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Iterates of positive linear operators on Bauer simplices

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Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. We consider positive linear operators acting on $C(K)$, where K is a metrizable Bauer simplex. For such an operator L we investigate the limit of the iterates L^m, when $m \to \infty$. Qualitative results and rates of convergence are obtained. The general results are illustrated by examples involving classical operators.

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1. Introduction

Iterates of positive linear operators were investigated in many papers and from several points of view. General criteria for their convergence can be found in [1], [2], [13], [14], [20], [21], [23]. Rates of convergence of the iterates were established in [6], [10], [16], [17], [18], [20], [21], [28]. The relationship with Korovkin theory is presented in [6], [7], [8], [22]. Iterates are essentially used for representing some strongly continuous semigroups of operators: see [7], [8], [17], [28]. Iterates for q-Bernstein operators are studied in [24]; the case of complex operators is considered in [11]. In the above papers analytical methods were used and also methods from probability theory. Results based on spectral theory can be found in [9]; fixed point theory is used in [3], [4], [30], [31], [32], [33].

This paper is devoted to the study of iterates of positive linear operators on Bauer simplices. General definitions and results are presented in this introduction; see also [5], [7], [8], [25].

Section 2 is devoted to the iterates of operators preserving affine functions. An example concerning a finite dimensional simplex is discussed in Section 3. Other examples are presented in Section 4.

Throughout the paper we shall use the following notions.

Let E be a real locally convex Hausdorff space and K a convex compact subset of E. Let $C(K)$ be the space of all continuous real-valued functions on K, endowed with the usual ordering and the supremum norm. By Hervé's theorem [5, Th.I.4.3], [7, p.57], $C(K)$ contains a strictly convex function if and only if K is metrizable. Throughout the paper we shall suppose that K is metrizable.

The set of all probability Radon measures on K will be denoted by $M_1^+(K)$. For each $x \in K$, ε_x stands for the probability Radon measure concentrated on $\{x\}$.

The *Choquet-Meyer* ordering $<$ on $M_1^+(K)$ is defined as follows: for every $\mu, \nu \in M_1^+(K)$, $\mu < \nu$ if $\mu(f) \leq \nu(f)$ for every convex function $f \in C(K)$. A measure μ which is maximal with respect to $<$ will be simply called a *maximal measure*.

Let $A(K)$ be the set of all affine functions for all $h \in C(K)$. The *barycenter* of $\mu \in M_1^+(K)$ is the point $r \in K$ for which $\mu(h) = h(r),\ h \in A(K)$; in this case $\mu(f) \geq f(r)$ for each convex function $f \in C(K)$.

There are several equivalent properties defining a Choquet simplex. We need the following one:

K is called a *Choquet simplex* if for every $x \in K$ there exists a unique maximal measure $\mu_x \in M_1^+(K)$ having x as barycenter.

The set of the extreme points of K will be denoted by $ex(K)$.

A Choquet simplex K such that $ex(K)$ is closed will be called a *Bauer simplex*. In this case μ_x is supported by $ex(K)$; moreover, if $\mu_x = \varepsilon_x$, then $x \in ex(K)$.

If K is a Bauer simplex, then the operator $P : C(K) \longrightarrow A(K)$ defined by

$$Pf(x) = \mu_x(f),\ f \in C(K),\ x \in K,$$

is linear, positive, and $Ph = h$ for all $h \in A(K)$.

P is called the *canonical projection* associated with the Bauer simplex K.

Let $L : C(K) \longrightarrow C(K)$ be a positive linear operator such that $Lh = h$, for every $h \in A(K)$. For each $x \in K$ let $\nu_x(f) := Lf(x),\ f \in C(K)$. Then $\nu_x \in M_+^1(K)$ and x is the barycenter of ν_x. In particular, if $x \in ex(K)$ then $\nu_x = \varepsilon_x$, so that

$$Lf(x) = f(x),\ x \in ex(K),\ f \in C(K). \tag{1.1}$$

Moreover, if $g \in C(K)$ is convex, then $\nu_x(g) \geq g(x),\ x \in K$, i.e.,

$$Lg \geq g. \tag{1.2}$$

We shall need the following result [26], [27], [7, Th.1.5.2].

Lemma 1.1. *Let $\mu \in M_1^+(K)$ with barycenter r and let u be a strictly convex function. If $\mu(u) = u(r)$, then $\mu = \varepsilon_r$.*

2. Iterates of positive linear operators preserving the affine functions

In the sequel, K will be a metrizable Bauer simplex.

Theorem 2.1. *Let $L : C(K) \longrightarrow C(K)$ be a positive linear operator such that $Lh = h$, $h \in A(K)$. Let $u \in C(K)$ be a strictly convex function. If*

$$\lim_{m\to\infty} L^m f = Pf,\ f \in C(K), \tag{2.1}$$

then

$$Lu(x) > u(x),\ x \in K \setminus ex(K). \tag{2.2}$$

Proof. Let $x \in K$. As in the preceding section, let $\nu_x(f) := Lf(x),\ f \in C(K)$. By (1.2), $Lu(x) \geq u(x)$. Suppose that $Lu(x) = u(x)$. Then $\nu_x(u) = u(x)$, and Lemma 1.1 yields $\nu_x = \varepsilon_x$, i.e., $Lf(x) = f(x),\ f \in C(K)$. By induction, $L^m f(x) = f(x)$, $f \in C(K)$. Now (2.1) shows that $Pf(x) = f(x),\ f \in C(K)$. This means that $\mu_x = \varepsilon_x$, which entails $x \in ex(K)$. □

For $K = [0,1]$ the above result was obtained in [29] and [12, Corollary 2].

We shall prove that the converse of Th. 2.1 is also true. Having applications in mind, let us consider a sequence of positive linear operators $L_n : C(K) \longrightarrow C(K)$ preserving the affine functions, i.e.,

$$L_n h = h,\ h \in A(K),\ n \geq 1. \tag{2.3}$$

Fix a strictly convex function $u \in C(K)$.
For $n \geq 1$ and $s \in (0, +\infty)$ define

$$a_n(s) := \max_K (Pu - u - ns(L_n u - u)). \tag{2.4}$$

For $x \in ex(K)$ we have $Pu(x) - u(x) = L_n u(x) - u(x) = 0$, so that $a_n(s) \geq 0$.

Lemma 2.2. *If* $ns \geq 1,\ m \geq 1$, *then*

$$0 \leq Pu - L_n^m u \leq a_n(s)\mathbf{1} + \left(1 - \frac{1}{ns}\right)^m (Pu - u), \tag{2.5}$$

where $\mathbf{1}$ *is the constant function of value* 1 *defined on* K.

Proof. Since P preserves the affine functions, we have $u \leq Pu$ by virtue of (1.2). Moreover, $Pu \in A(K)$, and so $L_n u \leq L_n(Pu) = Pu$. By induction we get $L_n^m u \leq Pu$, and this is the first inequality in (2.5).
From (2.4) we derive

$$a_n(s)\mathbf{1} \geq Pu - u - ns(L_n u - u).$$

This implies

$$\frac{1}{ns}(Pu - a_n(s)\mathbf{1}) + \left(1 - \frac{1}{ns}\right) u \leq L_n u.$$

Since $1 - \dfrac{1}{ns} \geq 0$, iterating over $m \geq 1$

$$\left(1 - \left(1 - \frac{1}{ns}\right)^m\right)(Pu - a_n(s)\mathbf{1}) + \left(1 - \frac{1}{ns}\right)^m u \leq L_n^m u.$$

This leads immediately to the second inequality in (2.5), and the lemma is proved. □

Lemma 2.3. *Let* $n \geq 1$ *be fixed, and suppose that for a given strictly convex function* $u \in C(K)$ *one has*

$$L_n u(x) > u(x),\ x \in K \setminus ex(K). \tag{2.6}$$

Then $\lim\limits_{s \to \infty} a_n(s) = 0$.

Proof. Since $a_n \geq 0$ and a_n is decreasing on $(0, +\infty)$, we have $l := \lim_{s\to\infty} a_n(s) \geq 0$. Suppose that $l > 0$. Let

$$A_s := \{x \in K : \ Pu(x) - u(x) - ns(L_n u(x) - u(x)) \geq l\}.$$

The sets A_s are closed and the family $(A_s)_{s>0}$ is descending. For each $s > 0$, A_s and $ex(K)$ are disjoint, so that (2.6) implies $\bigcap_{s>0} A_s = \emptyset$. Since K is compact, there exists $t > 0$ such that $A_t = \emptyset$. Then $a_n(t) < l$, a contradiction. $\square$

Theorem 2.4. *(i) Let* $0 < c < 1$. *Then*

$$0 \leq Pu - L_n^m u \leq a_n(m^c)\mathbf{1} + \left(1 - \frac{1}{nm^c}\right)^m (Pu - u), \tag{2.7}$$

for all $m, n \geq 1$.
(ii) If (2.6) holds for a given $n \geq 1$, *then*

$$\lim_{m\to\infty} L_n^m f = Pf, \ \ f \in C(K). \tag{2.8}$$

Proof. (i) is a consequence of (2.5), with $s = m^c$. From (2.7) and Lemma 2.2 we infer that $\lim_{m\to\infty} L_n^m u = Pu$. This fact, combined with Corollary 3.3.4 of [7], leads to (2.8). $\square$

In the sequel we shall suppose that the limit

$$T(t)f := \lim_{n\to\infty} L_n^{k(n)} f$$

exists in $C(K)$ for each $f \in C(K)$, each $t \geq 0$, and each sequence of positive integers $(k(n))_{n\geq 1}$ such that $\lim_{n\to\infty} \frac{k(n)}{n} = t$.

Denote $a(s) = \sup\{a_n(s) : n \geq 1\}, \ s > 0$.

Theorem 2.5. *(i) Let* $0 < c < 1$. *Then for all* $t > 0$,

$$0 \leq Pu - T(t)u \leq a(t^c)\mathbf{1} + (Pu - u)\exp(-t^{1-c}). \tag{2.9}$$

(ii) If $\lim_{s\to\infty} a(s) = 0$, *then*

$$\lim_{t\to\infty} T(t)f = Pf, \ \ f \in C(K). \tag{2.10}$$

Proof. Let $t > 0$ be fixed. If $nt^c \geq 1$, from (2.5) we get

$$0 \leq Pu - L_n^{k(n)} u \leq a(t^c)\mathbf{1} + \left(1 - \frac{1}{nt^c}\right)^{k(n)} (Pu - u).$$

Choosing $k(n)$ such that $\lim_{n\to\infty} \frac{k(n)}{n} = t$, we obtain (2.9).
If $\lim_{s\to\infty} a(s) = 0$, (2.9) yields

$$\lim_{t\to\infty} T(t)u = Pu.$$

Another application of [7, Cor. 3.3.4] concludes the proof. $\square$

3. An example and a quantitative result

Let K be the canonical simplex of $\mathbb{R}^d$, that is

$$K = \{x \in \mathbb{R}^d : x_1, \dots x_d \geq 0,\ x_1 + \dots + x_d \leq 1\}.$$

The canonical projection associated with K is defined, for every $f \in C(K)$ and $x \in K$, by

$$Pf(x) = (1 - x_1 - \dots - x_d)f(0) + x_1 f(v_1) + \dots + x_d f(v_d), \tag{3.1}$$

where $0,\ v_1 := (1, 0, \dots, 0), \dots, v_d := (0, \dots, 0, 1)$ are the vertices of K.
Let $f \in C(K)$; suppose that there exists a constant $Q_f > 0$ such that for all $x \in K$,

$$|f(x) - f(0)| \leq Q_f \sum_{i=1}^{d} x_i, \tag{3.2}$$

$$|f(x) - f(v_j)| \leq Q_f \left(1 - 2x_j + \sum_{i=1}^{d} x_i\right),\ j = 1, \dots, d. \tag{3.3}$$

Then, for $x \in K$,

$$\begin{aligned} |f(x) - Pf(x)| &= |f(x) - \left(1 - \sum_{i=1}^{d} x_i\right) f(0) - \sum_{i=1}^{d} x_i f(v_i)| \\ &= \left|\left(1 - \sum_{i=1}^{d} x_i\right)(f(x) - f(0)) + \sum_{i=1}^{d} x_i (f(x) - f(v_i))\right| \\ &\leq Q_f \left(\left(1 - \sum_{i=1}^{d} x_i\right) \sum_{i=1}^{d} x_i + \sum_{i=1}^{d} x_i \left(1 - 2x_i + \sum_{j=1}^{d} x_j\right)\right) \\ &= 2Q_f \left(\sum_{i=1}^{d} x_i - \sum_{i=1}^{d} x_i^2\right). \end{aligned}$$

Consider the strictly convex function $u \in C(K),\ u(x) = \sum_{i=1}^{d} x_i^2,\ x \in K$. Then

$$Pu(x) = \sum_{i=1}^{d} x_i,$$

so that for the above function f we have

$$|f(x) - Pf(x)| \leq 2Q_f(Pu(x) - u(x)),\ x \in K. \tag{3.4}$$

Let $L_n : C(K) \longrightarrow C(K)$ be a positive linear operator preserving affine functions. From (3.4) we get

$$|L_n^m f - Pf| \leq 2Q_f(Pu - L_n^m u). \tag{3.5}$$

Finally, (3.5) and (2.7) yield

$$|L_n^m f - Pf| \leq 2Q_f \left(a_n(m^c)\mathbf{1} + \left(1 - \frac{1}{nm^c}\right)^m (Pu - u)\right),$$

i.e.,

$$|L_n^m f(x) - Pf(x)| \leq 2Q_f \left[a_n(m^c) + \left(1 - \frac{1}{nm^c}\right)^m \sum_{i=1}^d x_i(1-x_i) \right]. \tag{3.6}$$

Moreover, in the context of Theorem 2.3 we derive from (3.4):

$$|T(t)f - Pf| \leq 2Q_f(Pu - T(t)u). \tag{3.7}$$

Combined with (2.9), this gives

$$|T(t)f(x) - Pf(x)| \leq 2Q_f \left[a(t^c) + (\exp(-t^{1-c})) \sum_{i=1}^d x_i(1-x_i) \right]. \tag{3.8}$$

Remark 3.1. If $f \in C^1(K)$, i.e., f has continuous partial derivatives on the interior of K which can be continuously extended on K, then (3.2) and (3.3) are satisfied with

$$Q_f := \max \left\{ \left\| \frac{\partial f}{\partial x_i} \right\|_\infty : i = 1, \dots, d \right\}.$$

4. Examples

In this section we present examples of sequences $(L_n)_{n\geq 1}$ of operators preserving affine functions and satisfying the fundamental condition (2.6).

Example 4.1. Let $B_n,\ n \geq 1$, be the *Bernstein-Schnabl* operators associated with the canonical projection P and the arithmetic mean *Toeplitz* matrix (see [7, p. 381]). Let $u \in C(K)$ be a strictly convex function. Suppose that for a given $n \geq 1$ and a given $x \in K$ one has $B_n u(x) = u(x)$. According to *Lemma 1.1*, we infer that $B_n f(x) = f(x)$, for every $f \in C(K)$. In particular, $B_n h^2(x) = h^2(x)$, for all $h \in A(K)$. Now [7, (6.1.16)] leads to $P(h^2)(x) = h^2(x),\ h \in A(K)$. From [7, Cor. 3.3.4 and Remark to Prop. 3.3.2] we deduce that $x \in ex(K)$. So (2.6) is satisfied for the operators B_n.

Example 4.2. Let $U_n,\ n \geq 1$, be the genuine *Bernstein-Durrmeyer* operators on a simples K in $\mathbb{R}^d$ (see [34], [19], [35]). If $u \in C(K)$ is strictly convex, then $U_n u \geq B_n u \geq u$ [19, Th.8]. If $U_n u(x) = u(x)$, then $B_n u(x) = u(x)$, and from Ex. 4.1 we know that $x \in ex(K)$. So (2.6) is satisfied for the operators U_n.

Example 4.3. It was proved in [28, Example 2.4] that (2.6) is satisfied for the classical Meyer-König and Zeller operators on $C[0,1]$.

Example 4.4. The case of the Bernstein-Schnabl operators on the unit interval, associated with a continuous selection of probability Borel measures on $[0,1]$, is considered in [28, Example 3.3]. The operators satisfy (2.6).

For all the operators presented in the above examples one can apply Lemma 2.2 and, consequently, one can obtain the corresponding quantitative results derived from Theorems 2.2 and 2.3.

Other examples and quantitative results can be found in [18], [28], [29].

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