Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 2, 261–268
DOI: 10.24193/subbmath.2017.2.11

Linear Weingarten factorable surfaces in isotropic spaces

Muhittin Evren Aydin and Alper Osman Ogrenmis

Abstract. In this paper, we deal with a certain factorable surface in the isotropic 3-space satisfying $aK+bH=c$, where K is the relative curvature, H the isotropic mean curvature and $a,b,c\in\mathbb{R}$. We obtain a complete classification for such surfaces. As a further study, we prove that a certain graph surface with $K=H^2$ is either a non-isotropic plane or a parabolic sphere.

Mathematics Subject Classification (2010): 53A35, 53A40, 53B25.

Keywords: Isotropic space, factorable surface, Weingarten surface, Euler inequality.

1. Introduction

Let M^2 be a regular surface of a Euclidean 3-space $\mathbb{R}^3$ and κ_1,κ_2 its principal curvatures. Then M^2 is called a *Weingarten surface* if the following non-trivial functional relation occurs:

$$\phi\left(\kappa_1,\kappa_2\right)=0 \tag{1.1}$$

for a smooth function ϕ of two variables. (1.1) immediately yields

$$\delta\left(K,H\right)=0, \tag{1.2}$$

where K and H are respectively the Gaussian and mean curvatures of M^2. (1.2) is equivalent to the vanishing of the corresponding Jacobian determinant, i.e. $\left|\partial\left(K,H\right)/\partial\left(u,v\right)\right|=0$ for a coordinate pair (u,v) on M^2. If M^2 is a surface that satisfies

$$aH+bK=c,\ \ a,b,c\in\mathbb{R},\ \ (a,b,c)\neq(0,0,0)\,, \tag{1.3}$$

then it is called a *linear Weingarten surface* (*LW-surface*). If $a=0$ or $b=0$ in (1.3), then the LW-surfaces reduce to the surfaces with constant curvature. Such surfaces have been extensively studied, see [7, 8], [13]-[17], [30].

On the other hand, a surface in $\mathbb{R}^3$ that is the graph of the function $z\left(x,y\right)=f\left(x\right)g\left(y\right)$ is said to be *factorable* or *homothetical.* In various ambient spaces, these surfaces have been desribed in terms of their curvatures and Laplace operator in [4, 9, 10, 12, 18, 19, 28, 29]. As distinct from the Euclidean case, a graph surface in the isotropic space $\mathbb{I}^3$ is said to be *factorable* if it is graph of either $z(x,y) = f(x)g(y)$ or $x(y,z) = f(y)g(z)$. We call them the *factorable surface of type 1* and *type 2*, respectively. Note that the factorable surface of one type cannot be carried into that of another type by the isometries of $\mathbb{I}^3$. These surfaces of both type in $\mathbb{I}^3$ with $K, H = const.$ were obtained in [1]-[3].

The main purpose of this paper is to obtain LW-factorable surfaces of type 1 in $\mathbb{I}^3$. As a further study, we classify the graph surfaces of the function $z = z(x,y)$ in $\mathbb{I}^3$ with $K = H^2$.

2. Preliminaries

For general references of the isotropic geometry, see [5], [23]-[27]. The isotropic 3-space $\mathbb{I}^3$ is a Cayley-Klein space defined from a 3-dimensional projective space $P(\mathbb{R}^3)$ with the absolute figure (ω, f_1, f_2), where ω is a plane in $P(\mathbb{R}^3)$ and f_1, f_2 are two complex-conjugate straight lines in ω. The homogeneous coordinates in $P(\mathbb{R}^3)$ are introduced in such a way that the *absolute plane* ω is given by $X_0 = 0$ and the *absolute lines* f_1, f_2 by $X_0 = X_1 + iX_2 = 0$, $X_0 = X_1 - iX_2 = 0$. The intersection point $F(0:0:0:1)$ of these two lines is called the *absolute point*. The affine coordinates in $P(\mathbb{R}^3)$ are given by $x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$, $x_3 = \frac{X_3}{X_0}$. The group of motions of $\mathbb{I}^3$ is defined by

$$(x_1, x_2, x_3) \longmapsto (x_1', x_2', x_3') : \begin{cases} x_1' = a_1 + x_1 \cos\phi - x_2 \sin\phi, \\ x_2' = a_2 + x_1 \sin\phi + x_2 \cos\phi, \\ x_3' = a_3 + a_4 x_1 + a_5 x_2 + x_3, \end{cases}$$

where $a_1, ..., a_5, \phi \in \mathbb{R}$.

Consider the points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *isotropic distance* $d_{\mathbb{I}}(x,y)$ of two points x and y is defined as

$$d_{\mathbb{I}}(x,y) = (y_1 - x_1)^2 + (y_2 - x_2)^2.$$

The lines in x_3-direction are called *isotropic* lines. The plane containing an isotropic line is called an *isotropic plane.* Other planes are *non-isotropic.*

Let M^2 be a graph surface immersed in $\mathbb{I}^3$ corresponding to a real-valued smooth function $z = z(x,y)$ on an open domain $D \subseteq \mathbb{R}^2$. Then it is parameterized as follows:

$$r : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{I}^3, \ \ (x,y) \longmapsto (x, y, z(x,y)). \tag{2.1}$$

It follows from (2.1) that M^2 is an admissible (i.e. without isotropic tangent planes). The metric on M^2 induced from $\mathbb{I}^3$ is given by $g_* = dx^2 + dy^2$. This implies that M^2 is always flat with respect to the induced metric g_* and thus its Laplacian is of the form $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The *relative* (or *isotropic Gaussian*) *curvature* K and the *isotropic mean curvature* H of M^2 are defined by

$$K = z_{xx} z_{yy} - (z_{xy})^2 \tag{2.2}$$

and

$$H=\frac{\triangle z}{2}=\frac{z_{xx}+z_{yy}}{2}. \tag{2.3}$$

M^2 is called *isotropic minimal* (resp. *isotropic flat*) if H (resp. K) vanishes.

3. LW-factorable surfaces of type 1

Let M^2 be a factorable surface of type 1 in $\mathbb{I}^3$, i.e., the graph of $z(x,y)=f(x)g(y)$. By (2.2) and (2.3), we get

$$K=(f''f)(g''g)-(f')^2(g')^2 \tag{3.1}$$

and

$$2H=f''g+fg'', \tag{3.2}$$

where $f'=\frac{df}{dx}$ and $g'=\frac{dg}{dy}$, etc. We mainly aim to classify the LW-factorable surfaces of type 1 in $\mathbb{I}^3$. For this, let M^2 satisfy the relation (1.3). Since at least one of a, b and c is nonzero in (1.3), without loss of generality, we may assume $b\neq 0$. By dividing both sides of (1.3) with b and putting $\frac{a}{b}=2m_0$ and $\frac{c}{b}=n_0$, we write

$$2m_0H+K=n_0,\ m_0,n_0\in\mathbb{R}. \tag{3.3}$$

If $m_0=0$, M^2 turns to be a factorable surface of type 1 in $\mathbb{I}^3$ with $K=const.$ however such surfaces were already provided in [1]. In our framework, it is meaningful to take $m_0\neq 0$. By $(3.1)-(3.3)$, we get

$$(f''f)(g''g)-(f')^2(g')^2+m_0(f''g+fg'')=n_0. \tag{3.4}$$

We have to distinguish several cases in order to solve (3.4). Remark that the roles of f and g are symmetric, so discussing on the cases based on f shall be sufficient. From now on, we use the notation c_i to denote nonzero constants and d_i to denote some constants, $i=1,2,3,...$

Case 1. $f(x)=f_0\in\mathbb{R}-\{0\}$. By (3.4), we find

$$g(y)=\frac{n_0}{2f_0m_0}y^2+d_1y+d_2. \tag{3.5}$$

Thereby, M^2 is isotropic flat factorable surface of type 1 with $H=\frac{n_0}{2m_0}$.

Case 2. f is a linear function, i.e. $f(x)=c_1x+d_3$. It follows from (3.4) that

$$m_0d_3g''-c_1^2(g')^2+(m_0c_1g'')x=n_0. \tag{3.6}$$

(3.6) implies that $g''=0$, namely $g(y)=c_2y+d_4$. In this case, M^2 is isotropic minimal factorable surface of type 1 with $K=-(c_1c_2)^2$.

Case 3. f is a non-linear function. From the symmetry, g is also a non-linear function. By dividing (3.4) with the product ff'', we get

$$g''g-\frac{(f')^2}{ff''}(g')^2+m_0\frac{g}{f}+m_0\frac{g''}{f''}=\frac{n_0}{ff''}. \tag{3.7}$$

By taking partial derivative (3.7) with respect to y and then dividing with $g'g''$, we deduce

$$1+\frac{gg'''}{g'g''}-2\frac{(f')^2}{ff''}+\left(\frac{m_0}{f}\right)\frac{1}{g''}+\left(\frac{m_0}{f''}\right)\frac{g'''}{g'g''}=0. \tag{3.8}$$

We have two cases:
Case 3.1. $g'''=0$, i.e.

$$g(y)=c_3y^2+d_5y+d_6. \tag{3.9}$$

Up to suitable translations of y, we may assume $d_5=d_6=0$. Then (3.8) reduces to

$$1-2\frac{(f')^2}{ff''}+\left(\frac{m_0}{2c_3}\right)\frac{1}{f}=0. \tag{3.10}$$

(3.10) can be rewritten as

$$\left(\frac{m_0}{2c_3}+f\right)f''-2(f')^2=0. \tag{3.11}$$

After solving (3.11), we find

$$f(x)=-\left(\frac{1}{c_4x+d_7}+\frac{m_0}{2c_3}\right). \tag{3.12}$$

Considering (3.9) and (3.12) into (3.4) gives the contradiction

$$x=-\frac{1}{c_4}\left(\frac{2m_0c_3}{n_0+m_0^2}+d_7\right)$$

due to the fact that x is an independent variable.
Case 3.2. $g'''\neq 0$. By taking partial derivatives of (3.8) with respect to x and y, we conclude

$$\left(\frac{f'}{f^2}\right)\frac{g'''}{(g'')^2}-\frac{f'''}{(f'')^2}\left(\frac{g'''}{g'g''}\right)'=0. \tag{3.13}$$

Due to $f'g'''\neq 0$, neither f''' nor $\left(\frac{g'''}{g'g''}\right)'$ can vanish in (3.13). Then (3.13) can be rewritten as

$$\frac{f'(f'')^2}{f^2f'''}=\frac{(g'')^2}{g'''}\left(\frac{g'''}{g'g''}\right)'. \tag{3.14}$$

Since the left side of (3.14) is a function of x, however the right side is a function of y. Then both sides have to be equal a nonzero constant, namely

$$\frac{f'(f'')^2}{f^2f'''}=c_5=\frac{(g'')^2}{g'''}\left(\frac{g'''}{g'g''}\right)'. \tag{3.15}$$

From the left side of (3.15), we write

$$\frac{f'''}{(f'')^2}=\frac{1}{c_5}\frac{f'}{f^2} \tag{3.16}$$

or, by taking once integral with respect to x,

$$f''=\frac{c_5f}{c_5d_8f+1}. \tag{3.17}$$

Likewise, by the right side of (3.15), we deduce

$$\frac{g'''}{g'g''} = \frac{-c_5}{g''} + d_9. \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.8) yields

$$\begin{aligned} &1 + (m_0 d_8 + g)\, d_9 - \frac{c_5\,(m_0 d_8 + g)}{g''} + \frac{m_0 d_9}{c_5 f} - \\ &-\frac{2\,(c_5 d_8 f + 1)\,(f')^2}{c_5 f^2} = 0. \end{aligned} \tag{3.19}$$

Taking partial derivative of (3.19) with respect to y and considering (3.18) leads to

$$g'' = -c_5\,(g + m_0 d_8)\,. \tag{3.20}$$

After substituting (3.20) into (3.19), we conclude

$$2 + m_0 d_8 d_9 + d_9 g + \frac{m_0 d_9}{c_5 f} - \frac{2\,(f')^2}{f f''} = 0,$$

which yields $d_9 = 0$ and $ff'' = (f')^2$. Solving this one gives $f(x) = c_6 \exp(c_7 x)$. By putting this in (3.4) we derive the polynomial equation on (f):

$$c_7^2 \left[g g'' - (g')^2 \right] f^2 + m_0 \left(c_7^2 g + g'' \right) f - n_0 = 0,$$

which implies that the coefficients must be zero; namely $n_0 = 0$,

$$g g'' - (g')^2 = 0 \text{ and } c_7^2 g + g'' = 0. \tag{3.21}$$

(3.21) leads to the contradiction $c_7^2 g^2 + (g')^2 = 0$ and therefore we have proved the following:

Theorem 3.1. *Let M^2 be a LW-factorable surface of type 1 which is the graph of $z(x,y) = f(x)\, g(y)$ in $\mathbb{I}^3$. Then we have either*
(A) $f(x) = f_0 \in \mathbb{R} - \{0\}\,,\ g(y) = c_6 y^2 + d_{10} y + d_{11}$;
(B) *or* $z(x,y) = (c_7 x + d_{12})(c_8 y + d_{13})\,.$

4. Graph surfaces with $K = H^2$

Let M^2 be a surface of the Euclidean 3-space $\mathbb{R}^3$. The Euler inequality for M^2 including the Gaussian and mean curvature follows

$$K \leq H^2. \tag{4.1}$$

The equality sign of (4.1) holds on M^2 if and only if it is totally umbilical, i.e. a part of a plane or a two sphere in $\mathbb{E}^3$. For more generalizations, see [6, 11], [20]-[22]. Now we are interested in the factorable surfaces of type 1 in $\mathbb{I}^3$ satisfying $K = H^2$. For this, let us reconsider (3.1) and (3.2). If $K = H^2$, then

$$(f''g - fg'')^2 + 4\,(f'g')^2 = 0. \tag{4.2}$$

(4.2) immediately implies that

$$f''g - fg'' = 0 \text{ and } f'g' = 0. \tag{4.3}$$

By (4.3) we conclude that either $f = const.$ and $g(y) = c_1 y + d_1$ or $g = const.$ and $f(x) = c_2 x + d_2$, which yields the following result:

Proposition 4.1. *The factorable surfaces of type 1 in* $\mathbb{I}^3$ *satisfying* $K = H^2$ *are only non-isotropic planes.*

As a generalization, we are able to investigate the graph surfaces of type 1 in $\mathbb{I}^3$ satisfying $K = H^2$. More precisely, let M^2 be a graph surface of $z = z(x, y)$ in $\mathbb{I}^3$. If $K = H^2$ on M^2, then we get

$$(z_{xx} - z_{yy})^2 + 4(z_{xy})^2 = 0, \tag{4.4}$$

which yields that

$$z_{xy} = 0 \tag{4.5}$$

and

$$z_{xx} = z_{yy}. \tag{4.6}$$

By (4.5), we derive

$$z(x, y) = \alpha(x) + \beta(y) \tag{4.7}$$

and considering (4.7) into (4.6) gives

$$\frac{d^2\alpha}{dx^2} = \frac{d^2\beta}{dy^2} = d_3, \ d_3 \in \mathbb{R}. \tag{4.8}$$

By solving (4.8), we find

$$\alpha(x) = \frac{d_3}{2}x^2 + d_4 x + d_5, \ \beta(y) = \frac{d_3}{2}y^2 + d_6 y + d_7. \tag{4.9}$$

(4.9) implies that M^2 is either a non-isotropic plane ($d_3 = 0$) or a parabolic sphere ($d_3 \neq 0$). Consequently, we have

Theorem 4.2. *A graph surface of a function* $z = z(x, y)$ *in* $\mathbb{I}^3$ *with* $K = H^2$ *is either (a piece of) a non-isotropic plane or (a piece of) a parabolic sphere given by*

$$z(x, y) = c_3\left(x^2 + y^2\right) + d_8 x + d_9 y + d_{10}.$$

References

[1] Aydin, M.E., Ergut, M., *Isotropic geometry of graph surfaces associated with product production functions in economics*, Tamkang J. Math., **47**(4)(2016), 433-443.

[2] Aydin, M.E., *Constant curvature factorable surfaces in 3-dimensional isotropic space*, arXiv:1612.02458 [math.DG].

[3] Aydin, M.E., Ogrenmis, A.O., *Homothetical and translation hypersurfaces with constant curvature in the isotropic space*, BSG Proceedings, **23**(2016), 1-10.

[4] Bekkar, M., Senoussi, B., *Factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces satisfying* $\triangle r_i = \lambda_i r_i$, J. Geom., **103**(2012), 17-29.

[5] Chen, B.Y., Decu, S., Verstraelen, L., *Notes on isotropic geometry of production models*, Kragujevac J. Math., **38**(2014), no. 1, 23-33.

[6] Chen, B.Y., *Mean curvature and shape operator of isometric immersions in real-space-forms*, Glasg. Math. J., **38**(1996), 87-97.

[7] Dillen, F., Kuhnel, W., *Ruled Weingarten surfaces in Minkowski 3-space*, Manuscripta Math., **98**(1999), 307-320.

[8] Galvez, J.A., Martinez, A., Milan, F., *Linear Weingarten surfaces in* R^3, Monatsh. Math., **138**(2003), 133-144.

[9] Goemans, W., Van de Woestyne, I., *Translation and homothetical lightlike hypersurfaces of semi-Euclidean space*, Kuwait J. Sci. Eng., **38**(2011), no. 2A, 35-42.

[10] Gray, A., Abbena, E., Salamon, S., *Modern differential geometry of curves and surfaces with mathematica*, Third Edition, Chapman and Hall/CRC Press, New York, 2006.

[11] Hou, Z.H., Ji, F., *Helicoidal surfaces with* $H^2 = K$ *in Minkowski 3-space*, J. Math. Anal. Appl., **325**(2007), 101-113.

[12] Jiu, L., Sun, H., *On minimal homothetical hypersurfaces*, Colloq. Math., **109**(2007), 239-249.

[13] Kim, M.H., Yoon, D.W., *Weingarten quadric surfaces in a Euclidean 3-space*, Turk. J. Math., **35**(2011), 479-485.

[14] Kuhnel, W., *Ruled W-surfaces*, Arch. Math., **62**(1994), 475-480.

[15] Lee, C.W., *Linear Weingarten rotational surfaces in pseudo-Galilean 3-space*, Int. J. Math. Anal., **9**(50)(2015), 2469-2483.

[16] Liu, H., Liu, G., *Weingarten rotation surfaces in 3-dimensional de Sitter space*, J. Geom., **79**(2004), 156-168.

[17] Lopez, R., *Rotational linear Weingarten surfaces of hyperbolic type*, Israel J. Math., **167**(2008), 283-301.

[18] Lopez, R., Moruz, M., *Translation and homothetical surfaces in Euclidean space with constant curvature*, J. Korean Math. Soc., **52**(2015), no. 3, 523-535.

[19] Meng, H., Liu, H., *Factorable surfaces in Minkowski space*, Bull. Korean Math. Soc., **46**(2009), no. 1, 155-169.

[20] Mihai, I., *On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms*, Nonlinear Analysis, **95**(2014), 714-720.

[21] Mihai, I., *On the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms*, Tohoku J. Math, to appear.

[22] Mihai, A., *Geometric inequalities for purely real submanifolds in complex space forms*, Results Math., **55**(2009), 457-468.

[23] Milin-Sipus, Z., *Translation surfaces of constant curvatures in a simply isotropic space*, Period. Math. Hung., **68**(2014), 160-175.

[24] Pottmann, H., Opitz, K., *Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces*, Comput. Aided Geom. Design, **11**(1994), 655-674.

[25] Pottmann, H., Grohs, P., Mitra, N.J., *Laguerre minimal surfaces, isotropic geometry and linear elasticity*, Adv. Comput. Math., **31**(2009), 391-419.

[26] Sachs, H., *Isotrope Geometrie des Raumes*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1990.

[27] Strubecker, K., *Differentialgeometrie des isotropen Raumes III*, Flachentheorie, Math. Zeitsch., **48**(1942), 369-427.

[28] Van de Woestyne, I., *Minimal homothetical hypersurfaces of a semi-Euclidean space*, Results Math., **27**(1995), 333-342.

[29] Yu, Y., Liu, H., *The factorable minimal surfaces*, Proceedings of The Eleventh International Workshop on Diff. Geom., **11**(2007), 33-39.

[30] Yoon, D.W., Tuncer, Y., Karacan, M.K., *Non-degenerate quadric surfaces of Weingarten type*, Annales Polonici Math., **107**(2013), 59-69.

Muhittin Evren Aydin
Firat University, Faculty of Science
Department of Mathematics
23200 Elazig, Turkey
e-mail: meaydin@firat.edu.tr

Alper Osman Ogrenmis
Firat University, Faculty of Science
Department of Mathematics
23200 Elazig, Turkey
e-mail: aogrenmis@firat.edu.tr