Stud. Univ. Babeş-Bolyai Math. 65(2020), No. 3, 365–371
DOI: 10.24193/subbmath.2020.3.05

On a certain class of harmonic functions and the generalized Bernardi-Libera-Livingston integral operator

Grigore Ştefan Sălăgean and Ágnes Orsolya Páll-Szabó

Abstract. In this paper we examine the closure properties of the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_c(f)$, $(c > -1)$ which is defined by $\mathcal{L}_c(f) = \mathcal{L}_c(h) + \overline{\mathcal{L}_c(g)}$, $f = h + \overline{g}$, h and g are analytic functions, where

$$\mathcal{L}_c(h)(z) = \frac{c+1}{z^c}\int_0^z (t^{c-1}h(t)dt \quad \text{and} \quad \mathcal{L}_c(g)(z) = \frac{c+1}{z^c}\int_0^z (t^{c-1}g(t)dt.$$

The obtained results are sharp and they improve known results.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Harmonic univalent functions, extreme points, varying arguments, Hadamard product, integral operator.

1. Preliminaries

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).
Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1.1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m \quad (0 \le b_1 < 1)\,,$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \quad (0 \le b_1 < 1)\,. \tag{1.2}$$

For functions $f \in \mathcal{H}$ given by (1.2) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=1}^{\infty} B_m z^m},\ (0 \le B_1 \le 1), \tag{1.3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \overline{\sum_{m=1}^{\infty} b_m B_m z^m} \quad (z \in \mathcal{U})\,. \tag{1.4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f * F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past.
In [8] a subclass of $\mathcal{H}$ denoted by $\mathcal{S}_{\mathcal{H}}(F;\gamma)$, for $0 \le \gamma < 1$, is defined and studied and it consists of functions of the form (1.1) satisfying the inequality:

$$\frac{\partial}{\partial \theta}\left(arg\left[(f * F)(z)\right]\right) > \gamma \tag{1.5}$$

$0 \le \theta < 2\pi$ and $z = re^{i\theta}$. Equivalently

$$Re\left\{\frac{z\left(h\left(z\right) * H\left(z\right)\right)' - \overline{z\left(g\left(z\right) * G\left(z\right)\right)'}}{h\left(z\right) * H\left(z\right) + \overline{g\left(z\right) * G\left(z\right)}}\right\} \ge \gamma \tag{1.6}$$

where $z \in \mathcal{U}$. We also let $\mathcal{V}_{\mathcal{H}}(F;\gamma) = S_{\mathcal{H}}(F;\gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ is the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [6], consisting of functions f of the form (1.1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\,\phi \equiv \pi\,(\text{mod } 2\pi)\,, \quad \delta_m + (m+1)\,\phi \equiv 0\,(\text{mod } 2\pi)\,, \quad m \ge 2, \tag{1.7}$$

where $\eta_m = arg\,(a_m)$ and $\delta_m = arg\,(b_m)$.
Some of the function classes emerge from the function class $S_{\mathcal{H}}(F;\gamma)$ defined above. Indeed, if we specialize the function $F(z)$ we can obtain, respectively, (see [8]) the class of functions defined using: the Wright's generalized operator on harmonic functions ([9], [13]), the Dziok-Srivastava operator on harmonic functions ([1]), the Carlson-Shaffer operator ([2]), the Ruscheweyh derivative operator on harmonic functions ([5], [7], [10]) , the Srivastava-Owa fractional derivative operator ([12]), the Sălăgean derivative operator for harmonic functions ([4], [11]).

In the following we suppose that $F(z)$ is of the form

$$F(z) = H(z) + \overline{G(z)} = z + \overline{z} + \sum_{m=2}^{\infty} C_m \left(z^m + \overline{z^m}\right), \tag{1.8}$$

where $C_m \geq 0 \, (m \geq 2)$.
In [8] the following characterization theorem is proved

Theorem 1.1. *Let $f = h + \overline{g}$ be given by (1.2) with restrictions (1.7) and $0 \leq b_1 < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$. Then $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ if and only if the inequality*

$$\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma} |a_m| + \frac{m+\gamma}{1-\gamma} |b_m| \right) C_m \leq 1 - \frac{1+\gamma}{1-\gamma} b_1 \tag{1.9}$$

holds true.

Theorem 1.2. [8] *Set $\lambda_m = \dfrac{1-\gamma}{(m-\gamma)C_m}$ and $\mu_m = \dfrac{1-\gamma}{(m+\gamma)C_m}$. Then for b_1 fixed, $0 \leq b_1 < \dfrac{1-\gamma}{1+\gamma}$ the extreme points for $\mathcal{V}_{\mathcal{H}}(F;\gamma)$, $0 \leq \gamma < 1$ are*

$$\left\{ z + \lambda_m x z^m + \overline{b_1 z} \right\} \cup \left\{ z + \overline{b_1 z + \mu_m x z^m} \right\}$$

where $m \geq 2$ and $x = 1 - \dfrac{1+\gamma}{1-\gamma} b_1$.

2. Main result

Now, we will examine the closure properties of the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_c(f)$, $(c > -1)$ which is defined by $\mathcal{L}_c(f) = \mathcal{L}_c(h) + \overline{\mathcal{L}_c(g)}$ where

$$\mathcal{L}_c(h)(z) = \frac{c+1}{z^c} \int_0^z (t^{c-1} h(t) dt \quad \text{and} \quad \mathcal{L}_c(g)(z) = \frac{c+1}{z^c} \int_0^z (t^{c-1} g(t) dt.$$

Theorem 2.1. *Let $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$. Then $\mathcal{L}_c(f) \in \mathcal{V}_{\mathcal{H}}(F;\delta(\gamma))$ where*

$$\delta(\gamma) = \frac{(2+\gamma)(c+2)(1-b_1) - 2(c+1)\left[(1-\gamma) - (1+\gamma) b_1\right]}{(2+\gamma)(c+2)(1+b_1) + (c+1)\left[(1-\gamma) - (1+\gamma) b_1\right]} > \gamma.$$

The result is sharp.

Proof. Since $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ we have

$$\frac{\displaystyle\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma} |a_m| + \frac{m+\gamma}{1-\gamma} |b_m| \right) C_m}{1 - \dfrac{1+\gamma}{1-\gamma} b_1} \leq 1. \tag{2.1}$$

We know from Theorem 1.1 that $\mathcal{L}_c(f) \in \mathcal{V}_{\mathcal{H}}(F; \delta(\gamma))$ if and only if

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|a_m| + \frac{m+\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|b_m|\right)C_m}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1} \leq 1. \tag{2.2}$$

We note that the inequalities

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|a_m| + \frac{m+\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|b_m|\right)C_m}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1}$$

$$\leq \frac{\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m| + \frac{m+\gamma}{1-\gamma}|b_m|\right)C_m}{1-\frac{1+\gamma}{1-\gamma}b_1} \tag{2.3}$$

imply (2.2). It is sufficient to determine $\delta(\gamma)$ such that

$$\frac{\frac{m-\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1} \leq \frac{\frac{m-\gamma}{1-\gamma}}{1-\frac{1+\gamma}{1-\gamma}b_1} \tag{2.4}$$

and

$$\frac{\frac{m+\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1} \leq \frac{\frac{m+\gamma}{1-\gamma}}{1-\frac{1+\gamma}{1-\gamma}b_1}. \tag{2.5}$$

holds true. (2.4) is equivalent to

$$\frac{m-\delta(\gamma)}{1-\delta(\gamma)-b_1-\delta(\gamma)b_1}\frac{c+1}{c+m} \leq \frac{m-\gamma}{(1-\gamma)-(1+\gamma)b_1}$$

$$\delta(\gamma) \leq \frac{(m-\gamma)(c+m)(1-b_1) - m(c+1)\left[(1-\gamma)-(1+\gamma)b_1\right]}{(m-\gamma)(c+m)(1+b_1) - (c+1)\left[(1-\gamma)-(1+\gamma)b_1\right]}. \tag{2.6}$$

Relation (2.5) is equivalent to

$$\frac{m+\delta(\gamma)}{1-\delta(\gamma)-b_1-\delta(\gamma)b_1}\frac{c+1}{c+m} \leq \frac{m+\gamma}{(1-\gamma)-(1+\gamma)b_1}$$

$$\delta(\gamma) \leq \frac{(m+\gamma)(c+m)(1-b_1) - m(c+1)\left[(1-\gamma)-(1+\gamma)b_1\right]}{(m+\gamma)(c+m)(1+b_1) + (c+1)\left[(1-\gamma)-(1+\gamma)b_1\right]}. \tag{2.7}$$

From (2.6) and (2.7) we choose the smaller one:

$$\frac{(m-\gamma)\,(c+m)\,(1-b_1)-m\,(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}{(m-\gamma)\,(c+m)\,(1+b_1)-(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}$$
$$>\frac{(m+\gamma)\,(c+m)\,(1-b_1)-m\,(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}{(m+\gamma)\,(c+m)\,(1+b_1)+(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}$$

or equivalently

$$\frac{2(c+1)\Delta^2 m(m-1)}{[(m-\gamma)\,(c+m)\,(1+b_1)-(c+1)\,\Delta]\,[(m+\gamma)\,(c+m)\,(1+b_1)+(c+1)\,\Delta]}>0,$$

where $\Delta=[(1-\gamma)-(1+\gamma)\,b_1]>0$ which is true. So

$$\delta(\gamma)\leq\frac{(m+\gamma)\,(c+m)\,(1-b_1)-m\,(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}{(m+\gamma)\,(c+m)\,(1+b_1)+(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}. \tag{2.8}$$

Let us consider the function $E:[2;\infty)\to\mathbb{R}$

$$E(x)=\frac{(x+\gamma)\,(c+x)\,(1-b_1)-x\,(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}{(x+\gamma)\,(c+x)\,(1+b_1)+(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]};$$

then its derivative is:

$$E'(x)=\frac{(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]\left[(1+b_1)x^2+2x(1-b_1)+2\gamma+b_1-1\right]}{\{(x+\gamma)\,(c+x)\,(1+b_1)+(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]\}^2}>0.$$

$E(x)$ is an increasing function. In our case we need $\delta\,(\gamma)\leq E\,(m)\,,\forall m\geq 2$ and for this reason we choose

$$\delta\,(\gamma)=E(2)=\frac{(2+\gamma)\,(c+2)\,(1-b_1)-2\,(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}{(2+\gamma)\,(c+2)\,(1+b_1)+(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}.$$

We must check $\delta\,(\gamma)>\gamma$ that is equivalent to

$$\frac{[(1-\gamma)-(1+\gamma)\,b_1]\,(2+\gamma)\,[(c+2)-(c+1)]}{(2+\gamma)\,(c+2)\,(1+b_1)+(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}>0$$

which is true.
The result is sharp, because if

$$f(z)=z+\overline{b_1 z+\frac{1-\gamma}{(2+\gamma)C_2}\left(1-\frac{1+\gamma}{1-\gamma}b_1\right)z^2}$$

then

$$\mathcal{L}_c(f)(z)=z+\overline{b_1 z+\frac{1-\gamma}{(2+\gamma)C_2}\left(1-\frac{1+\gamma}{1-\gamma}b_1\right)z^2\frac{c+1}{c+2}}$$
$$=z+\overline{b_1 z+\frac{1-\delta\,(\gamma)}{(2+\delta\,(\gamma))C_2}\left(1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1\right)z^2}$$
$$\Leftrightarrow\frac{1-\gamma}{(2+\gamma)}\frac{c+1}{c+2}\frac{1-\gamma-(1+\gamma)b_1}{1-\gamma}=\frac{1-\delta\,(\gamma)}{(2+\delta\,(\gamma))}\frac{1-\delta(\gamma)-(1+\delta(\gamma))b_1}{1-\delta(\gamma)}$$
$$\Leftrightarrow\delta(\gamma)=\frac{(2+\gamma)\,(c+2)(1-b_1)-2\,(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}{(2+\gamma)\,(c+2)(1+b_1)+(c+1)\,[(1-\gamma)-(1+\gamma)\,b_1]}$$

this is the (2.7) inequality. □

Acknowledgement. The present work has received financial support through the project: Entrepreneurship for innovation through doctoral and postdoctoral research, POCU/360/6/13/123886 co-financed by the European Social Fund, through the Operational Program for Human Capital 2014- 2020.

References

[1] Al-Kharsani, H.A., Al-Khai, R.A., *Univalent harmonic functions*, J. Inequal. Pure Appl. Math., **8**(2007), no. 2, Art. 59, 8 pp.

[2] Carlson, B.C., Shaffer, S.B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15**(1984), no. 4, 737-745.

[3] Clunie, J., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A.I. Math., **9**(1984), 3-25.

[4] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Sălăgean-type harmonic univalent functions*, Southwest J. Pure Apll. Math., (2002), no. 2, 77-82.

[5] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Starlikeness of harmonic functions defined by Ruscheweyh derivatives* J. Indian Acad. Math., **26**(2004), no. 1, 191-200.

[6] Jahangiri, J.M., Silverman, H., *Harmonic univalent functions with varying arguments*, Int. J. Appl. Math., **8**(2002), no. 3, 267-275.

[7] Murugusundaramoorthy, G., *A class of Ruscheweyh-type harmonic univalent functions with varying arguments*, Southwest J. Pure Appl. Math., **2**(2003), 90-95.

[8] Murugusundaramoorthy, G., Sălăgean, G.S., *On a certain class of harmonic functions associated with a convolution structure*, Mathematica, **54**(77)(2012), Special Issue, 131-142.

[9] Murugusundaramoorthy, G., Vijaya, K., *A subclass of harmonic functions associated with Wright hypergeometric functions*, Adv. Stud. Contemp. Math. (Kyungshang), **18**(2009), no. 1, 87-95.

[10] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[11] Sălăgean, G.S., *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), 362-372, Lecture Notes in Math., 1013, Springer, Berlin, 1983.

[12] Srivastava, H.M., Owa, S., *Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions*, Nagoya Math. J., **106**(1987), 1-28.

[13] Wright, E.M., *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc., **46**(1940), no. 2, 389-408.

Grigore Ştefan Sălăgean
Babeş-Bolyai University,
Faculty of Mathematics and Computer Sciences,
1, Kogălniceanu Street,
400084 Cluj-Napoca, Romania
e-mail: `salagean@math.ubbcluj.ro`

Ágnes Orsolya Páll-Szabó
Babeş-Bolyai University,
Faculty of Mathematics and Computer Sciences,
1, Kogălniceanu Street,
400084 Cluj-Napoca, Romania
e-mail: pallszaboagnes@math.ubbcluj.ro