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Numerical optimal control for satellite attitude profiles

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Abstract. Many modern science satellites are 3-axis stabilized. The construction of attitude profiles therefore play a central role in satellite control. Besides the dynamical properties numerous constraints need to be fulfilled. In [6] a generic way for calculating such attitudes is given. Other options to design slews connecting two attitudes have been published in various papers (e.g. [3, 11]) including approaches using optimal control techniques (e.g. [4, 8, 11]).
In this paper we will present a new approach for optimal control of slews and attitude profiles. After the description of a set of the considered Hamiltonian functions and the respective slew maneuvers some analytical consequences of the choices are given. A comparison with the actual operational Euler angle slew in [6] is given and shows a close match. The performed numerical investigations of direct solutions help to gain a clearer picture on the underlaying analytical problem. By applying the Pontryagin maximum principle to the Hamiltonian equation, a family of closed dynamics ordinary differential equation for the direct optimal control problem is presented and their solutions and properties are investigated.

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1. Introduction

1.1. Dynamic Optimization

For the numerical solution of optimal control problems there are two fundamentally different approaches. Formulating the solution of the optimization problem and then using a discretization method to approximate the solution is called indirect approach [2]. In the so called direct approach the problem is first discretized and

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then optimization methods are used to find an approximate solution [9]. The well known indirect methods are the Hamilton Jacobi Bellmann equation and the Hamilton equations together with the Pontryagin maximum principle. Direct methods have been popular in the recent past. There are several reasons that support the direct approach: To a limited extend realtime applications are possible and it is rather easy and straight forward to incorporate constraints into the procedure. We will consider the second indirect approach in this paper. The well known result from the calculus of variations is given by:

Theorem 1.1. [5, 2] *Let $F\colon I\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ and $L\colon I\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ be differentiable. Then the variation of the Hamiltonian $H(t,x,u,\lambda) = L(t,x,\dot{x}) + \lambda\cdot F(t,x,u)$ with respect to the independent variables $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, results in the equations*

$$\begin{aligned} \dot{\lambda}(t) &= -\tfrac{\partial}{\partial x} H(t,x(t),u(t),\lambda(t))\,, \\ \dot{x}(t) &= \tfrac{\partial}{\partial \lambda} H(t,x(t),u(t),\lambda(t))\,, \\ 0 &= \tfrac{\partial}{\partial u} H(t,x(t),u(t),\lambda(t))\,. \end{aligned}$$

The first two equations are differential equations for $x(t)$ and $\lambda(t)$, the so called **Hamiltonian equations**. The last one is the optimality condition, an algebraic equation for $u(t)$, which is valid for all t. The generalization of the optimality condition for the optimal trajectory $\lambda^*(t), x^*(t)$ is:

$$H(t,x^*(t),u^*(t),\lambda^*(t)) = \max_u H(t,x^*(t),u(t),\lambda^*(t))\,.$$

This equation is often referred to as the **Pontryagin maximum principle**. Since we want to prescribe the initial and final values x_{ini} and x_{fin} of our state variables, we will end up with a two-point boundary value problem of the following kind, where u^* is the optimal control to be determined:

$$\begin{aligned} \dot{x} &= \tfrac{\partial}{\partial \lambda} H(t,x,u^*,\lambda)\,, \quad x(t_{\text{ini}}) = x_{\text{ini}}\,, \\ \dot{\lambda} &= -\tfrac{\partial}{\partial x} H(t,x,u^*,\lambda)\,, \quad \lambda(t_{\text{fin}}) = \lambda_{\text{fin}}\,. \end{aligned}$$

Remark 1.2. The exact list of state variables will depend on the exact statement of the problem, e.g. we will have to add an integral constraint to the state variables in order to be able to enforce further constraints on the solution trajectory.

1.2. Numerical Dynamic Optimization

There are numerous ways in order to solve two-point boundary value problems numerically. There are many standard schemes, but with the desire to be able to solve real-time problems time critical approaches have surfaced in the recent years. The numerical simulations for this paper where undertaken by three different schemes.

- A single shooting method using symbolic differentiation, symbolic solvers and standard ordinary differential equation integrators (of Runge-Kutta and Adams type) and the derivative free optimization method of Nelder-Mead.

- A fast direct approach using the CasADi tool with algorithmic differentiation, symbolic ordinary differential equation solver and nonlinear optimization techniques [1].
- A commercial software package with built in boundary value problem solvers. Here the exact solution approach is undisclosed.

Besides the obvious difference in time consumption of the different approaches, there have been no inconsistencies in the respective results. Further the analytical results presented in this paper match the characteristic of the numerical solutions.

2. Optimal Slews

Unit quaternions provide a mathematical way for representing orientations in 3-space. We will denote the field of quaternions by $\mathbb{H}$ and the quaternions themselves by q. The quaternion multiplication is written as $*$. In the following sections vectors x of the $\mathbb{R}^3$ are embedded in $\mathbb{H} \approx \mathbb{R}^4$ in the canonical way by setting the scalar part to 0. With $\overline{q}$ we denote the complex conjugate quaternion of q and for all q_1, q_2 and q_3 we have $(q_1 * q_2) * q_3 = q_1 * (q_2 * q_2)$ and $\overline{q_1 * q_2} = \overline{q_2} * \overline{q_1}$. Further we can explicitly express $*$ by

$$q_1 * q_2 = \begin{pmatrix} \mathrm{Re}(q_1)\,\mathrm{Re}(q_2) - \mathrm{Im}\,m(q_1) \cdot \mathrm{Im}(q_2) \\ \mathrm{Im}(q_1) \times \mathrm{Im}(q_2) + \mathrm{Re}(q_1)\,\mathrm{Im}(q_2) + \mathrm{Re}(q_2)\,\mathrm{Im}(q_1) \end{pmatrix},$$

where $\mathrm{Re}(q)$ and $\mathrm{Im}(q)$ denote the real- and imaginary part of q.

2.1. Eigenaxis Slews

An attitude slew is a time profile $q(t)\colon [t_0, t_1] \to \mathbb{H}$ connecting two orientations in 3-space. The rotation slew of a rigid body has therefore the state vector $x = q = (q_s, q_x, q_y, q_z) \in \mathbb{H}$. q_s is the scalar part of the quaternion and q_x, q_y and q_z indicate the vector parts. As control variable u the angular velocity $\omega = (\omega_x, \omega_y, \omega_z)$ is chosen. The kinematic equation of the rotational movement can be written as

$$\dot{x} = \dot{q} = \tfrac{1}{2}\omega * q = F\,(\omega\,,\,q)\,.$$

A constant of integration is given namely by the length of the quaternion q:

Lemma 2.1. [4] *Let $\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}^3 \subset \mathbb{H})$ be given. Then for the solution $q \in \mathcal{C}^1(\mathbb{R}, \mathbb{H})$ of the differential equation $\dot{q} = \frac{1}{2}\,\omega * q$ we get $\|q(t)\| = \|q(t_0)\|\ \forall t$.*

This does exclude $\|q\|^2 = 1$ from the design as a cost term for reducing the duration of the slew – it is automatically built in. The cost function we choose is therefore $L = \|\omega\|^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$. This results in the Hamiltonian

$$H \;=\; L + \lambda^\top F = \|\omega\|^2 + \tfrac{1}{2}\,\lambda^\top \omega * q$$

and the Hamilton equations are (see also [4])

$$\begin{aligned} \dot{q} &= +\tfrac{\partial H}{\partial \lambda} = +\tfrac{1}{2} \cdot \omega * q \\ \dot{\lambda} &= -\tfrac{\partial H}{\partial q} = -\tfrac{1}{2} \cdot \overline{\omega} * \lambda\,. \end{aligned}$$

From the Pontyagin maximum principle follows

$$
\begin{aligned}
& 0=\tfrac{\partial H}{\partial u}=\tfrac{\partial H}{\partial \omega}=2\omega+\tfrac{1}{2}\cdot\lambda*\overline{q} \\
\Rightarrow\quad & \omega=-\tfrac{1}{4}\cdot\lambda*\overline{q} \\
\Rightarrow\quad & \dot{\omega}=-\tfrac{1}{4}\left(\dot{\lambda}*\overline{q}+\cdot\lambda*\dot{\overline{q}}\right) \\
& \quad=+\tfrac{1}{8}\left([\overline{\omega}*\lambda]*\overline{q}-\lambda*\overline{[\omega*q]}\right)=\tfrac{1}{8}\left(\overline{\omega}*[\lambda*\overline{q}]-[\lambda*\overline{q}]*\overline{\omega}\right) \\
& \quad=-\tfrac{1}{2}\left(\overline{\omega}*\omega-\omega*\overline{\omega}\right)=-\tfrac{1}{2}\left(\|\omega\|^2-\|\omega\|^2\right)=0 \quad \text{i.e.} \quad \boxed{\dot{\omega}=0}.
\end{aligned}
$$

Lemma 2.2. *The unconstraint optimal control slew connecting two attitudes q_1 and q_2 is an eigenaxis slew with constant angular velocity.*

In [7] the same result can be found, formulated in the language of Lie theory. For now i.e. in this paper we will not make use of this formalism, since we are in the end interested in numerical solution schemes and do not see the benefit at this point. Nevertheless with respect to the the Euler-Poincaré equations it could be beneficial to consider this in the future. Although the analytic solution can be explicitly stated, it is interesting to note that the numerical integrators do preserve the constant of integration $\|q\|$ flawlessly.

2.2. Geometric Optimal Slews

Geometric and dynamic constraints often lead to cost terms that contradict each other. This can be easily demonstrated by the means of examples. Therefore they shall not be mixed as optimization terms. A rather stepwise approach by first constructing a geometrically optimal path and then use e.g. weight functions like in [8] for optimizing the dynamics and speed is suggested. This idea is related to engineering solutions where the relative slow motion of the celestial bodies is completely neglected. So we consider the rotational motion of a rigid body with the state vector as

$$x=(q,\omega)=(q_s,q_x,q_y,q_z,\omega_x,\omega_y,\omega_z)\in\mathbb{R}^7 \; .$$

As a control u a torque term $T=(t_x,t_y,t_z)^\top$ is used and the kinematic equation is:

$$\dot{x}=(\dot{q},\dot{\omega})=F\left(\omega\;,\;q\right)=\left(\tfrac{1}{2}\cdot\omega*q,T\right)$$

The cost function chosen is $L=\|T\|^2=t_x^2+t_y^2+t_z^2$. Then

$$H=L+\lambda^\top F=\|T\|^2+\lambda_1^\top\cdot\tfrac{1}{2}\cdot\omega*q+\lambda_2^\top\cdot T$$

and the respective Hamilton equations are:

$$
\begin{aligned}
\begin{pmatrix}\dot{q}\\ \dot{\omega}\end{pmatrix} &= +\frac{\partial H}{\partial \lambda}=\begin{pmatrix}\frac{1}{2}\cdot\omega*q\\ T\end{pmatrix}\\
\begin{pmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{pmatrix} &= -\frac{\partial H}{\partial x}=\begin{pmatrix}-\frac{1}{2}\cdot\overline{\omega}*\lambda_1\\ -\frac{1}{2}\cdot\lambda_1*\overline{q}\end{pmatrix}
\end{aligned}
$$

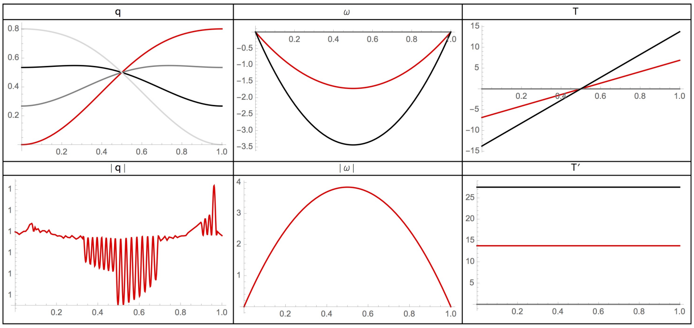
From the Pontyagin maximum principle follows

$$
\begin{aligned}
& 0 = \tfrac{\partial H}{\partial u} = \tfrac{\partial H}{\partial T} = 2T + \lambda_2 \\
\Rightarrow \quad & T = -\tfrac{1}{2} \cdot \lambda_2 = \dot{\omega} \\
\Rightarrow \quad & \ddot{\omega} = -\tfrac{1}{2} \cdot \dot{\lambda}_2 = \tfrac{1}{4} \lambda_1 * \overline{q} \\
\Rightarrow \quad & \dddot{\omega} = +\tfrac{1}{4} \left(\dot{\lambda}_1 * \overline{q} + \lambda_1 * \overline{\dot{q}} \right) \\
& \quad = -\tfrac{1}{8} \left([\overline{\omega} * \lambda_1] * \overline{q} - \lambda_1 * \overline{[\omega * q]} \right) = -\tfrac{1}{8} \left(\overline{\omega} * [\lambda_1 * \overline{q}] - [\lambda_1 * \overline{q}] * \overline{\omega} \right) \\
& \quad = +\tfrac{1}{4} \left(\overline{\omega} * \dot{\lambda}_2 - \dot{\lambda}_2 * \overline{\omega} \right) = -\tfrac{1}{2} \left(\overline{\omega} * \ddot{\omega} - \ddot{\omega} * \overline{\omega} \right) .
\end{aligned}
$$

Theorem 2.3 (**ω - ode**)**.** *The unconstraint optimal control slew connecting two attitudes q_1, ω_1 and q_2, ω_2 is governed by the an angular velocity ω for which*

$$
\boxed{\dddot{\omega} = -\tfrac{1}{2} \left(\overline{\omega} * \ddot{\omega} - \ddot{\omega} * \overline{\omega} \right)} \quad \textit{or equivalent} \quad \boxed{\dddot{\omega} = \omega \times \ddot{\omega}} \quad \textit{holds.}
$$

Example 2.4. Shown is a geometric slew connecting the initial and final state $(q(0), \omega(0)) = (\frac{1}{\sqrt{14}}(0,1,2,3)^\top, 0)$ and $(q(1), \omega(1)) = (\frac{1}{\sqrt{14}}(3,2,1,0)^\top, 0)$:



2.3. Constraint Optimal Slews

If we add integral terms to the dynamics of the slew, additional constraints can be considered. For the motion of a rigid body, the state vector then becomes

$$
x = (q, \omega, c) = (q_s, q_x, q_y, q_z, \omega_s, \omega_x, \omega_y, \omega_z, c) \in \mathbb{R}^{8+m}
$$

$m = 1$ or 2. As control we again consider a torque $T = (t_x, t_y, t_z)^\top$ and the kinematic equation is:

$$
\dot{x} = (\dot{q}, \dot{\omega}, \dot{c}) = F(\omega\,,\, q) = (\tfrac{1}{2} \cdot \omega * q, T, C(q, \omega))
$$

As cost function we choose $L = c^2 + \|T\|^2 = c^2 + t_s^2 + t_x^2 + t_y^2 + t_z^2$. Then

$$
H = L + \lambda^\top F = c^2 + \|T\|^2 + \lambda_1^\top \cdot \tfrac{1}{2} \cdot \omega * q + \lambda_2^\top \cdot T + \lambda_3^\top \cdot C(q, \omega)
$$

and the Hamilton equations are:

$$\begin{pmatrix} \dot{q} \\ \dot{\omega} \\ \dot{c} \end{pmatrix} = +\frac{\partial H}{\partial \lambda} = \begin{pmatrix} \frac{1}{2} \cdot \omega * q \\ T \\ C(q,\omega) \end{pmatrix}$$

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = -\frac{\partial H}{\partial x} = \begin{pmatrix} -\frac{1}{2} \cdot \overline{\omega} * \lambda_1 - \lambda_3^\top \cdot \frac{\partial}{\partial q} C(q,\omega) \\ -\frac{1}{2} \cdot \lambda_1 * \overline{q} - \lambda_3^\top \cdot \frac{\partial}{\partial \omega} C(q,\omega) \\ 2c \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \cdot \overline{\omega} * \lambda_1 - C_q \\ -\frac{1}{2} \cdot \lambda_1 * \overline{q} - C_\omega \\ 2c \end{pmatrix}$$

From the Pontyagin maximum principle follows with $\dot{\lambda}_2 = -\frac{1}{2}\lambda_1 * \overline{q} - C_\omega$ and $\dot{\omega} = -\frac{1}{2}\lambda_2$:

$$\begin{aligned} 0 &= \tfrac{\partial H}{\partial u} = \tfrac{\partial H}{\partial T} = 2T + \lambda_2 \\ &\Rightarrow\ T = -\tfrac{1}{2} \cdot \lambda_2 = \dot{\omega} \\ &\Rightarrow\ \ddot{\omega} = -\tfrac{1}{2} \cdot \dot{\lambda}_2 = \tfrac{1}{4}\lambda_1 * \overline{q} + \tfrac{1}{2} C_\omega \\ &\Rightarrow\ \dddot{\omega} = +\tfrac{1}{4}\left(\dot{\lambda}_1 * \overline{q} + \lambda_1 * \dot{\overline{q}}\right) + \tfrac{1}{2}\dot{C}_\omega \\ &\qquad = -\tfrac{1}{8}\left(\overline{\omega} * [\lambda_1 * \overline{q}] + 2C_q * \overline{q} - [\lambda_1 * \overline{q}] * \overline{\omega}\right) + \tfrac{1}{2}\dot{C}_\omega \\ &\qquad = +\tfrac{1}{8}\left(\overline{\omega} * [2C_\omega + 2\dot{\lambda}_2] - 2C_q * \overline{q} - [2C_\omega + 2\dot{\lambda}_2] * \overline{\omega}\right) + \tfrac{1}{2}\dot{C}_\omega \\ &\qquad = +\tfrac{1}{4}\left(\overline{\omega} * [C_\omega + \dot{\lambda}_2] - C_q * \overline{q} - [C_\omega + \dot{\lambda}_2] * \overline{\omega}\right) + \tfrac{1}{2}\dot{C}_\omega \\ &\qquad = -\tfrac{1}{2}\left(\overline{\omega} * \ddot{\omega} - \ddot{\omega} * \overline{\omega}\right) + \tfrac{1}{4}\left(\overline{\omega} * C_\omega - C_q * \overline{q} - C_\omega * \overline{\omega}\right) + \tfrac{1}{2}\dot{C}_\omega \end{aligned}$$

Theorem 2.5. *With the above definitions we have*

$$\boxed{\dddot{\omega} = \tfrac{1}{2}\left(\overline{\omega} * \ddot{\omega} - \ddot{\omega} * \overline{\omega}\right) + \tfrac{1}{4}\left(\overline{\omega} * C_\omega - C_q * \overline{q} - C_\omega * \overline{\omega}\right) + \tfrac{1}{2}\dot{C}_\omega}$$

and for $C(q,\omega) = C(q)$ *we have* $\boxed{\dddot{\omega} = -\tfrac{1}{2}\left(\overline{\omega} * \ddot{\omega} - \ddot{\omega} * \overline{\omega}\right) - \tfrac{1}{4} C_q * \overline{q}}$.

We want to derive a geometric form of the disturbance term $C(q)$ and to this end we need the following well known fact:

Lemma 2.6. [10] *For a orthogonal matrix* $A \in \mathbb{R}^{3\times 3}$ *and the respective quaternion* $q_A \in \mathbb{H}$ *and a vector* $x \in \mathbb{R}^3$ *we have for the coordinate change from inertial coordinates* x_{in} *to satellite coordinates* x_{sc} :

$$x_{sc} = A \cdot x_{in} = \overline{q}_A * x_{in} * q_A.$$

Theorem 2.7. *Let* $a_{sc}(t) = a_{sc}$ *be fixed in spacecraft frame, and* $b_{in}(t) = b_{in}$ *be fixed in inertial frame. For* $C(q) = \langle a_{in}, b_{in}\rangle - c_0 = \langle a_{sc}, b_{sc}\rangle - c_0$ *we have:*

$$\boxed{C_q * \overline{q} = -2\, b_{in} * a_{in} = 2\begin{pmatrix} \langle a_{in}\,,\, b_{in}\rangle \\ a_{in} \times b_{in} \end{pmatrix} = 2\begin{pmatrix} \langle a_{sc}\,,\, b_{sc}\rangle \\ a_{in} \times b_{in} \end{pmatrix}}\ .$$

Proof.

$$\begin{aligned} C_{q_i} &= \tfrac{\partial}{\partial q_i}[\langle a_{\rm sc}, b_{\rm sc}\rangle - c_0] = \tfrac{\partial}{\partial q_i}[a_{\rm sc}\cdot(\overline{q} * b_{\rm in} * q)] = a_{\rm sc}\cdot \tfrac{\partial}{\partial q_i}[\overline{q} * b_{\rm in} * q] \\ &= [\tfrac{\partial}{\partial q_i}\overline{q} * b_{\rm in} * q + \overline{q} * b_{\rm in} * \tfrac{\partial}{\partial q_i} q]\cdot a_{\rm sc} \\ &= [\overline{e}_i * b_{\rm in} * q]\cdot a_{\rm sc} + [\overline{q} * b_{\rm in} * e_i]\cdot a_{\rm sc} \end{aligned}$$

With the notation

$$\begin{aligned} R(q) &:= \big(\overline{e}_1 * q, \overline{e}_2 * q, \overline{e}_3 * q, \overline{e}_4 * q\big)^\top \quad \text{and} \\ L(q) &:= \big(\overline{q} * e_1, \overline{q} * e_2, \overline{q} * e_3, \overline{q} * e_4\big)^\top \end{aligned}$$

we can write $R(q_1)\cdot q_2 = q_1 * \overline{q}_2$ and $L(\overline{q_1})\cdot q_2 = q_1 * q_2$. Since the complex conjugate of a vector in 3-space is $\overline{a}_{\rm sc} = -a_{\rm sc}$ and $\overline{b}_{\rm in} = -b_{\rm in}$ we finally have

$$\begin{aligned} C_q * \overline{q} &= [\begin{pmatrix} \overline{e}_1 * b_{\rm in} * q \\ \overline{e}_2 * b_{\rm in} * q \\ \overline{e}_3 * b_{\rm in} * q \\ \overline{e}_4 * b_{\rm in} * q \end{pmatrix}\cdot a_{\rm sc} + \begin{pmatrix} \overline{q} * b_{\rm in} * e_1 \\ \overline{q} * b_{\rm in} * e_2 \\ \overline{q} * b_{\rm in} * e_3 \\ \overline{q} * b_{\rm in} * e_4 \end{pmatrix}\cdot a_{\rm sc}\ ] * \overline{q} \\ &= [R(b_{\rm in} * q)\cdot a_{\rm sc} + L(\overline{\overline{q} * b_{\rm in}})\cdot a_{\rm sc}\ ] * \overline{q} \\ &= [b_{\rm in} * q * \overline{a}_{\rm sc} + \overline{\overline{q} * b_{\rm in}} * a_{\rm sc}\ ] * \overline{q} \\ &= [b_{\rm in} * q * \overline{a}_{\rm sc} + \overline{b}_{\rm in} * q * a_{\rm sc}\ ] * \overline{q} = -2\ b_{\rm in} * q * a_{\rm sc} * \overline{q} \\ &= -2\ b_{\rm in} * a_{\rm in}. \end{aligned}$$

□

Example 2.8. A slew with the prescribed constraint $\langle a_{\rm sc}, b_{\rm sc}\rangle = 0$ or $a_{sc} \perp b_{sc}$ and ω, $a_{\rm in}$, and $b_{\rm in} \in \mathbb{R}^3$ will have the following dynamics:

$$\begin{aligned} \dddot{\omega} &= -\tfrac{1}{2}\,(\overline{\omega} * \ddot{\omega} - \ddot{\omega} * \overline{\omega}) - \tfrac{1}{4} C_q * \overline{q} = -\tfrac{1}{2}\,(\overline{\omega} * \ddot{\omega} - \ddot{\omega} * \overline{\omega}) + \tfrac{1}{2}\ b_{\rm in} * a_{\rm in} \\ &= \tfrac{1}{2}[\begin{pmatrix} \langle \ddot{\omega}\,,\,\overline{\omega}\rangle \\ \ddot{\omega} \times \overline{\omega} \end{pmatrix} - \begin{pmatrix} \langle \overline{\omega}\,,\,\ddot{\omega}\rangle \\ \overline{\omega} \times \ddot{\omega} \end{pmatrix}] + \tfrac{1}{2}\begin{pmatrix} \langle a_{\rm sc}\,,\,b_{\rm sc}\rangle \\ a_{\rm in} \times b_{\rm in} \end{pmatrix} = \omega \times \ddot{\omega} + \tfrac{1}{2}\begin{pmatrix} \langle a_{\rm sc}\,,\,b_{\rm sc}\rangle \\ a_{\rm in} \times b_{\rm in} \end{pmatrix}. \end{aligned}$$

And if $\langle a_{\rm sc}, b_{\rm sc}\rangle = 0$ for $t \geq t_0$, then the dynamic simplifies to

$$\boxed{\dddot{\omega} = \omega \times \ddot{\omega} + \tfrac{1}{2} a_{\rm in} \times b_{\rm in}}\ .$$

This is a new ω-ode $\dddot{\omega} = \omega \times \ddot{\omega} + c(t)$ which is similar to the second constant of integration $\ddot{\omega} = \omega \times \dot{\omega} + c$.

2.4. Comparison with a Euler Angle Slew

In [6] a description of a slew maneuver using an appropriate reference frame and then perform three successive rotations is given:

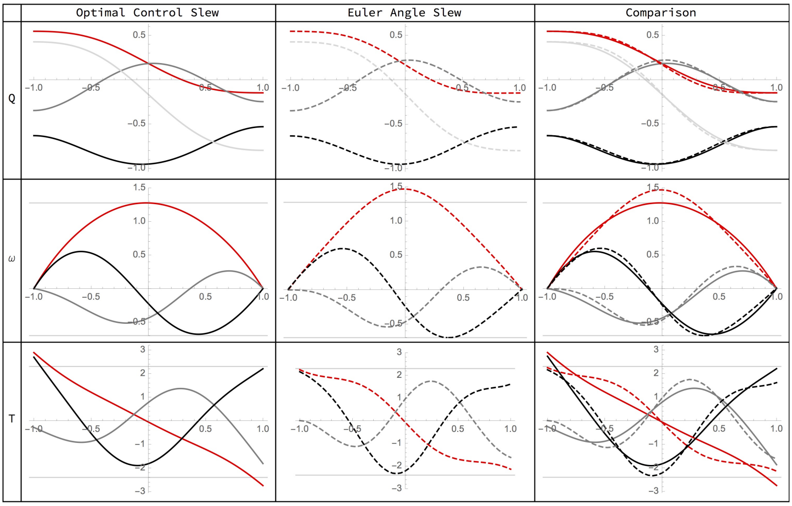
1. Rotation around the reference e_1-axis, i.e. the sun direction $s_{\rm in}$. Here the $y_{\rm sc}$-axis stays orthogonal to the sun line.
2. Rotation around the new reference e_2-axis, i.e. around the axis of the solar arrays $y_{\rm sc}$. Here the $y_{\rm sc}$-axis stays orthogonal to the sun line.

3. Rotation around the new reference e_3-axis. Here inconsistencies of the boundary values may compromise the orthogonality of the y_{sc}-axis with respect to the sun line.

The boundary values then determine the Euler angles and their derivative the slew is given by three cubic splines for these Euler angles and the respective rotation $R_3(\eta_3(t)) \cdot R_2(\eta_2(t)) \cdot R_1(\eta_1(t))$.

Example 2.9. The following graphs show a comparison of the constraint optimal slew ($s_{\text{in}} \perp y_{\text{sc}}$) and the Euler angle slew. The values that have been used are the fixed inertial sun direction $s_{\text{in}} = (-0.930975, -0.344742, -0.120159)$ and

$$q_{\text{ini}} = (0.546232, -0.34778, -0.631268, 0.426827)\,, \quad \omega_{\text{ini}} = (0, -0.000012, 0)\,,$$
$$q_{\text{fin}} = (0.148181, \quad 0.24793, \quad 0.530584, 0.796904)\,, \quad \omega_{\text{fin}} = (0, \quad 0.000012, 0)\,.$$



The constraint optimal slew has overall lower rates, but higher torques at both ends of the interval. Note for the constraint optimal slew additional constraints could still be added.

3. Dynamics of the Angular Velocity

In this section we perform further investigations of dynamics of angular velocity. Two types of solution families are described. The relation of these solutions to the two following constants of integration is described.

3.1. Analytic Solutions

Theorem 3.1. *Let the following initial value problem (IVP)*

$$\boxed{\dddot{\omega}(t) = \omega(t) \times \ddot{\omega}(t)}\,, \quad \omega(t_0) = \omega_1, \quad \dot{\omega}(t_0) = \omega_2, \quad \ddot{\omega}(t_0) = \omega_3$$

with $\omega(t) \in \mathbb{R}^3$ and $t \in [t_0, t_1] \subset \mathbb{R}$ be given. Then there exist the following two constants of integration:

1. $\forall\, t \in [t_0, t_1]\colon\ \ \|\ddot{\omega}(t)\| = \|\ddot{\omega}(t_0)\|$ *and*
2. $\forall\, t \in [t_0, t_1]\colon\ \ \ddot{\omega}(t) - \omega(t) \times \dot{\omega}(t) = \ddot{\omega}(t_0) - \omega(t_0) \times \dot{\omega}(t_0)$

Proof. 1. $\frac{d}{dt}\|\ddot{\omega}(t)\|^2 = 2\ddot{\omega}(t) \cdot \dddot{\omega}(t) = 2\ddot{\omega}(t) \cdot (\omega(t) \times \ddot{\omega}(t)) = 0.$

2. $\frac{d}{dt}[\ddot{\omega}(t) - \omega(t) \times \dot{\omega}(t)] = \dddot{\omega}(t) - \underbrace{\dot{\omega}(t) \times \dot{\omega}(t)}_{=0} - \omega(t) \times \ddot{\omega}(t) = 0.$ □

Theorem 3.2 (Quadratic Solutions). *For the IVP* $\dddot{\omega}(t) = \omega(t) \times \ddot{\omega}(t)$ *with* $\omega(t_0) = \omega_1$, $\dot{\omega}(t_0) = \omega_2$, $\ddot{\omega}(t_0) = \omega_3$ *and* $\omega(t) \in \mathbb{R}^3$, $t \in [t_0, t_1] \subset \mathbb{R}$ *we have:*

1. *If the initial values* ω_1, ω_2 *and* ω_3 *are colinear, i.e.* $c_1 \cdot \omega_1 = c_2 \cdot \omega_2 = c_3 \cdot \omega_3$ *for* c_1, c_2 *and* $c_3 \in \mathbb{R}$, *then* $\dddot{\omega}(t) \equiv 0$ *and the* $\omega_i(t)$ *stay for* $t \in [t_0, t_1] \subset \mathbb{R}$ *colinear. Further the solution of the ordinary differential equation in this case is given by*
$$\omega(t) := \omega_1 + \omega_2(t - t_0) + \frac{\omega_3}{2}(t - t_0)^2 \ .$$
2. *If two components of a solution of the ordinary differential equation are linear dependent, so is the third component. And therefore this is a quadratic solution.*

Proof. 1. Since the differential equation is Lipschitz continuous and apparently $\omega(t)$ as given above is a solution of the IVP with $\omega(t_0) = \omega_1$, $\dot{\omega}(t_0) = \omega_2$, $\ddot{\omega}(t_0) = \omega_3$ and $\dddot{\omega}(t) \equiv 0$ the claim follows.

2. Let $d \in \mathbb{R}$ be given, without loss of generality we assume
$$\begin{pmatrix} \omega_x(t) \\ d\,\omega_x(t) \\ \omega_z(t) \end{pmatrix} \times \begin{pmatrix} \ddot{\omega}_x(t) \\ d\,\ddot{\omega}_x(t) \\ \ddot{\omega}_z(t) \end{pmatrix} = \begin{pmatrix} d\,[\omega_x(t)\ddot{\omega}_z(t) - \omega_z(t)\ddot{\omega}_x(t)] \\ -[\omega_x(t)\ddot{\omega}_z(t) - \omega_z(t)\ddot{\omega}_x(t)] \\ 0 \end{pmatrix} = \begin{pmatrix} \dddot{\omega}_x(t) \\ d\,\dddot{\omega}_x(t) \\ \dddot{\omega}_z(t) \end{pmatrix}$$
and get from $\dddot{\omega}_z(t) = 0$ and $\dddot{\omega}_x(t) = -d^2\dddot{\omega}_x(t)$ solutions of the form
$$\omega_x(t) = a_0 + a_1\,t + a_2\,t^2 \text{ and } \omega_z(t) = b_0 + b_1\,t + b_2\,t^2.$$ □

Remark 3.3. Observe that the quadratic functions for ω become cubic when the integration to a quaternion profile is considered.

Example 3.4. A non zero solution of the ω - ode with boundary values zero at $t = 1$ and $t = 2$ is given by $\omega(t) = (0, 0, 0)^\top + (1, 2, 3)^\top[t - 1] - (2, 4, 6)^\top \frac{[t-1]^2}{2}$.

Theorem 3.5 (Periodic Solutions). 1. *For* a_0, a_1, $a_2 \in \mathbb{R}$ *the differential equation* $\dddot{\omega}(t) = \omega(t) \times \ddot{\omega}(t)$ *has the following periodic solutions on* $[t_0, t_1] \subset \mathbb{R}$*:*
$$\omega_1(t) = \begin{pmatrix} a_1 \\ a_0\cos(a_1 t + a_2) \\ a_0\sin(a_1 t + a_2) \end{pmatrix}, \quad \omega_{-1}(t) = \begin{pmatrix} -a_1 \\ a_0\sin(a_1 t + a_2) \\ a_0\cos(a_1 t + a_2) \end{pmatrix},$$
$$\omega_2(t) = \begin{pmatrix} a_0\sin(a_1 t + a_2) \\ a_1 \\ a_0\cos(a_1 t + a_2) \end{pmatrix}, \quad \omega_{-2}(t) = \begin{pmatrix} a_0\cos(a_1 t + a_2) \\ -a_1 \\ a_0\sin(a_1 t + a_2) \end{pmatrix},$$
$$\omega_3(t) = \begin{pmatrix} a_0\cos(a_1 t + a_2) \\ a_0\sin(a_1 t + a_2) \\ a_1 \end{pmatrix}, \quad \omega_{-3}(t) = \begin{pmatrix} a_0\sin(a_1 t + a_2) \\ a_0\cos(a_1 t + a_2) \\ -a_1 \end{pmatrix}.$$

2. *For all these solutions the second constant of integration has the form*

$$\ddot{\omega} = \omega \times \ddot{\omega} \pm a_0^2 a_1 e_i.$$

3. *There are no other solutions of the form*

$$\omega(t) = \begin{pmatrix} \cos(f(t)) \\ g(t) \\ \sin(f(t)) \end{pmatrix}.$$

Proof. A straight forward calculation shows 1. and 2. by inspection. To show 3. note first that

$$\dddot{\omega} - \omega \times \ddot{\omega} = \begin{pmatrix} * \\ \dddot{g}(t) + \ddot{f}(t) \\ * \end{pmatrix},$$

so that $f(t) = -\dot{g}(t) + c_0 + c_1 t$. With

$$\omega(t) = \begin{pmatrix} \cos(-\dot{g}(t) + c_0 + c_1 t) \\ g(t) \\ \sin(-\dot{g}(t) + c_0 + c_1 t) \end{pmatrix}$$

and

$$\begin{aligned} f_1(t) &:= \dddot{g}(t)\,[g(t) + 3(c_1 - \ddot{g}(t))] \\ f_2(t) &:= g(t)\,(c_1 - \ddot{g}(t))^2 + (c_1 - \ddot{g}(t))^3 + \ddot{g}(t) + \ddddot{g}(t) \end{aligned}$$

we get

$$\begin{aligned} \dddot{\omega} - \omega \times \ddot{\omega} &= \begin{pmatrix} f_1(t)\cos(-\dot{g}(t) + c_0 + c_1 t) + f_2(t)\sin(-\dot{g}(t) + c_0 + c_1 t) \\ 0 \\ f_1(t)\sin(-\dot{g}(t) + c_0 + c_1 t) - f_2(t)\cos(-\dot{g}(t) + c_0 + c_1 t) \end{pmatrix} \\ &= \begin{pmatrix} f_1(t)\cos(\ldots) + f_2(t)\sin(\ldots) \\ 0 \\ f_1(t)\sin(\ldots) - f_2(t)\cos(\ldots) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

From the last equation it can be seen, that $f_2(t) = f_1(t)\frac{\sin(\ldots)}{\cos(\ldots)}$ and therefore:

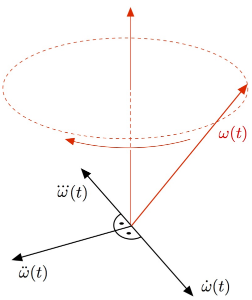
$$0 = f_1(t)\cos(\ldots)^2 + f_2(t)\sin(\ldots)\cos(\ldots) = f_1(t)[\cos(\ldots)^2 + \sin(\ldots)^2] = f_1(t)\,.$$

The differential equation $f_1(1) = \dddot{g}(t)\,[g(t) + 3(c_1 - \ddot{g}(t))] = 0$ has the two solutions $g_1(t) = b_0 + b_1 t + b_2 t^2$ and $g_2(t) = -3c_1 + b_0 e^{\frac{t}{\sqrt{3}}} + b_1 e^{-\frac{t}{\sqrt{3}}}$. The solution g_1 implies $f_2(t) = 0$ for $c_1 = 0$ and $b_2 = 0$ which results in quadratic solutions. g_2 is not a solution of the second differential equation. □

Example 3.6. With the initial values

$$\omega(t_0) = (0,1,1)^\top,\ \dot{\omega}(t_0) = (1,0,0)^\top,\ \ddot{\omega}(t_0) = -(0,0,1)^\top$$

a periodic motion is performed by the solution of the ω-ode. The vectors $\dot{\omega}(t)$, $\ddot{\omega}(t)$ and $\dddot{\omega}(t)$ lie in a plane and are rotated 90° each and $\dot{\omega}(t)$ points towards $-\dddot{\omega}(t)$.



$\omega(t)$ moves on a 45° cone and all three derivatives rotate in the common plane.

3.2. Properties of the Solutions

Now we look at a more general type of solutions, where yet no analytic representation is known to the author.

Theorem 3.7 (Step Response Solution). *If for the first constant of integration holds*

$$\ddot{\omega}(t_0) - \omega(t_0) \times \dot{\omega}(t_0) = 0,$$

then we get with $\hat{\omega} = \frac{\omega}{\|\omega\|}$*:*

1. *We have* $\dot{\omega}(t) \cdot \ddot{\omega}(t) = 0$ *and the vectors* $\omega(t)$, $\ddot{\omega}(t)$ *and* $\dddot{\omega}(t)$ *form an orthogonal basis i.e.* $\forall\ t \in [t_0, t_1] :\ \omega(t) \cdot \ddot{\omega}(t) = \dddot{\omega}(t) \cdot \ddot{\omega}(t) = \dddot{\omega}(t) \cdot \omega(t) = 0$.
2. *We have* $\|\dot{\omega}(t)\| \equiv \|\dot{\omega}(t_0)\| = c_1$, $\|\ddot{\omega}(t)\| \equiv \|\ddot{\omega}(t_0)\| = c_2$ *and*
$$\|\dddot{\omega}(t)\| = c_2 \cdot \|\omega(t)\|\ \forall\ t \in [t_0, t_1].$$
3. $\|\omega(t)\| = \sqrt{(c_1 \cdot (t \pm c))^2 + \left(\frac{c_2}{c_1}\right)^2}, \quad c = c(\|\omega(t_0)\|) \in \mathbb{R}.$
4. $|1 - (\hat{\omega} \cdot \hat{\dot{\omega}})^2| \leq O(\frac{1}{\|\omega(t)\|^2})$ *for* $t \to \infty$ *i.e.* $\measuredangle(\hat{\omega}, \hat{\dot{\omega}}) \to 0$ *for* $t \to \infty$.

Proof. First we get

1. $\ddot{\omega} = \omega \times \dot{\omega} \quad \Rightarrow \quad \omega \cdot \ddot{\omega} = \dot{\omega} \cdot \ddot{\omega} = \frac{1}{2} \cdot \frac{d}{dt}\|\dot{\omega}(t)\|^2 = 0$ and $\|\dot{\omega}(t)\| = c_1$.
2. $\dddot{\omega} = \omega \times \ddot{\omega} \quad \Rightarrow \quad \omega \cdot \dddot{\omega} = \ddot{\omega} \cdot \dddot{\omega} = \frac{1}{2} \cdot \frac{d}{dt}\|\ddot{\omega}(t)\|^2 = 0$ and $\|\ddot{\omega}(t)\| = c_2$.

For the Norms of the derivatives of ω then holds

1. $\|\omega(t)\|^2 =: f^2(t)$.
2. $\|\dot{\omega}(t)\|^2 = \|\dot{f}\hat{\omega} + f\dot{\hat{\omega}}\|^2 = \dot{f}^2\|\hat{\omega}\|^2 + 2f\dot{f} \cdot \hat{\omega} \cdot \dot{\hat{\omega}} + f^2\|\dot{\hat{\omega}}\|^2 = \dot{f}^2 + f^2\|\dot{\hat{\omega}}\|^2$.
3. $\|\ddot{\omega}(t)\|^2 = \|\omega \times \dot{\omega}\|^2 = f^4\|\hat{\omega} \times \dot{\hat{\omega}}\|^2 = f^4\|\hat{\omega}\|^2\|\dot{\hat{\omega}}\|^2 = f^4\|\dot{\hat{\omega}}\|^2$.
4. $\|\dddot{\omega}(t)\|^2 = \|\omega \times \ddot{\omega}\|^2 = \|\omega\|^2\|\ddot{\omega}\|^2 = f^6 \cdot \|\dot{\hat{\omega}}\|^2$, da $\omega \cdot \ddot{\omega} = 0$.

For the unit vector $\hat{\omega}(t)$ we get further

$$\hat{\omega}(t) = \begin{pmatrix} \sin(\vartheta(t))\cos(\varphi(t)) \\ \sin(\vartheta(t))\sin(\varphi(t)) \\ \cos(\vartheta(t)) \end{pmatrix},$$

$$\partial_\vartheta \hat{\omega}(t) = \begin{pmatrix} \cos(\vartheta(t))\cos(\varphi(t)) \\ \cos(\vartheta(t))\sin(\varphi(t)) \\ -\sin(\vartheta(t)) \end{pmatrix},$$

$$\partial_\varphi \hat{\omega}(t) = \begin{pmatrix} -\sin(\vartheta(t))\sin(\varphi(t)) \\ \sin(\vartheta(t))\cos(\varphi(t)) \\ 0 \end{pmatrix}$$

with

$$\hat{\omega}(t) \perp \partial_\vartheta \hat{\omega}(t) \perp \partial_\varphi \hat{\omega}(t) \perp \hat{\omega}(t)$$

and

$$\|\partial_\vartheta \hat{\omega}(t)\| = 1,\ \|\partial_\varphi \hat{\omega}(t)\| = \sin(\vartheta(t))^2.$$

And finally

$$\begin{aligned} \Phi(t) &:= \|\dot{\hat{\omega}}\|^2 = \|\partial_\varphi \hat{\omega}\cdot\dot{\varphi} + \partial_\vartheta\hat{\omega}\cdot\dot{\vartheta}\|^2 \\ &= \|\partial_\varphi\hat{\omega}\|^2\cdot\dot{\varphi}^2 + \underbrace{\|\partial_\vartheta\hat{\omega}\|^2}_{=1}\cdot\dot{\vartheta}^2 + \underbrace{\partial_\varphi\hat{\omega}\cdot\partial_\vartheta\hat{\omega}}_{=0}\cdot\dot{\varphi}\,\dot{\vartheta} \\ &= \sin(\vartheta(t))^2\cdot\dot{\varphi}^2 + \dot{\vartheta}^2\,. \end{aligned}$$

From the equations

$$\|\dot{\omega}(t)\|^2 = \dot{f}^2 + f^2\|\dot{\hat{\omega}}\|^2 = c_1^2$$

and

$$\|\ddot{\omega}(t)\|^2 = f^4\cdot\|\dot{\hat{\omega}}\|^2 = c_2^2$$

we construct the differential equation

$$\dot{f}(t)^2 + \frac{c_2^2}{f(t)^2} = c_1^2$$

and using standard techniques we get

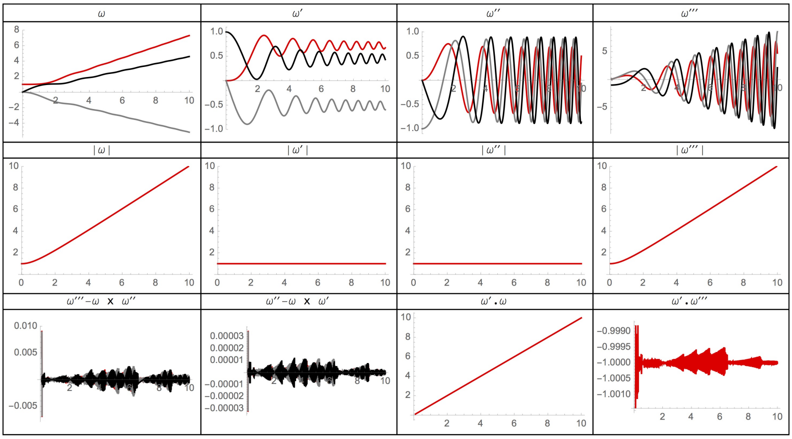
$$f(t) = \sqrt{c_1^2(t-t_0)^2 + \left(\tfrac{c_2}{c_1}\right)^2}.$$

Finally

$$\begin{aligned} \|\hat{\omega}\times\dot{\hat{\omega}}\|^2 &= \|\omega\times\dot{\omega}\|^2\frac{1}{\|\omega\|^2}\frac{1}{\|\dot{\omega}\|^2} = \frac{1}{\|\omega\|^2}\frac{\|\ddot{\omega}\|^2}{\|\dot{\omega}\|^2} = \frac{1}{\|\omega\|^2}\cdot\frac{c_2^2}{c_1^2} \\ &= \left[\|\omega\|^2\|\dot{\omega}\|^2 - (\omega\cdot\dot{\omega})^2\right]\frac{1}{\|\omega\|^2}\frac{1}{\|\dot{\omega}\|^2} = 1-(\hat{\omega}\cdot\hat{\dot{\omega}})^2 \\ &\Rightarrow\ 1-(\hat{\omega}\cdot\hat{\dot{\omega}})^2 = \frac{1}{\|\omega\|^2}\cdot\frac{c_2^2}{c_1^2} = \sin^2[\measuredangle(\hat{\omega},\hat{\dot{\omega}})] \end{aligned}$$

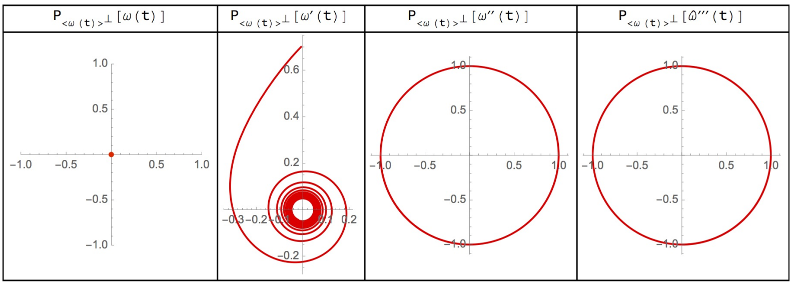
□

Example 3.8 (Step Response Solution). In this example the statements of the above theorem are confirmed. And it is clearly recognizable, that the norms of ω and its derivatives exhibit the expected and tranquil behavior.



$$t_0 = 0\ ,\ \ t_1 = 10\ ,\ \ \omega(t_0) = \begin{pmatrix}1\\0\\0\end{pmatrix}\ ,\ \ \dot\omega(t_0) = \begin{pmatrix}0\\0\\1\end{pmatrix}\ ,\ \ \ddot\omega(t_0) = -\begin{pmatrix}0\\1\\0\end{pmatrix}$$

Example 3.9 (Step Response Solution). A better representation of this type of solutions can be given by projecting all vectors to $\omega(t)^{\perp}$. But one has to beware that this is a moving frame and all calculations in this frame need the respective corrections.



$$t_0 = 0\ ,\ \ t_1 = 10\ ,\ \ \omega(t_0) = \begin{pmatrix}0\\1\\0\end{pmatrix}\ ,\ \ \dot\omega(t_0) = \begin{pmatrix}0\\1\\1\end{pmatrix}\ ,\ \ \ddot\omega(t_0) = -\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Remark 3.10. The linearization of the ordinary differential equation has the characteristic polynomial

$$\boxed{\chi(x) = x^3\ (x^3 + \|\omega\, x - \ddot\omega\|^2)^2}$$

and thus the eigenvalues are $(0,0,0,\pm\sqrt{\xi_1},\pm\sqrt{\xi_2},\pm\sqrt{\xi_3})$, where ξ_1,ξ_2 and ξ_3 are the solutions of the equation

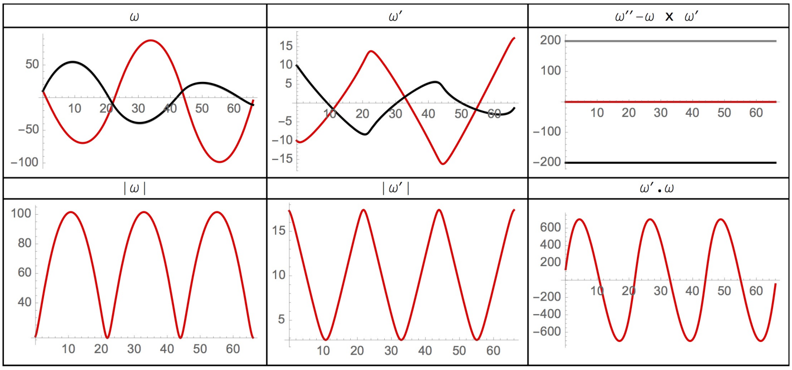
$$x^3 + \|\omega\, x - \ddot\omega\|^2 = x^3 + \|\omega\|^2\ x^2 - 2\,\omega\cdot\ddot\omega\ x + \|\ddot\omega\|^2 = 0$$

and are therefore determined by the terms $\|\omega\|$, $\|\ddot\omega\|$ and $\omega\cdot\ddot\omega$. Since

$$\frac{\|\dddot\omega\|^2}{\|\omega\|^2} = \|\ddot\omega\|^2 - \frac{\frac{1}{2}\frac{d}{dt}\|\dot\omega\|^2 - \frac{1}{2}\frac{d}{dt}\|\dot\omega(t_0)\|^2}{\|\omega\|^2}$$

holds, we have for $\frac{d}{dt}\|\dot\omega\| \approx 0$ that $\|\omega(t)\|\|\ddot\omega(t)\| \approx \|\dddot\omega(t)\|$, if $\|\omega(t)\| \to \infty$.

Example 3.11 (Quasi Periodic Solution). A particular type of solution is given when the second constant of integration is e.g. set to $(\delta\,, +\Delta\,, -\Delta)$, where $0 \leq \delta \ll \Delta$. Then the second and third component are close to each other (appearing almost identical) and stay this way over time, as can be seen in the figure. Since by Theorem 3.2 all three components must be pairwise different over time, these solutions appear therefore somewhat paradox. At the same time the values of $|\omega|$, $|\dot{\omega}|$ and $\dot{\omega} \cdot \omega$ exhibit a periodic behavior.



$$t_0 = 0\ ,\ t_1 = 66\ ,\ \omega(t_0) = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}\ ,\ \dot{\omega}(t_0) = \begin{pmatrix} -10 \\ 10 \\ 10 \end{pmatrix}\ ,\ \ddot{\omega}(t_0) = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

4. Summary

There are still several open questions. It has not yet been investigated, if it is possible and practical to build the gyroscopic term into the slew by $C(q, \omega)$. Unbalanced boundary conditions may prevent a corresponding solution. This is a very important topic for the practical usability of the whole approach. The details of the numerical methods especially the direct method using [1] have not been described. All this are topics for future work.

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