$$A_n^0(f,x) = A_n(f \circ g^{-1}, g(x)),$$

where g is the function defined by $g(x) = x/2$, $x \geq 0$.

The author of [5] studied the operators A_n^ν in polynomial and exponential weight spaces (see also [6]), but did not point out how well behave these operators compared to the Szász-Mirakyan operators.

In this paper, we show that A_n are King-type operators [12] preserving the functions e_0 and e_2 and so extending the class of Szász-Mirakyan type operators which preserve some polynomial functions [2, 18]. We also prove that the error of approximation of a function f by $A_n f$ is smaller than the error of approximation by the classical Szász-Mirakyan operators. In the final part of the paper, we present some approximation properties of (A_n), showing what functions can be uniformly approximated by these operators and what is the order of the convergence by giving a quantitative Voronovskaya theorem. A similar study for Bernstein operators was done recently in [4, 9] and for Baskakov operators in [10].

2. Some properties of the operators

Let us notice first that the operators A_n preserve the functions e_0 and e_2 (we denote as usual $e_k(x) = x^k$). From the relation (1.1) we can easily see that

$$A_n(e_0, x) = e_0(x) = 1.$$

From the following relation

$$\sum_{k=0}^{\infty} s_{n,k}^2(x) \cdot \frac{k^2}{n^2} = e^{-2nx} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2} \cdot \frac{k^2}{n^2} = x^2 e^{-2nx} \sum_{k=1}^{\infty} \frac{(nx)^{2k-2}}{[(k-1)!]^2}$$

$$= x^2 e^{-2nx} \sum_{i=0}^{\infty} \frac{(nx)^{2i}}{(i!)^2} = x^2 \sum_{i=0}^{\infty} s_{n,i}^2(x).$$

we deduce that $A_n(e_2, x) = e_2(x) = x^2$, for every $x \geq 0$. In fact, only for $\nu = 0$, the general operators A_n^ν do preserve the function e_2. This can be seen from the following relation obtained in [5]

$$A_n^\nu(e_2,x) = x^2 \cdot \frac{I_{\nu+2}(nx)}{I_\nu(nx)} + \frac{2x}{n} \cdot \frac{I_{\nu+1}(nx)}{I_\nu(nx)}$$

and the recurrence relation (9.6.26) of [1]

$$I_{\nu-1}(t) - I_{\nu+1}(t) = \frac{2\nu}{t} I_\nu(t).$$

We have

$$A_n^\nu(e_2,x) = x^2 - \frac{2x\nu}{n} \cdot \frac{I_{\nu+1}(nx)}{I_\nu(nx)}.$$

So, $A_n^\nu(e_2) = e_2$ if and only if $\nu = 0$.

Let us denote

$$\mu_{n,k}(x) = A_n((e_1 - x)^k, x), \quad k = 0, 1, 2, \dots$$

the central moments of the operators A_n, which will be very important in our study.

Next let us observe that

$$\mu_{n,2}(x) = -2x\mu_{n,1}(x). \tag{2.1}$$

Indeed,

$$\mu_{n,2}(x) = A_n(e_2,x) - 2xA_n(e_1,x) + x^2 A_n(e_0,x) = 2x^2 - 2xA_n(e_1,x) = -2x\mu_{n,1}(x).$$

Lemma 2.1. *For every $x \in (0,\infty)$ we have*

$$\lim_{n\to\infty} 4n \cdot \mu_{n,1}(x) = -1 \tag{2.2}$$

$$\lim_{n\to\infty} 2n \cdot \mu_{n,2}(x) = x. \tag{2.3}$$

Proof. Because of the relation (2.1) it suffices to prove (2.2).
Let us denote

$$K_n(x) = \sum_{k=0}^{\infty} s_{n,k}^2(x). \tag{2.4}$$

The function K_n was expressed [15] in terms of the modified Bessel function I_0 by

$$K_n(x) = e^{-2nx} I_0(2nx). \tag{2.5}$$

Using the well-known relation

$$s'_{n,k}(x) = s_{n,k}(x) \cdot \frac{k - nx}{x}$$

we have

$$2n \cdot \mu_{n,1}(x) = 2n \left(\frac{\sum_{k=0}^{\infty} s_{n,k}^2(x) \cdot \frac{k}{n}}{K_n(x)} - x \right) = \frac{2 \sum_{k=0}^{\infty} s_{n,k}^2(x)(k - nx)}{K_n(x)}$$

$$= \frac{2x \sum_{k=0}^{\infty} s_{n,k}(x) s'_{n,k}(x)}{K_n(x)} = \frac{x K'_n(x)}{K_n(x)} = \frac{2nx[I'_0(2nx) - I_0(2nx)]}{I_0(2nx)}.$$

We have obtained a formula expressing the central moment of order 1 in terms of the modified Bessel function I_0:

$$\mu_{n,1}(x) = x \left(\frac{I'_0(2nx)}{I_0(2nx)} - 1 \right). \tag{2.6}$$

For every $x \in (0, \infty)$ the quantity $t = 2nx$ grows to infinity when n tends to infinity. Using the asymptotic relations (9.7.1) and (9.7.3) from Abramowitz and Stegun [1]

$$I_0(t) \sim \frac{e^t}{\sqrt{2\pi t}} \left(1 + \frac{1}{8t} + \frac{9}{2(8t)^2} + \dots \right) \quad (t \to \infty) \tag{2.7}$$

$$I'_0(t) \sim \frac{e^t}{\sqrt{2\pi t}} \left(1 - \frac{3}{8t} - \frac{15}{2(8t)^2} - \dots \right) \quad (t \to \infty)$$

we obtain

$$\mu_{n,1}(x) \sim -\frac{1}{4n} - \frac{1}{32n^2 x} - \frac{15}{1024 n^3 x^2} - \dots \quad (n \to \infty)$$

which proves (2.2). □

Lemma 2.2. *The sequence* $(n \cdot \mu'_{n,1}(x))$ *converges to zero for every* $x > 0$.

Proof. Computing the derivative of $\mu_{n,1}$ we obtain

$$\mu'_{n,1}(x) = \frac{I'_0(2nx)}{I_0(2nx)} - 1 + 2nx \cdot \frac{I''_0(2nx) I_0(2nx) - [I'_0(2nx)]^2}{[I_0(2nx)]^2}.$$

Using the relation $tI''_0(t) + I'_0(t) - tI_0(t) = 0$ (see (9.6.1) from [1]), we have

$$\mu'_{n,1}(x) = 2nx - 1 - 2nx \frac{[I'_0(2nx)]^2}{[I_0(2nx)]^2}.$$

The asymptotic relations (2.7) show that

$$\mu'_{n,1}(x) \sim -\frac{29}{128(2nx)^2} + \frac{31}{1024(2nx)^3} + \dots \quad (n \to \infty)$$

and this proves the assertion stated in the lemma. □

Lemma 2.3. *For every* $x \geq 0$ *we have*

$$\mu_{n,2}(x) \leq S \cdot \frac{x}{n}, \tag{2.8}$$

where S is defined by

$$S = \sup_{x>0} \left(x - \frac{x^2}{\frac{1}{2} + \sqrt{x^2 + \frac{9}{4}}} \right) = 0.67038\ldots$$

Proof. Using (2.6) and (2.1) the central moment of order 2 can be expressed by

$$\mu_{n,2}(x) = 2x^2 \left(1 - \frac{I_0'(2nx)}{I_0(2nx)} \right).$$

To prove (2.8) it is enough to prove that

$$t \left(1 - \frac{I_0'(t)}{I_0(t)} \right) < S, \quad t > 0.$$

Using inequality (73) of [16] we have

$$\frac{tI_0'(t)}{I_0(t)} > \frac{t^2}{\frac{1}{2} + \sqrt{\frac{9}{4} + t^2}}.$$

But this proves that

$$t \left(1 - \frac{I_0'(t)}{I_0(t)} \right) < t - \frac{t^2}{\frac{1}{2} + \sqrt{\frac{9}{4} + t^2}} \le S. \qquad \square$$

Remark 2.4. Because the second central moment of the usual Szász-Mirakyan operators is $\frac{x}{n}$, inequality (2.8) proves that the central moment of order 2 of the operators (1.1) is smaller than the classical Szász-Mirakyan operators. In addition, we use the estimation

$$|L_n(f,x) - f(x)| \le (1 + n\mu_{n,2}(x)) \cdot \omega \left(f, \frac{1}{\sqrt{n}} \right),$$

which is valid for every sequence of positive linear operators (L_n) preserving constants functions and for every uniformly continuous function f. This estimation proves that the error by approximating f with $A_n f$ is smaller than the error of approximation by the classical Szász-Mirakyan operators.

We prove in the next Lemma that A_n satisfy a differential equation. This equation is similar to the relation satisfied by the so called exponential type operators (see [13, 11]).

Lemma 2.5. *For every $f \in C[0,1]$ and $x \in (0,1)$ we have*

$$(A_n(f,x))' = \frac{2n}{x} \left[A_n(f \cdot (e_1 - xe_0), x) - A_n(e_1 - xe_0, x) \cdot A_n(f,x) \right]. \qquad (2.9)$$

Proof. Using again

$$s_{n,k}'(x) = s_{n,k}(x) \cdot \frac{k - nx}{x}$$

we get

$$\left(\frac{s_{n,k}^2(x)}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)}\right)' = \frac{2s_{n,k}(x)s_{n,k}'(x)}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)} - \frac{2s_{n,k}^2(x)\sum\limits_{i=0}^{n} s_{n,i}(x)s_{n,i}'(x)}{\left(\sum\limits_{i=0}^{n} s_{n,i}^2(x)\right)^2}$$

$$= \frac{2s_{n,k}^2(x)}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)} \cdot \left(\frac{k-nx}{x} - \frac{\sum\limits_{i=0}^{n} s_{n,i}^2(x)\frac{i-nx}{x}}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)}\right)$$

$$= \frac{2s_{n,k}^2(x)}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)} \cdot \left(\frac{k}{x} - \frac{\sum\limits_{i=0}^{n} s_{n,i}^2(x)\frac{i}{x}}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)}\right)$$

$$= \frac{2n}{x} \cdot \frac{s_{n,k}^2(x)}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)} \cdot \left(\frac{k}{n} - \frac{\sum\limits_{i=0}^{n} s_{n,i}^2(x)\frac{i}{n}}{\sum\limits_{i=0}^{n} s_{n,i}^2(x)}\right).$$

We obtain

$$(A_n(f,x))' = \frac{2n}{x} \cdot A_n(f \cdot (e_1 - A_n(e_1,x)), x)$$

which is equivalent with (2.9). □

Lemma 2.6. *We have for every $x > 0$*

$$\lim_{n\to\infty} (2n)^2 \cdot \mu_{n,4}(x) = 3x^2.$$

Proof. Using Lemma 2.2 and (2.1) the following limit holds true for every $x > 0$

$$\lim_{n\to\infty} 2n \cdot \mu_{n,2}'(x) = \lim_{n\to\infty} -4n\mu_{n,1}(x) - 4nx\mu_{n,1}'(x) = 1.$$

In relation (2.9) we take $f = (e_1 - xe_0)^k$ and we obtain the recurrence relation

$$(\mu_{n,k}(x))' + k \cdot \mu_{n,k-1}(x) = \frac{2n}{x} \cdot \left[\mu_{n,k+1}(x) - \mu_{n,1}(x) \cdot \mu_{n,k}(x)\right], \tag{2.10}$$

which is similar to the relation (2.7) of Ismail and May [11]. Using (2.10) we get

$$2n\mu_{k+1}(x) = x\mu_{n,k}'(x) + kx\mu_{n,k-1}(x) + 2n\mu_{n,1}(x)\mu_{n,k}(x), \quad k = 1, 2, \ldots$$

For $k = 2$ we have

$$2n\mu_3(x) = x\mu_{n,2}'(x) + 2x\mu_{n,1}(x) + 2n\mu_{n,1}(x)\mu_{n,2}(x).$$

Multiplying this equality with $2n$ and using the relations (2.2) and (2.3), we have for every x

$$\lim_{n\to\infty} 4n^2 \cdot \mu_{n,3}(x) = -\frac{x}{2}.$$

For $k = 3$, the recurrence (2.10) becomes

$$\mu'_{n,3}(x) + 3\mu_{n,2}(x) = \frac{2n}{x} \cdot [\mu_{n,4}(x) - \mu_{n,1}(x)\mu_{n,3}(x)] .$$

Multiplying with $2n$ and letting n tend to infinity we get

$$\lim_{n\to\infty} 4n^2 \cdot \mu_{n,4}(x) = 3x^2,$$

for every $x > 0$, if $2n\mu'_{n,3}(x) \to 0$. We prove this convergence.

Applying the derivative to the relation (2.10) for $k = 2$ we get

$$\begin{aligned} 2n\mu'_{n,3}(x) &= 2n\mu_{n,1}(x)\mu'_{n,2}(x) + 2n\mu_{n,2}(x)\mu'_{n,1}(x) \\ &\quad + \mu'_{n,2}(x) + x\mu''_{n,2}(x) + 2x\mu'_{n,1}(x) + 2\mu_{n,1}(x). \end{aligned}$$

It remains to prove that $\mu''_{n,2}$ converges to zero.

Applying the derivative twice to the relation (2.1), the sequence $(\mu''_{n,2})$ converges to zero if and only if the sequence $\mu''_{n,1}$ converges to zero. But applying the derivative to the relation (2.10) for $k = 1$ we obtain

$$2n\mu'_{n,2}(x) = 4n\mu_{n,1}(x)\mu'_{n,1}(x) + \mu'_{n,1}(x) + x\mu''_{n,1}(x) + 1.$$

Using that $2n\mu'_{n,2}(x) \to 1$ we obtain that $\mu''_{n,1} \to 0$ and the lemma is proved. □

3. Some approximation results

In order to give some approximation results for the operators A_n, let us introduce some notation.

For $\alpha \geq 0$, we denote by $C_{\theta,\alpha}$ the space of all continuous functions defined on the positive half-line $f : (0, \infty) \to \mathbb{R}$ with the property that exists a constant $M > 0$ such that $|f(x)| \leq Me^{\alpha\theta(x)}$, for every $x > 0$. We denote with C_θ the union of all spaces $C_{\theta,\alpha}$.

Let us observe that for $\theta(x) = x$, the functions $A_n f$ exist for every $f \in C_{\theta,\alpha}$. To prove this, it is enough to prove that $A_n(e^{\alpha t})$ exist. We will prove more in the next lemma.

Lemma 3.1. *The sequence $A_n(e^{\alpha t}, x)$ converges pointwise to the function $e^{\alpha x}$.*

Proof. We have

$$A_n(e^{\alpha t}, x) = \frac{I_0\left(2nxe^{\frac{\alpha}{2n}}\right)}{I_0(2nx)}.$$

For a fixed $x \in (0, \infty)$ we use the asymtotic relation (2.7) and we obtain

$$A_n(e^{\alpha t}, x) \sim \frac{e^{2nxe^{\frac{\alpha}{2n}}}}{\sqrt{2\pi \cdot 2nxe^{\frac{\alpha}{2n}}}} \cdot \frac{\sqrt{2\pi \cdot 2nx}}{e^{2nx}} \sim e^{2nx(e^{\frac{\alpha}{2n}}-1)} \sim e^{\alpha x} \quad (n \to \infty). \qquad \square$$

Remark 3.2. The Lemma 3.1 implies that for a fixed $x > 0$ we have

$$A_n(\max(e^{\alpha t}, e^{\alpha x}), x) \leq M_\alpha(x), \tag{3.1}$$

for every $n \in \mathbb{N}$. Indeed, for $x > 0$, there is n_0 such that

$$\left| A_n(e^{\alpha t}, x) - e^{\alpha x} \right| \leq 1, \quad \text{for every } n \geq n_0.$$

We obtain for every $n \geq n_0$

$$A_n(\max(e^{\alpha t}, e^{\alpha x}), x) \leq A_n(e^{\alpha t} + e^{\alpha x}, x) \leq 1 + 2e^{\alpha x}.$$

The inequality (3.1) is true for

$$M_\alpha(x) = 1 + 2e^{\alpha x} + \max_{n \leq n_0} A_n(\max(e^{\alpha t}, e^{\alpha x}), x).$$

Remark 3.3. As was pointed out in Remark 7.2.1 of [6], the function $A_n f$ does not necessarily belong to the space $C_{\theta,\alpha}$ when f belong to the space $C_{\theta,\alpha}$, for $\theta(x) = x$. We prove that for $\theta(x) = \sqrt{x}$, this condition is satisfied as in the case of the classical Szász-Mirakyan operators (see [7]).

Lemma 3.4. *There is a constant $M_\alpha > 0$ not depending on n or x such that*

$$A_n(e^{\alpha\sqrt{t}}, x) \leq M_\alpha e^{\alpha\sqrt{x}}, \tag{3.2}$$

for every $x > 0$, $\alpha \geq 0$ and $n \in \mathbb{N}$.

Proof. We need to prove that $A_n(e^{\alpha(\sqrt{t}-\sqrt{x})}, x)$ is bounded.
Starting from the inequality

$$\sqrt{t} - \sqrt{x} = \frac{t-x}{\sqrt{t}+\sqrt{x}} \leq \frac{t-x}{\sqrt{x}}, \quad x > 0 \tag{3.3}$$

we obtain that

$$A_n(e^{\alpha(\sqrt{t}-\sqrt{x})}, x) \leq A_n(e^{\frac{\alpha(t-x)}{\sqrt{x}}}, x) = \frac{A_n(e^{\frac{\alpha t}{\sqrt{x}}}, x)}{e^{\alpha\sqrt{x}}} = \frac{I_0\left(2nxe^{\frac{\alpha}{2n\sqrt{x}}}\right)}{I_0(2nx) \cdot e^{\alpha\sqrt{x}}}.$$

Using again (2.7) we deduce the existence of a constant $t_0 > 0$ such that

$$\frac{e^t}{2\sqrt{2\pi t}} < I_0(t) < \frac{3e^t}{2\sqrt{2\pi t}}, \quad \text{for every } \ t > t_0.$$

So, for $x > \frac{t_0}{2n}$ and $n \in \mathbb{N}$

$$\begin{aligned} A_n(e^{\alpha(\sqrt{t}-\sqrt{x})}, x) &\leq 3\frac{e^{2nxe^{\frac{\alpha}{2n\sqrt{x}}}}}{\sqrt{2\pi \cdot 2nxe^{\frac{\alpha}{2n\sqrt{x}}}}} \cdot \frac{\sqrt{2\pi \cdot 2nx}}{e^{2nx} \cdot e^{\alpha\sqrt{x}}} \\ &\leq 3\exp\left(2nx(e^{\frac{\alpha}{2n\sqrt{x}}} - 1) - \alpha\sqrt{x}\right). \end{aligned}$$

Using the inequality $e^u - 1 \leq u + u^2 e^u$, $u \geq 0$, we obtain

$$\begin{aligned} A_n(e^{\alpha(\sqrt{t}-\sqrt{x})}, x) &\leq 3\exp\left(2nx \cdot \frac{\alpha}{2n\sqrt{x}} + 2nx \cdot \frac{\alpha^2}{4n^2 x} e^{\frac{\alpha}{2n\sqrt{x}}} - \alpha\sqrt{x}\right) \\ &= 3\exp\left(\frac{\alpha^2}{2n} e^{\frac{\alpha}{2n\sqrt{x}}}\right) \leq 3\exp\left(\frac{\alpha^2}{2} e^{\frac{\alpha}{\sqrt{2t_0}}}\right). \end{aligned}$$

Consider now the case when x is smaller than $\frac{t_0}{2n}$. In this case, we need only prove that $A_n(e^{\alpha\sqrt{t}}, x)$ is bounded. Because $\sqrt{k} \le k$, for every $k = 0, 1, 2, \ldots$ and $I_0(2nx) \ge 1$ we obtain

$$A_n(e^{\alpha\sqrt{t}}, x) \le A_n(e^{t\alpha\sqrt{n}}, x) = \frac{I_0(2nxe^{\frac{\alpha}{2\sqrt{n}}})}{I_0(2nx)} \le I_0\left(2nxe^{\frac{\alpha}{2\sqrt{n}}}\right) \le I_0\left(t_0 e^{\frac{\alpha}{2}}\right). \qquad \square$$

We need the following general result.

Theorem 3.5 ([8])**.** *Let m be a nonnegative integer and let $f \in C_{\theta,\alpha}$ such that f is m times continuously differentiable with $f^{(m)} \in C_{\theta,\alpha}$. Then*

$$\left| L_n(f, x) - \sum_{k=0}^{m} \frac{f^{(k)}(x)}{k!} \cdot \mu_{n,k}(x) \right| \le \frac{1}{m!}\left(A_{n,m}(x) + \frac{B_{n,m}(x)}{\delta_n}\right) \omega_{\varphi,\theta,\alpha}\left(f^{(m)}, \delta_n\right)$$

where

$$A_{n,m}(x) = L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)|t - x|^m, x\right)$$

$$B_{n,m}(x) = L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)|t - x|^m \cdot |\varphi(t) - \varphi(x)|, x\right)$$

$$\omega_{\varphi,\theta,\alpha}(f, \delta) = \sup_{\substack{x,t \in I \\ |\varphi(t)-\varphi(x)| \le \delta}} \frac{|f(t) - f(x)|}{\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)}$$

and φ is a continuous and strictly increasing function on I such that $\theta \circ \varphi^{-1}$ is uniformly continuous on $\varphi(I)$.

Theorem 3.6. *Let $\theta(x) = \varphi(x) = \sqrt{x}$. For every $f \in C_{\theta,\alpha}$ there is a constant $M > 0$ independent of n and x such that*

$$|A_n(f, x) - f(x)| \le Me^{\alpha\sqrt{x}} \cdot \omega_{\varphi,\theta,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

for every $x > 0$ and $n \in \mathbb{N}$.

Proof. We apply Theorem 3.5 for $m = 0$ and $\delta_n = \frac{1}{\sqrt{n}}$. Using inequality (3.2) we easily obtain that $A_{n,0}(x) \le C_1 e^{\alpha\sqrt{x}}$, for every $x > 0$, for some constant $C_1 > 0$. Using the Cauchy-Schwarz inequality for positive linear operators the quantity $B_{n,0}(x)$ is bounded by

$$B_{n,0}(x) \le \sqrt{A_n(\max(e^{2\alpha\sqrt{t}}, e^{2\alpha\sqrt{x}}), x)} \cdot \sqrt{A_n(|\varphi(t) - \varphi(x)|^2, x)}.$$

Using inequalities (3.3) and (2.8) we have for $x > 0$

$$A_n(|\varphi(t) - \varphi(x)|^2, x) \le \frac{1}{x} \cdot \mu_{n,2}(x) \le \frac{S}{n}.$$

Using again (3.2), the inequality

$$\sqrt{n} \cdot B_{n,0}(x) \le C_2$$

is true for every $x > 0$ and $n \ge 1$, where C_2 is some constant independent of n and x. $\square$

Corollary 3.7. *For every function f such that $g(x) = e^{-x}f(x^2)$ is uniformly continuous on $(0,\infty)$ we have*

$$\lim_{n\to\infty} \sup_{x>0} \; e^{-\alpha\sqrt{x}} \left|A_n(f,x) - f(x)\right| = 0.$$

Proof. Because g is uniformly continuous, $\omega_{\varphi,\theta,\alpha}(f, 1/\sqrt{n}) \to 0$ when $n \to \infty$ (see [8]). □

Theorem 3.8. *For $\alpha \geq 0$, $\theta(x) = x$ and $\varphi(x) = x$ let $f \in C_{\theta,\alpha}$ be a twice continuously differentiable function such that $f'' \in C_{\theta,\alpha}$. Then*

$$\left|A_n(f,x) - f(x) - \mu_{n,1}(x)f'(x) - \frac{\mu_{n,2}(x)}{2}f''(x)\right|$$
$$\leq \frac{1}{2}\left(\sqrt{\mu_{n,4}(x)M_{2\alpha}(x)} + \sqrt{n}\cdot\sqrt[4]{M_{4\alpha}(x)}\cdot\sqrt[4]{[\mu_{n,4}(x)]^3}\right)\cdot\omega_{\varphi,\theta,\alpha}\left(f'', \frac{1}{\sqrt{n}}\right),$$

for every $x > 0$ and $n \in \mathbb{N}$.

Proof. We use Theorem 3.5 for $m = 2$ and $\delta_n = \frac{1}{\sqrt{n}}$. We have

$$A_{n,2}(x) \leq \sqrt{A_n(\max(e^{2\alpha t}, e^{2\alpha x}), x)}\cdot\sqrt{A_n(|t-x|^4, x)} \leq \sqrt{\mu_{n,4}(x)M_{2\alpha}(x)}.$$

Using Hölder inequality for $p = 4$ and $q = 4/3$ we obtain

$$\begin{aligned}B_{n,2}(x) &= A_n(\max(e^{\alpha t}, e^{\alpha x})\,|t-x|^3, x)\\ &\leq \left(A_n(\max(e^{4\alpha t}, e^{4\alpha x}), x)\right)^{\frac{1}{4}}\cdot\left(A_n\left(|t-x|^4, x\right)\right)^{\frac{3}{4}}\\ &\leq \sqrt[4]{M_{4\alpha}(x)}\cdot\sqrt[4]{[\mu_{n,4}(x)]^3}.\end{aligned}$$

□

Corollary 3.9. *For every $f \in C_{\theta,\alpha}$, with $\theta(x) = x$ such that f'' exists and*

$$g(x) = e^{-x}f''(x)$$

is uniformly continuous on $(0,\infty)$ and for every $x > 0$

$$\lim_{n\to\infty} n[A_n(f,x) - f(x)] = -\frac{1}{4}\cdot f'(x) + \frac{x}{4}\cdot f''(x).$$

Proof. Because g is uniformly continuous on $(0,\infty)$, the quantity $\omega_{\varphi,\theta,\alpha}\left(f'', \frac{1}{\sqrt{n}}\right)$ tends to zero as n goes to infinity. We multiply with n the inequality proved in Theorem 3.8 and we take the limit as n tends to infinity, using Lemma 2.6 and the relations (2.2) and (2.3). The right-hand side of this inequality is 0. □

Problem 3.10. We propose the reader to study the general operators

$$L_n(f,x) = \frac{\displaystyle\sum_{k=0}^{\infty} g(s_{n,k}(x))f\left(\frac{k}{n}\right)}{\displaystyle\sum_{k=0}^{\infty} g(s_{n,k}(x))}, \quad x \geq 0,\; n = 1, 2, \ldots.$$

For $g(x) = x$ we obtain the classical Szász-Mirakyan operators. For $g(x) = x^2$ we have the operators studied in this paper. It would be interesting to study the operators for $g(x) = x^m$, related to the Rényi entropy and for $g(x) = x \ln x$, related to the Shannon entropy.

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