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Notes on the norm of pre-Schwarzian derivatives of certain analytic functions

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Abstract. In this paper, we obtain sharp bounds for the norm of pre-Schwarzian derivatives of certain analytic functions. Initially this problem was handled by H. Rahmatan, Sh. Najafzadeh and A. Ebadian [Stud. Univ. Babeş-Bolyai Math. **61**(2016), no. 2, 155-162]. We pointed out that their proofs are incorrect and present correct proofs.

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1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc on the complex plane $\mathbb{C}$. Let $\mathcal{H}$ be the family of all analytic functions and $\mathcal{A} \subset \mathcal{H}$ be the family of all normalized functions in Δ. We denote by $\mathcal{U}$ the class of all univalent functions in Δ and denote by $\mathcal{LU} \subset \mathcal{H}$ the class of all locally univalent functions in Δ. For a $f \in \mathcal{LU}$, we consider the following norm

$$||f|| = \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|,$$

where the quantity f''/f is often referred to as pre-Schwarzian derivative of f such that in the theory of Teichmüller spaces is considered as element of complex Banach spaces. We remark that $||f|| < \infty$ if, and only if, f is uniformly locally univalent in Δ. We also notice that, $||f|| \le 6$ if f is univalent in Δ and, conversely, f is univalent in Δ if $||f| \le 1$. Both of these bounds are sharp, see [1]. For more geometric properties of the function f relating the norm, see [2, 4, 9] and the references therein.

We say that a function f is subordinate to g, written by $f(z) \prec g(z)$ or $f \prec g$ where f and g belonging to the class $\mathcal{A}$, if there exists a Schwarz function $w(z)$ is analytic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that $f(z)=g(w(z))$ for all $z\in\Delta$.

Here are two certain subclasses of analytic and normalized functions $\mathcal{A}$ functions defined. First, we say that a function $f\in\mathcal{A}$ belongs to the class $\mathcal{S}(\alpha,\beta)$ if it satisfies the following two-sided inequality

$$\alpha<\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<\beta\quad(z\in\Delta),$$

where $0\le\alpha<1$ and $\beta>1$. The class $\mathcal{S}(\alpha,\beta)$ was introduced by Kuroki and Owa (cf. [7]) and generalized by Kargar et al. [6]. We also say that a function $f\in\mathcal{A}$ belongs to the class $\mathcal{V}(\alpha,\beta)$ if

$$\alpha<\mathrm{Re}\left\{\left(\frac{z}{f(z)}\right)^2 f'(z)\right\}<\beta\quad(z\in\Delta).$$

The class $\mathcal{V}(\alpha,\beta)$ was first introduced by Kargar et al., see [5].
Since the convex univalent function

$$P_{\alpha,\beta}(z)=1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right)\quad(z\in\Delta),\tag{1.1}$$

where

$$\phi:=\frac{2\pi(1-\alpha)}{\beta-\alpha},\tag{1.2}$$

maps Δ onto the domain $\Omega=\{\omega:\alpha<\mathrm{Re}\{\omega\}<\beta\}$ conformally, thus we have.

Lemma 1.1. ([7, Lemma 1.3]) *Let $\alpha\in[0,1)$ and $\beta\in(1,\infty)$. Then $f\in\mathcal{S}(\alpha,\beta)$ if, and only if,*

$$\frac{zf'(z)}{f(z)}\prec 1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right)\quad(z\in\Delta),$$

where ϕ is defined in (1.2).

Lemma 1.2. ([5, Lemma 1.1]) *Let $\alpha\in[0,1)$ and $\beta\in(1,\infty)$. Then $f\in\mathcal{V}(\alpha,\beta)$ if, and only if,*

$$\left(\frac{z}{f(z)}\right)^2 f'(z)\prec 1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right)\quad(z\in\Delta),$$

where ϕ is defined in (1.2).

Rahmatan, Najafzadeh and Ebadian (see [10]) estimated the norm of pre-Schwarzian derivatives of the function f where f belongs to the classes $\mathcal{S}(\alpha,\beta)$ and $\mathcal{V}(\alpha,\beta)$. Both their estimates and proofs are incorrect. Indeed, the results that were wrongly proven by them are as follows:

Theorem A. *For $0\le\alpha<1<\beta$, if $f\in\mathcal{S}(\alpha,\beta)$, then*

$$||f||\le\frac{2(\beta-\alpha)}{\pi}\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right).$$

Theorem B. *For $0\le\alpha<1<\beta$, if $f\in\mathcal{V}(\alpha,\beta)$, then*

$$||f||\le\frac{3(\beta-\alpha)}{\pi}\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right).$$

We first note that both the above bounds are complex numbers!

In this paper we give the best estimate for $||f||$ when $f \in \mathcal{S}(\alpha,\beta)$ and disprove the Theorem B. However, we show that $||f|| < \infty$ when $f \in \mathcal{V}(\alpha,\beta)$.

2. Main results

The correct version of Theorem A is as follows.

Theorem 2.1. *Let $\alpha \in [0,1)$ and $\beta \in (1,\infty)$. If a function f belongs to the class $\mathcal{S}(\alpha,\beta)$, then*

$$||f|| \leq \frac{2(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)+2\pi^2} - \frac{4\sin(\phi/2)}{\sqrt{4\sin^2(\phi/2)+2\pi^2}}, \tag{2.1}$$

where ϕ is defined in (1.2). *The result is sharp.*

Proof. Let that $\alpha \in [0,1)$, $\beta \in (1,\infty)$ and ϕ be given by (1.2). If $f \in \mathcal{S}(\alpha,\beta)$, by Lemma 1.1, then we have

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right) \quad (z \in \Delta). \tag{2.2}$$

The above subordination relation (2.2) implies that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right) \quad (z \in \Delta),$$

or equivalently

$$\log\left\{\frac{zf'(z)}{f(z)}\right\} = \log\left\{1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right\} \quad (z \in \Delta), \tag{2.3}$$

where $w(z)$ is the well-known Schwarz function. From (2.3), differentiating on both sides, after simplification, we obtain

$$\begin{aligned}\frac{f''(z)}{f'(z)} = \frac{(\beta-\alpha)i}{\pi}\Bigg[&\frac{1}{z}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right) \\ &+ \frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\Bigg].\end{aligned} \tag{2.4}$$

It is well-known that $|w(z)| \leq |z|$ (cf. [3]) and also by the Schwarz-Pick lemma, for a Schwarz function the following inequality

$$|w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \quad (z \in \Delta), \tag{2.5}$$

holds (see [8]). We also know that if log is the principal branch of the complex logarithm, then we have

$$\log z = \ln|z| + i\arg z \quad (z \in \Delta\setminus\{0\}, -\pi < \arg z \leq \pi). \tag{2.6}$$

Therefore, by the above equation (2.6), it is well-known that if $|z| \geq 1$, then

$$|\log z| \leq \sqrt{|z-1|^2+\pi^2}, \tag{2.7}$$

while for $0<|z|<1$, we have

$$|\log z| \leq \sqrt{\left|\frac{z-1}{z}\right|^2+\pi^2}. \tag{2.8}$$

Thus, it is natural to distinguish the following cases.
Case 1. $\left|\frac{1-e^{i\phi}w(z)}{1-w(z)}\right| \geq 1$.
By (2.7), we have

$$\begin{aligned}\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right| &\leq \sqrt{\left|\frac{1-e^{i\phi}w(z)}{1-w(z)}-1\right|^2+\pi^2}\\ &= \frac{\sqrt{|1-e^{i\phi}|^2|w(z)|^2+\pi^2|1-w(z)|^2}}{|1-w(z)|}\\ &\leq \frac{\sqrt{4\sin^2(\phi/2)|w(z)|^2+\pi^2(1+|w(z)|^2)}}{1-|w(z)|}\\ &\leq \frac{\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}{1-|z|} \end{aligned} \tag{2.9}$$

for all $z\in\Delta$. We note that the above inequality is well defined also for $z=0$. Thus from (2.4), (2.5) and (2.9), we get

$$\begin{aligned} &\left|\frac{f''(z)}{f'(z)}\right|\\ =\ &\left|\frac{(\beta-\alpha)i}{\pi}\left[\frac{1}{z}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right.\right.\\ &\left.\left.+\frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\right]\right|\\ \leq\ &\frac{(\beta-\alpha)}{\pi}\left[\frac{1}{|z|}\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right|\right.\\ &\left.+\frac{\left|1-e^{i\phi}\right||w'(z)|}{|1-w(z)|\,|1-e^{i\phi}w(z)|\left(1-\frac{(\beta-\alpha)}{\pi}\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right|\right)}\right]\\ \leq\ &\frac{(\beta-\alpha)}{\pi}\left[\frac{1}{|z|}\left\{\frac{\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}{1-|z|}\right\}\right.\\ &\left.+\frac{2\sin(\phi/2)}{1-|z|-\frac{(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}\cdot\frac{1+|z|}{1-|z|^2}\right]. \end{aligned}$$

However, we obtain

$$\begin{aligned}
||f|| &= \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \\
&\le \sup_{z\in\Delta}\left\{\frac{(\beta-\alpha)}{\pi}\left[\frac{1+|z|}{|z|}\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}\right.\right. \\
&\quad \left.\left. + \frac{2\sin(\phi/2)(1+|z|)}{1-|z|-\frac{(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}\right]\right\} \\
&= \frac{2(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)+2\pi^2} - \frac{4\sin(\phi/2)}{\sqrt{4\sin^2(\phi/2)+2\pi^2}}
\end{aligned}$$

concluding the inequality (2.1).

Case 2. $\left|\frac{1-e^{i\phi}w(z)}{1-w(z)}\right| < 1$.

By (2.8), we have

$$\begin{aligned}
\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right| &\le \sqrt{\left|\frac{\frac{1-e^{i\phi}w(z)}{1-w(z)}-1}{\frac{1-e^{i\phi}w(z)}{1-w(z)}}\right|^2+\pi^2} \\
&= \frac{\sqrt{|1-e^{i\phi}|^2|w(z)|^2+\pi^2|1-e^{i\phi}w(z)|^2}}{|1-e^{i\phi}w(z)|} \\
&\le \frac{\sqrt{4\sin^2(\phi/2)|w(z)|^2+\pi^2(1+|w(z)|^2)}}{1-|w(z)|} \qquad (|e^{i\phi}|=1) \\
&\le \frac{\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}{1-|z|}.
\end{aligned}$$

Since in both cases 1 and 2 we have the equal estimates for

$$\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right|,$$

therefore, in this case also, the desired result will be achieved. For the sharpness, consider the function $f_{\alpha,\beta}(z)$ as follows

$$\begin{aligned}
f_{\alpha,\beta}(z) &= z\exp\left\{\frac{(\beta-\alpha)i}{\pi}\int_0^z \frac{1}{\xi}\log\left(\frac{1-e^{i\phi}\xi}{1-\xi}\right)\mathrm{d}\xi\right\} \\
&= z + \frac{(\beta-\alpha)i}{\pi}\left(1-e^{i\phi}\right)z^2+\cdots,
\end{aligned}$$

where ϕ is defined in (1.2), $0\le\alpha<1$ and $\beta>1$. A simple calculation, gives us

$$\frac{zf'_{\alpha,\beta}(z)}{f_{\alpha,\beta}(z)} = 1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right) \quad (z\in\Delta)$$

and thus $f_{\alpha,\beta}(z) \in \mathcal{S}(\alpha,\beta)$. With the same proof as above we get the desired result. The result also is sharp for a rotation of the function $f_{\alpha,\beta}(z)$ as follows:

$$\mathfrak{f}_{\alpha,\beta}(z) = z\exp\left\{\frac{(\beta-\alpha)i}{\pi}\int_0^z \frac{1}{\xi}\log\left(\frac{1-e^{i\phi}\xi}{1-e^{-i\phi}\xi}\right)\mathrm{d}\xi\right\}.$$

This is the end of proof. □

Remark 2.2. In Theorem B, the authors of [10] estimated the norm $||f||$ when $f \in \mathcal{V}(\alpha,\beta)$. But in the proof of this theorem [10, p. 160], wrongly, they used from the following equation

$$\frac{zf'(z)}{f(z)} = P_{\alpha,\beta}(w(z)),$$

where $P_{\alpha,\beta}$ is defined in (1.1). This means that f, simultaneously, belonging to the class $\mathcal{S}(\alpha,\beta)$ and $\mathcal{V}(\alpha,\beta)$.
Next, we show that the best estimate for $||f||$ when $f \in \mathcal{V}(\alpha,\beta)$ does not exist.

Theorem 2.3. *Let $\alpha \in [0,1)$ and $\beta \in (1,\infty)$. If a function f belongs to the class $\mathcal{V}(\alpha,\beta)$, then $||f|| < \infty$.*

Proof. Let $\alpha \in [0,1)$ and $\beta \in (1,\infty)$ and $f \in \mathcal{V}(\alpha,\beta)$. Then by Lemma 1.2 and by use of definition of subordination, we have

$$\left(\frac{z}{f(z)}\right)^2 f'(z) = P_{\alpha,\beta}(w(z)) = 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right), \tag{2.10}$$

where w is Schwarz function and ϕ is defined in (1.2). Taking logarithm on both sides of (2.10) and differentiating, we get

$$\begin{aligned}\frac{f''(z)}{f'(z)} &= 2\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) + \frac{(\beta-\alpha)i}{\pi} \qquad (2.11)\\ &\quad\times\left[\frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\right].\end{aligned}$$

With a simple calculation, (2.10) implies that

$$\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) = \frac{f(z)}{z}\left(\frac{P_{\alpha,\beta}(w(z))}{z} - 1\right). \tag{2.12}$$

Combining (2.11) and (2.12), give us

$$\begin{aligned}\frac{f''(z)}{f'(z)} &= 2\left(\frac{f(z)}{z}\left(\frac{P_{\alpha,\beta}(w(z))}{z} - 1\right)\right)\\ &\quad + \frac{(\beta-\alpha)i}{\pi}\left[\frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\right]\end{aligned}$$

It was proved in ([5, Theorem 2.2]) that if $f \in \mathcal{V}(\alpha,\beta)$ where $0 < \alpha \le 1/2$ and $\beta > 1$, then

$$1 - \frac{1}{\alpha} < \operatorname{Re}\left\{\frac{f(z)}{z}\right\} < \infty \quad (z \in \Delta).$$

Since $\mathrm{Re}\{z\} \leq |z|$, the last two-sided inequality means that $|f(z)/z| < \infty$ when $f \in \mathcal{V}(\alpha, \beta)$. Thus from the above we deduce that

$$\left|\frac{f''(z)}{f'(z)}\right| < \infty \quad (z \in \Delta)$$

concluding the proof. □

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