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# A refinement of an inequality due to Ankeny and Rivlin

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**Abstract.** Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree $n$,

$$M(p, R) := \max_{|z|=R\geq 0} |p(z)|, \text{ and } M(p, 1) := M(p).$$

Then by well-known result due to Ankeny and Rivlin [1], we have

$$M(p.R) \leq \left(\frac{R^n + 1}{2}\right) M(p), \; R \geq 1.$$

In this paper, we sharpen and generalizes the above inequality by using a result due to Govil [5].

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## 1. Introduction

Let $\mathcal{P}_n := \left\{ p(z); p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \right\}$ be a class of polynomial of degree $n$. Let $\max_{|z|=R} |p(z)| = M(p, R)$ and $M(p, 1) = M(p)$.Then from maximum modulus principle, $M(p, R)$ is a strictly increasing function and for $0 \leq R < \infty$. Also, it is a simple deduction from the maximum modulus principle (see [10, p. 158, Problem 269]) that for $R \geq 1$,

$$M(p, R) \leq R^n M(p). \tag{1.1}$$

The result is best possible and equality holds if and only if $p(z) = \lambda z^n$, where $\lambda$ being a complex number.
For $p \in \mathcal{P}_n$ not vanishing in the interior of unit circle, Ankeny and Rivlin [1] sharpened inequality (1.1), by proving following result.

**Theorem 1.1.** *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right) M(p), \;\; R \geq 1. \tag{1.2}$$

*The above inequality is sharp and equality holds for polynomial*

$$p(z) = \alpha + \beta z^n, \;\; |\alpha| = |\beta|.$$

Since the equality in (1.2) holds only for $p(z) = \alpha + \beta z^n$, which satisfy

$$|\beta| = \frac{1}{2} M(p), \tag{1.3}$$

therefore it should possible to improve the bound (1.2) for the polynomial not satisfying (1.3). Govil [5] solve this problem by proving the following result.

**Theorem 1.2.** *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \;\; \leq \;\; & \left(\frac{R^n+1}{2}\right) M(p) - \frac{n}{2}\left(\frac{M(p)^2 - 4|a_n|^2}{M(p)}\right)\left\{\frac{(R-1)M(p)}{M(p)+2|a_n|}\right. \\ & \left. - \ln\left(1 + \frac{(R-1)M(p)}{M(p)+2|a_n|}\right)\right\}. \end{aligned} \tag{1.4}$$

*The result is best possible and the equality holds for $p(z) = (\lambda + \mu z^n)$, $\lambda$ and $\mu$ being complex numbers with $|\lambda| = |\mu|$.*

The other extension and generalization of Theorem 1.1 has been mentioned in the various article,e.g Aziz [2], Aziz and Mohammad [3], Milovanović, Mitrinović and Rassias [8], Govil [6], Govil, Qazi and Rahman [7] and Rahman and Schmeisser [12], Tripathi [13] etc.

## 2. Main results

In this paper, we prove the following improved generalization of Theorem 1.2 for the class of Lacunary type of polynomial

$$p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu.$$

**Theorem 2.1.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$ is a polynomial of degree $n$ and $p(z) \neq 0$ for $|a| < k, k \geq 1$, then for $R > r \geq 1$,*

$$\begin{aligned} |\{p(Re^{i\theta})\}^s| \leq & \frac{(R^{ns} - r^{ns})}{1+k^\mu}\{M(p)\}^s - \frac{n}{1+k^\mu}\{M(p)\}^s \left(1 - \frac{(1+k^\mu)|a_n|}{M(p)}\right) h(n) \\ & + |\{p(re^{i\theta})\}^s|, \end{aligned} \tag{2.1}$$

*where*

$$h(n) = \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1}\left(\frac{R^{n-k} - r^{n-k}}{n-k}\right)(-1)^k\left(\frac{(1+k^\mu)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{k-1}$$
$$+ (-1)^n\left(\frac{(1+k^\mu)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{n-1}\ln\left(\frac{R(M(p)) + (1+k^\mu)|a_n|}{r(M(p)) + (1+k^\mu)|a_n|}\right)$$

*for* $n \geq 1$ *and* $h(0) = 0$.

On taking $s = 0$, $\mu = 1$, $r = 1$ and $k = 1$, we have the following application of above Theorem 2.1.

**Corollary 2.2.** *If* $p \in \mathcal{P}_n$ *and* $p(z) \neq 0$ *for* $|z| < 1$, *then for* $R \geq 1$,

$$|p(Re^{i\theta})| \leq \frac{(R^n + 1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(n), \tag{2.2}$$

*where*

$$h(n) = \left(\frac{R^n - 1}{n}\right) + \sum_{k=1}^{n-1}\left(\frac{R^{n-k} - 1}{n-k}\right)(-1)^k\left(\frac{2|a_n|}{M(p)} + 1\right)\left(\frac{2|a_n|}{M(p)}\right)^{k-1}$$
$$+ (-1)^n\left(\frac{2|a_n|}{M(p)} + 1\right)\left(\frac{2|a_n|}{M(p)}\right)^{n-1}\ln\left(1 + \frac{(R-1)M(p)}{M(p) + 2|a_n|}\right)$$

*for* $n \geq 1$ *and* $h(0) = 0$.

**Remark 2.3.** From Lemma 3.7, we get $0 \leq h(n)$. Using this in Corollary 2.2, we get

$$|p(Re^{i\theta})| \leq \frac{(R^n + 1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(n) \leq \frac{(R^n + 1)}{2}M(p),$$

which shows that Corollary 2.2, clearly refines Theorem 1.1 due to Ankeny and Rivlin [1].

**Remark 2.4.** From Lemma 3.7, we have $h(1) \leq h(n)$. Using this inequality in Corollary 2.2, we get

$$\begin{aligned} |p(Re^{i\theta})| &\leq \frac{(R^n + 1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(n) \\ &\leq \frac{(R^n + 1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(1), \end{aligned} \tag{2.3}$$

and,

$$h(1) = (R-1) - \left(1 + \frac{2|a_n|}{M(p)}\right)\ln\left(1 + \frac{(R-1)M(p)}{M(p) + 2|a_n|}\right). \tag{2.4}$$

Substitute the value of $h(1)$ in (2.3), we get

$$\begin{aligned} |p(Re^{i\theta})| \leq{}& \left(\frac{R^n + 1}{2}\right)M(p) - \frac{n}{2}\left(\frac{M(p)^2 - 4|a_n|^2}{M(p)}\right)\left\{\frac{(R-1)M(p)}{M(p) + 2|a_n|}\right. \\ &\left. - \ln\left(1 + \frac{(R-1)M(p)}{M(p) + 2|a_n|}\right)\right\}, \end{aligned}$$

which is Theorem 1.2 due to Govil [5].

By taking $\mu = 1$ in inequality (2.1), we obtain the following results.

**Corollary 2.5.** *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $|z| < k, k \geq 1$, then for $R > r \geq 1$,*

$$\begin{aligned} |\{p(Re^{i\theta})\}^s| \quad \leq \quad & \frac{(R^{ns} - r^{ns})}{1+k}\{M(p)\}^s - \frac{n}{1+k}\{M(p)\}^s \left(1 - \frac{(1+k)|a_n|}{M(p)}\right) h(n) \\ & + |\{p(re^{i\theta})\}^s|, \end{aligned} \tag{2.5}$$

*where*

$$\begin{aligned} h(n) = & \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1} \left(\frac{R^{n-k} - r^{n-k}}{n-k}\right)(-1)^k \left(\frac{(1+k)|a_n|}{M(p)} + 1\right) \left(\frac{(1+k)|a_n|}{M(p)}\right)^{k-1} \\ & + (-1)^n \left(\frac{(1+k)|a_n|}{M(p)} + 1\right) \left(\frac{(1+k)|a_n|}{M(p)}\right)^{n-1} \ln\left(\frac{R(M(p)) + (1+k)|a_n|}{r(M(p)) + (1+k)|a_n|}\right) \end{aligned}$$

*for $n \geq 1$ and $h(0) = 0$.*

**Remark 2.6.** We also have some other application Theorem 2.1, by taking $s = 0$, $k = 1$ and $r = 1$ respectively.

## 3. Lemmas

For the proof of theorem, we need the following lemmas. Our first lemma is a well-known generalization of Schwarz's lemma (see for example [9, p. 167]).

**Lemma 3.1.** *If $f(z)$ is analytic inside and on the circle $|z| = 1$, $f(0) = a$, where $|a| < f$, then*

$$|f(z)| \leq M(f) \left(\frac{M(f)|z| + |a|}{|a||z| + M(f)}\right). \tag{3.1}$$

**Lemma 3.2.** *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$, then for $|z| = R \geq 1$,*

$$|p(z)| \leq \left(\frac{|a_n|R + M(p)}{M(p)R + |a_n|}\right) M(p)R^n. \tag{3.2}$$

The proof follows easily on applying Lemma 3.1 to the function $T(z) = z^n p(1/z)$ and noting that $M(T) = M(p)$ (for details see [12, Lemma 2]).

From Lemma 3.2, one immediately gets:

**Lemma 3.3.** *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$, then for $|z| = R \geq 1$,*

$$|p(z)| \leq \left(1 - \frac{(M(p) - |a_n|)(R-1)}{M(p)R + |a_n|}\right) M(p)R^n. \tag{3.3}$$

The following result is due to Chan and Malik [4].

**Lemma 3.4.** *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ *is a polynomial of degree* $n$*, and* $p(z) \neq 0$ *for* $|z| < k, k \geq 1$*, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^{\mu}} M(p). \tag{3.4}$$

**Lemma 3.5.** *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* $n$*, and let* $r \geq 1$*, then*

$$\left(1 - \frac{(x-|a_n|)(r-1)}{rx + n|a_n|}\right) x \tag{3.5}$$

*is an increasing function of* $x$*, for* $x > 0$*.*

The proof of above lemma is straight forward using derivative test, so we omit the detail proof.

**Lemma 3.6.** *Let*

$$h(n) = \int_r^R \frac{(t-1)(t^{n-1})}{t+a} dt \text{ for } n \geq 1.$$

*Then*

$$h(n) = \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1} \left(\frac{R^{n-k} - r^{n-k}}{n-k}\right) (-1)^k (a+1) a^{k-1}$$
$$+ (-1)^n (a+1) a^{n-1} \ln\left(\frac{R+a}{r+a}\right).$$

*Proof.* We define the function $f(n) = \int_r^R \frac{t^n}{t+a} dt$ for $n \geq 0$. It is easy to see that

$$h(n) = f(n) - f(n-1) \text{ for } n \geq 1.$$

We can obtain

$$f(n) + af(n-1) = \int_r^R \frac{t^n + at^{n-1}}{t+a} dt$$
$$= \int_r^R \frac{t^{n-1}(t+a)}{t+a} dt = \frac{R^n - r^n}{n} = g(n), \quad \text{(say)}.$$

Then

$$f(n) = g(n) - af(n-1). \tag{3.6}$$

Solving the recurrence relation (3.6), we get

$$f(n) = \sum_{k=0}^{n-1} g(n-k)(-1)^k a^k + (-1)^n a^n f(0), \tag{3.7}$$

where

$$f(0) = \int_1^R \frac{1}{r+a} dr = \ln\left(\frac{R+a}{r+a}\right).$$

Now, Substituting the value of $f(0)$ in (3.7), we get

$$f(n) \quad = \quad \sum_{k=0}^{n-1} g(n-k)(-1)^k a^k + (-1)^n a^n \ln\left(\frac{R+a}{r+a}\right), n \geq 0. \qquad (3.8)$$

Using $h(n) = f(n) - f(n-1)$ and value of $g(n)$, we have Lemma 3.6 for $n \geq 1$. □

**Lemma 3.7.** *Let*

$$h(n) = \int_r^R \frac{(t-1)(t^{n-1})}{t+a} dt \text{ for } n \geq 1.$$

*Then $h(n)$ is a non-negative increasing function of $n$ for $n \geq 1$.*

*Proof.* Let

$$f(n) = \int_r^R \frac{r^n}{r+a} dr \text{ for } n \geq 0.$$

It is easy to see that $h(n) = f(n) - f(n-1)$ for $n \geq 1$. For $n \geq 1$,

$$f(n) - f(n-1) = \int_1^R \frac{(r-1)(r^{n-1})}{r+a} dr \geq \int_1^R \frac{(r-1)(r^{n-2})}{r+a} dr = f(n-1) - f(n-2)$$

as $r^{n-1} \geq r^{n-2}$ for $r \geq 1$. Therefore,

$$h(n) = f(n) - f(n-1) \geq f(n-1) - f(n-2) = h(n-1).$$

Therefore, $h(n)$ is an increasing function of $n$ for $n \geq 1$.
Also, $h(n) = f(n) - f(n-1) \geq 0$ for $n \geq 0$ as

$$\int_r^R \frac{(t-1)(t^{n-1})}{t+a} dr \geq 0$$

for $n \geq 1$ and $h(0) = 0$. Therefore, $h(n) \geq 0$ and is an increasing function of $n$ for $n \geq 0$. □

## 4. Proof of the Theorem

*Proof of Theorem* 2.1. For each $\theta$, $0 \leq \theta < 2\pi$, we have

$$|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s| = \left|\int_r^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt\right| \leq \int_r^R s|\{p(te^{i\theta})\}^{s-1}||p'(te^{i\theta})|dt,$$

$$\leq \{M(p)\}^{s-1} \int_r^R t^{ns-n} s|p'(te^{i\theta})|dt$$

$$|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s|$$
$$\leq \{M(p)\}^{s-1} \int_r^R st^{ns-1} \left\{1 - \frac{(M(p') - n|a_n|)(t-1)}{n|a_n| + tM(p')}\right\} M(p')dt, \qquad (4.1)$$

by using Lemma 3.3 for the polynomial $p'(z)$, which is of degree $n-1$. We can see, from Lemma 3.5, the integrand in (4.1) is an increasing function of $M(p')$.

Now, applying Lemma 3.4 to inequality (4.1), we get for $0 \leq \theta < 2\pi$,

$$|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s|$$
$$\leq \{M(p)\}^{s-1} \int_r^R st^{sn-1} \left\{1 - \frac{(\frac{n}{1+k^\mu} M(p) - n|a_n|)(t-1)}{n|a_n| + t\frac{n}{1+k^\mu} M(p)}\right\} \frac{n}{1+k^\mu} M(p)dt$$
$$= \frac{(R^{ns} - r^{ns})}{1+k^\mu}\{M(p)\}^s - \frac{n}{1+k^\mu}\{M(p)\}^s(1-a)\int_r^R \frac{(t-1)(t^{n-1})}{t+a}dt, \tag{4.2}$$

by taking $a = \dfrac{(1+k^\mu)|a_n|}{M(p)}$.

Using Lemma 3.6 in inequality (4.2), and substituting the value of $a$, we get

$$|\{p(Re^{i\theta})\}^s| \leq \frac{(R^{ns} - r^{ns})}{1+k^\mu}\{M(p)\}^s - \frac{n}{1+k^\mu}\{M(p)\}^s \left(1 - \frac{(1+k^\mu)|a_n|}{M(p)}\right) h(n)$$
$$+|\{p(re^{i\theta})\}^s|, \tag{4.3}$$

where

$$h(n) = \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1}\left(\frac{R^{n-k} - r^{n-k}}{n-k}\right)(-1)^k \left(\frac{(1+k^\mu)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{k-1}$$
$$+ (-1)^n \left(\frac{(1+k^\mu)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{n-1} \ln\left(\frac{R(M(p)) + (1+k^\mu)|a_n|}{r(M(p)) + (1+k^\mu)|a_n|}\right)$$

for $n \geq 1$ and $h(0) = 0$. □

## 5. Computation

For the polynomial $p(z) = (z-2)^2$, $p(z) \neq 0$ for $|z| < 1$ and $M(p) = 9$. Then, for $R = 3$, exact value of $M(p, R)$ is 25. Using Theorem 1.2,

$$M(p, R) \leq 45 - 7 * (2 - 11/9 \log(29/11)) = 39.29 \tag{5.1}$$

Using Corollary 2.2 of Theorem 2.1,

$$M(p, R) \leq 45 - 7 * (4 - 22/9 + 22/81 \log(29/11)) = 32.26 \tag{5.2}$$

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