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Meromorphic close-to-convex functions satisfying a differential inequality

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Abstract. In the present paper, we study the differential inequality

$$-\Re\left[(1-\alpha)z^2 f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \beta, (z \in \mathbb{E})$$

where $f \in \Sigma$ and notice that the members of class Σ which satisfy the above inequality are meromorphic close-to-convex.

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1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

which are analytic in the punctured open unit disc $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where

$$\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}.$$

A function $f \in \Sigma$ is said to be meromorphic starlike of order α if and only if

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, (z \in \mathbb{E})$$

for some real α $(0 \leq \alpha < 1)$. The class of such functions is denoted by $\mathcal{MS}^*(\alpha)$. Write $\mathcal{MS}^* = \mathcal{MS}^*(0)$, the class of meromorphic starlike functions i.e. meromorphic functions which satisfy the condition

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, (z \in \mathbb{E}).$$

A function $f \in \Sigma$ is said to be meromorphic close-to-convex of order α if there exists a meromorphic starlike function $g \in \mathcal{MS}^*$ such that

$$-\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, (z \in \mathbb{E}).$$

The class of such functions is denoted by $\mathcal{MC}(\alpha)$. Write $\mathcal{MC} = \mathcal{MC}(0)$, the class of meromorphic close-to-convex functions i.e. meromorphic functions which satisfy the condition

$$-\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, (z \in \mathbb{E}) \tag{1.1}$$

where $g \in \mathcal{MS}^*$.

A little calculation yields that the function $g(z) = \dfrac{1}{z}$ is a member of class $\mathcal{MS}^*$. Therefore, the condition (1.1) reduces to the following condition

$$-\Re(z^2 f'(z)) > 0, (z \in \mathbb{E}).$$

Therefore, $f \in \mathcal{MC}$ if $-\Re(z^2 f'(z)) > 0$.

In the literature of meromorphic functions, many authors obtained the conditions for meromorphic close-to-convex functions. Some of the results from literature are given below:

Jing and Li [4] have proved the following results:

Theorem 1.1. *For any $f \in \Sigma$, suppose that for arbitrary α, f satisfies $-z^2 f'(z) \neq \alpha$ and the following inequalities:*

(i) *For the case $0 < \alpha < \frac{1}{2}$*

$$2 + \Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{\alpha}{2(1-\alpha)},$$

(ii) *For the case $\frac{1}{2} \leq \alpha < 1$*

$$2 + \Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1-\alpha}{2\alpha},$$

then $f \in \mathcal{MC}(\alpha)$.

Theorem 1.2. *Let $f \in \Sigma$, suppose that for arbitrary α, f satisfies $-z^2 f'(z) \neq \alpha$ and the following inequality:*

$$1 + \Re\left(\frac{zf''(z)}{f'(z)}\right) \geq \frac{3\alpha - 2}{2(1-\alpha)},$$

then $f \in \mathcal{MC}(\alpha)$.

Goyal and Prajapat [1] proved the following results:

Theorem 1.3. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - z^2 f'(z) + 1\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \qquad (0 \leq \alpha < 1),$$

then $f \in \mathcal{MC}(\alpha)$.

Theorem 1.4. *If $f \in \Sigma$ satisfies the following inequality*
$$\left|\frac{zf''(z)}{f'(z)} - z^2 f'(z) + 1\right| < \frac{3}{2},$$
then $f \in \mathcal{MC}$.

Theorem 1.5. *If $f \in \Sigma$ satisfies the following inequality*
$$\Re[z^2\{f'(z)(z^2 f'(z) - 1) - zf''(z)\}] > -\frac{1}{2},$$
then $f \in \mathcal{MC}$.

Recently Wang and Guo [3] proved the following results:

Theorem 1.6. *Let $f \in \Sigma$ and suppose that there exists a meromorphic starlike function g such that*
$$\Re\left\{\frac{zf'(z)}{g(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right)\right\} > \frac{1}{2}\left(1 + \left|\frac{zf'(z)}{g(z)}\right|^2\right),$$
then $f \in \mathcal{MC}$.

Theorem 1.7. *Let $f \in \Sigma$ and suppose that there exists a meromorphic starlike function g such that*
$$\Re\left\{\frac{zf'(z)}{g(z)}\left(-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)}\right)\right\} > -\frac{1}{4}\left(1 + \left|\frac{zf'(z)}{g(z)}\right|^2\right),$$
then $f \in \mathcal{MC}(\frac{1}{2})$.

Theorem 1.8. *For $f \in \Sigma$, suppose that there exists a meromorphic starlike function g such that*
$$\Re\left\{\frac{zf'(z)}{g(z)}\left(-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)}\right)\right\} > -\frac{1}{2}(1-\alpha), (0 \le \alpha < 1)$$
then $f \in \mathcal{MC}(\alpha)$.

2. Preliminaries

We shall need the following lemma of Miller and Mocanu [2] to prove our main result.

Lemma 2.1. *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let ϕ satisfy the following conditions:*
(i) $\phi(u, v)$ is continuous in $\mathbb{D}$;
(ii) $(1, 0) \in \mathbb{D}$ and $\Re\phi(1, 0) > 0$; and
(iii) $\Re\ \{\phi(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \le -(1 + u_2^2)/2$.
Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the unit disc $\mathbb{E}$ such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If
$$\Re[\phi(p(z), zp'(z))] > 0, z \in \mathbb{E},$$
then $\Re\ p(z) > 0,\ z \in \mathbb{E}$.

3. Main theorem

Theorem 3.1. *Let α and β be real numbers such that $\alpha \leq \beta < 1$. If $f \in \Sigma$ satisfies*

$$-\Re\left[(1-\alpha)z^2f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \beta, z \in \mathbb{E}, \tag{3.1}$$

then $-\Re(z^2f'(z)) > 0$ in $\mathbb{E}$. So, f is meromorphic close-to-convex in $\mathbb{E}$. The result is sharp in the sense that the constant β on the right hand side of (3.1) cannot be replaced by a real number smaller than α.

Proof. Define a function p by $p(z) = -z^2f'(z)$ where p is analytic in $\mathbb{E}$. Then,

$$\begin{aligned} &-\left[(1-\alpha)z^2f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] \\ &= -\left[(1-\alpha)(-p(z))+\alpha\left(-1+\frac{zp'(z)}{p(z)}\right)\right] \end{aligned} \tag{3.2}$$

Thus, condition (3.1) is equivalent to

$$\Re\left[\frac{1-\alpha}{1-\beta}p(z)-\frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)}+\frac{\alpha-\beta}{1-\beta}\right] > 0, z \in \mathbb{E}. \tag{3.3}$$

If $\mathbb{D} = (\mathbb{C}\setminus\{0\})\times\mathbb{C}$, define $\phi(u, v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\phi(u,v) = \frac{1-\alpha}{1-\beta}u-\frac{\alpha}{1-\beta}\frac{v}{u}+\frac{\alpha-\beta}{1-\beta}.$$

Then $\phi(u, v)$ is continuous in $\mathbb{D}$, $(1, 0) \in D$ and $\Re(\phi(1, 0)) = 1 > 0$. Further, in view of (3.3),

$$\Re[\phi(p(z), zp'(z))] > 0, \ z \in \mathbb{E}.$$

Let $u = u_1 + iu_2, v = v_1 + iv_2(u_1, u_2, v_1, v_2$ are real numbers). Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leq -\frac{1+u_2^2}{2}$, we have

$$\Re[\phi(iu_2, v_1)] = \Re\left[\frac{1-\alpha}{1-\beta}iu_2-\frac{\alpha}{1-\beta}\frac{v_1}{iu_2}+\frac{\alpha-\beta}{1-\beta}\right] = \frac{\alpha-\beta}{1-\beta} \leq 0.$$

In view of Lemma 2.1, proof now follows.

To show that the constant β on the right side of (3.1) cannot be replaced by a real number smaller than α, we consider the function

$$f_0(z) = \frac{-z-2\log(1-z)}{z^2},$$

which belongs to the class Σ. A simple calculation gives

$$\begin{aligned} &-\left[(1-\alpha)z^2f_0'(z)+\alpha\left(1+\frac{zf_0''(z)}{f_0'(z)}\right)\right] \\ &= -(1-\alpha)\left[\frac{-z^2+3z+4(1-z)\log(1-z)}{z(1-z)}\right] \\ &-\alpha\left[\frac{-z^3+10z^2-7z-8(1-z)^2\log(1-z)}{z^3-4z^2+3z+4(1-z)^2\log(1-z)}\right] \end{aligned}$$

Using Mathematica 7.0, we plot in Figure 3.1, the image of the unit disc $\mathbb{E}$ under the operator

$$-\left[(1-\alpha)z^2 f_0'(z) + \alpha\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right)\right]$$

taking $\alpha = -1$.

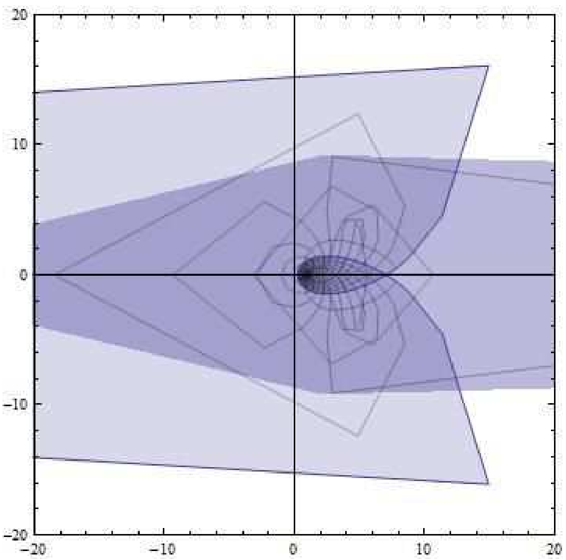

Figure 3.1

From Figure 3.1, we observe that minimum real part of

$$-\left[(1-\alpha)z^2 f_0'(z) + \alpha\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right)\right] \text{ for } \alpha = -1$$

is smaller than -1 (the chosen value of α).
In Figure 3.2, we plot the image of unit disc $\mathbb{E}$ under the function $-z^2 f_0'(z)$.
It is obvious that $-\Re(z^2 f_0'(z)) \not> 0$ for all z in $\mathbb{E}$.
Moreover, the point $z = 0.9$ is an interior point of $\mathbb{E}$, but at this point

$$-\Re(z^2 f_0'(z)) = -10.766... < 0.$$

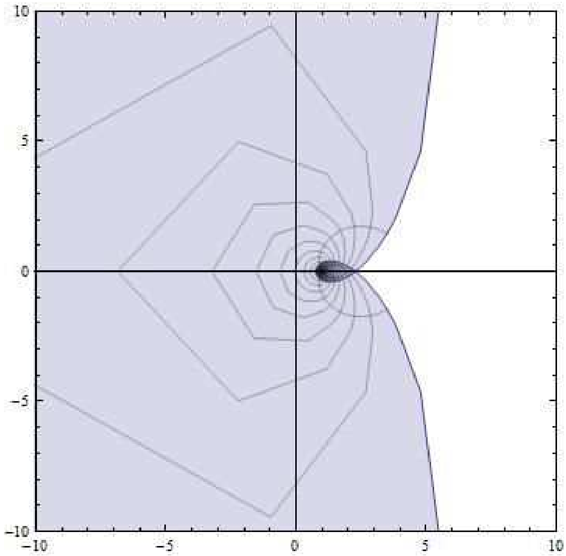

Figure 3.2

This justifies our claim. □

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