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Modified Kantorovich-Stancu operators (II)

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Abstract. In this paper, we introduce a new kind of Bernstein-Kantorovich-Stancu operators. These operators generalize the operators introduced in the paper [2] by V. Gupta, G. Tachev and A.M. Acu.

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1. Introduction

For $f \in C([0,1])$, the Bernstein operator of degree n is defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ x \in [0,1]$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ k = 0, 1, \dots, n$$

and $p_{n,k}(x) = 0$ if $k < 0$ or $k > n$.
In [3], H. Khosravian-Arab, M. Delghan and M.R. Eslahchi, starting from well-known equalities

$$p_{n,k}(x) = (1-x)p_{n-1,k}(x) + x p_{n-1,k-1}(x)$$

and

$$p_{n,k}(x) = (1-x)^2 p_{n-2,k}(x) + 2x(1-x)p_{n-2,k-1}(x) + x^2 p_{n-2,k-2}(x),\ 0 < k < n$$

have introduced modified Bernstein operators:

(i) $B_n^{M,1}$ defined by

$$B_n^{M,1}(f;x) = \sum_{k=0}^{n} p_{n,k}^{M,1}(x) f\left(\frac{k}{n}\right),\ x \in [0,1] \tag{1.1}$$

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where

$$p_{n,k}^{M,1} = a(x,n)p_{n-1,k}(x) + a(1-x,n)p_{n-1,k-1}(x)$$

and

$$a(x,n) = a_1(n)x + a_0(n),\ n = 0,1,\dots$$

(ii) $B_n^{M,2}$ defined by

$$B_n^{M,2}(f;x) = \sum_{k=0}^{n} p_{n,k}^{M,2}(x) f\left(\frac{k}{n}\right),\ x \in [0,1] \tag{1.2}$$

where

$$p_{n,k}^{M,2}(x) = b(x,n)p_{n-2,k}(x) + d(x,n)p_{n-2,k-1}(x) + b(1-x,n)p_{n-2,k-2}(x)$$

and

$$b(x,n) = b_2(n)x^2 + b_1(n)x + b_0(n),$$
$$d(x,n) = d_0(n)x(1-x),\ n = 0,1,\dots$$

$a_0(n)$, $a_1(n)$, $b_0(n)$, $b_1(n)$, $b_2(n)$ and $d_0(n)$ are the unknown sequences which are determined in appropriate way for each forms.

V. Gupta, G. Tachev and A.M. Acu ([2]) have considered the operators:

$$K_n^{M,1}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{M,1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s)ds \tag{1.3}$$

and

$$K_n^{M,2}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{M,1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s)ds. \tag{1.4}$$

Here, they have discussed a uniform convergence estimate for these modified operators. In 1968, D.D. Stancu ([5]) has introduced the linear positive operators

$$P_n^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$$

defined by

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right)$$

where α, β are two fixed real numbers such that $0 \le \alpha \le \beta$.

In 2004, D. Bărbosu ([1]) has introduced Kantorovich-Stancu operators

$$K_n^{(\alpha,\beta)} : L_1([0,1]) \to C([0,1])$$

defined by

$$K_n^{(\alpha,\beta)}(f;x) = (n+\beta+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(s)ds.$$

Regarding the previously modified operators, we note the following:

(a) The operators $B_n^{M,1}$ and $B_n^{M,2}$ are linear combinations of the operators $P_{n-1}^{(0,1)}$ and $P_{n-1}^{(1,1)}$, respectively of the operators $P_{n-2}^{(0,2)}$, $P_{n-2}^{(1,2)}$ and $P_{n-2}^{(2,2)}$, more precisely

$$B_n^{M,1}(f;x) = a(x,n)P_{n-1}^{(0,1)}(f;x) + a(1-x,n)P_{n-1}^{(1,1)}(f;x)$$

and

$$B_n^{M,2}(f;x) = b(x,n)P_{n-2}^{(0,2)}(f;x) + d(x,n)P_{n-2}^{(1,2)}(f;x) + b(1-x,n)P_{n-2}^{(2,2)}(f;x);$$

(b) The operators $K_n^{M,1}$ and $K_n^{M,2}$ are linear combinations of the operators $K_{n-1}^{(0,1)}$ and $K_{n-1}^{(1,1)}$, respectively of the operators $K_{n-2}^{(0,2)}$, $K_{n-2}^{(1,2)}$ and $K_{n-2}^{(2,2)}$, therefore

$$K_n^{M,1}(f;x) = a(x,n)K_{n-1}^{(0,1)}(f;x) + a(1-x,n)K_{n-1}^{(1,1)}(f;x)$$

and

$$K_n^{M,2}(f;x) = b(x,n)K_{n-2}^{(0,2)}(f;x) + d(x,n)K_{n-2}^{(1,2)}(f;x) + b(1-x,n)K_{n-2}^{(2,2)}(f;x).$$

From the above reasons, in this paper we introduce, for any $\alpha, \beta \in \mathbb{R}$, $0 \leq \alpha \leq \beta$ the operators

$$\overline{K}_n^{(\alpha,\beta)}(f;x) = (n+\beta+1)\sum_{k=0}^{n} p_{m,k}^{M,1}(x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(s)ds \tag{1.5}$$

and

$$\overline{\overline{K}}_n^{(\alpha,\beta)}(f;x) = (n+\beta+1)\sum_{k=0}^{n} p_{m,k}^{M,2}(x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(s)ds. \tag{1.6}$$

We mention that the Kantorovich-Stancu type operators $\overline{K}_n^{(\alpha,\beta)}$ was studied in a recent paper submitted for publication ([4]).

2. Auxiliary results

Lemma 2.1. *The central moments of $K_n^{(\alpha,\beta)}$ are given by:*

$$K_n^{(\alpha,\beta)}((t-x)^s;x) = \frac{1}{s+1}\sum_{k=0}^{n} p_{n,k}(x)\left\{\sum_{i=0}^{s+1}\binom{s+1}{i}\left(\frac{k+1}{n+1}-x\right)^i \frac{1}{(n+\beta+1)^{s-i}}\right.$$
$$\left.\times\left[\sum_{j=1}^{s+1-i}(-1)^{j+1}\binom{s+1-i}{j}\left(\alpha-\beta\frac{k+1}{n+1}\right)^{s+1-i-j}\right]\right\}. \tag{2.1}$$

Proof.

$$K_n^{(\alpha,\beta)}((t-x)^s;x) = (n+\beta+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} (t-x)^s dt$$

$$= (n+\beta+1)\frac{1}{s+1}\sum_{k=0}^{n} p_{n,k}(x)\left[\left(\frac{k+\alpha+1}{n+\beta+1}-x\right)^{s+1} - \left(\frac{k+\alpha}{n+\beta+1}-x\right)^{s+1}\right] \tag{2.2}$$

Because

$$\frac{k+\alpha+1}{n+\beta+1}-x = \frac{k+1}{n+1}-x+\frac{k+\alpha+1}{n+\beta+1}-\frac{k+1}{n+1}$$
$$= \frac{k+1}{n+1}-x+\left(\alpha-\beta\frac{k+1}{n+1}\right)\frac{1}{n+\beta+1},$$

we have

$$\left(\frac{k+\alpha+1}{n+\beta+1}-x\right)^{s+1}$$
$$=\sum_{i=0}^{s+1}\binom{s+1}{i}\left(\frac{k+1}{n+1}-x\right)^{i}\left(\alpha-\beta\frac{k+1}{n+1}\right)^{s+1-i}\frac{1}{(n+\beta+1)^{s+1-i}}$$

and

$$\left(\frac{k+\alpha}{n+\beta+1}-x\right)^{s+1}$$
$$=\sum_{i=0}^{s+1}\binom{s+1}{i}\left(\frac{k+1}{n+1}-x\right)^{i}\left(\alpha-1-\beta\frac{k+1}{n+1}\right)^{s+1-i}\frac{1}{(n+\beta+1)^{s+1-i}}.$$

So, (2.2) becomes

$$K_n^{(\alpha,\beta)}((t-x)^s;x)=\frac{1}{s+1}\sum_{k=0}^{n}p_{n,k}(x)\left\{\sum_{i=0}^{s+1}\binom{s+1}{i}\left(\frac{k+1}{n+1}-x\right)^{i}\frac{1}{(n+\beta+1)^{s-i}}\right.$$
$$\left.\times\left[\left(\alpha-\beta\frac{k+1}{n+1}\right)^{s+1-i}-\left(\alpha-1-\beta\frac{k+1}{n+1}\right)^{s+1-i}\right]\right\}$$
$$=\frac{1}{s+1}\sum_{k=0}^{n}p_{n,k}(x)\left\{\sum_{i=0}^{s+1}\binom{s+1}{i}\left(\frac{k+1}{n+1}-x\right)^{i}\frac{1}{(n+\beta+1)^{s-i}}\right.$$
$$\left.\times\left[\sum_{j=1}^{s+1-i}(-1)^{j+1}\binom{s+1-i}{j}\left(\alpha-\beta\frac{k+1}{n+1}\right)^{s+1-i-j}\right]\right\}.$$

Remark 2.2. For $s=\overline{1,6}$ we have

$$K_n^{(\alpha,\beta)}(t-x;x)=-\frac{\beta+1}{n+\beta+1}x+\frac{2\alpha+1}{2(n+\beta+1)},$$
$$K_n^{(\alpha,\beta)}((t-x)^2;x)=\frac{n-(2\alpha+1)(\beta+1)}{(n+\beta+1)^2}x(1-x)$$
$$+\frac{(\beta-2\alpha)(\beta+1)}{(n+\beta+1)^2}x^2+\frac{3\alpha^2+3\alpha+1}{3(n+\beta+1)^2},$$
$$K_n^{(\alpha,\beta)}((t-x)^3;x)=-\frac{(3\beta+5)n-(\beta+1)^3}{(n+\beta+1)^3}x^2(1-x)$$
$$+\frac{(12\alpha+10)n-6(2\alpha+1)(\beta+1)^2+4(\beta+1)^3}{4(n+\beta+1)^3}x(1-x)$$
$$-\frac{4(3\alpha^2+3\alpha+1)(\beta+1)-6(2\alpha+1)(\beta+1)^2+4(\beta+1)^3}{4(n+\beta+1)^3}x$$
$$+\frac{4\alpha^3+6\alpha^2+4\alpha+1}{4(n+\beta+1)^3},$$
$$K_n^{(\alpha,\beta)}((t-x)^4;x)=\frac{3n^2-2(3+4(\beta+1)+3(\beta+1)^2)n}{(n+\beta+1)^4}(x(1-x))^2$$

$$-\frac{[4(2\alpha+1)+2(6\alpha+1)(\beta+1)-6(\beta+1)^2]n-2(2\alpha+1)(\beta+1)^3}{(n+\beta+1)^4}x^2(1-x)$$

$$+\frac{(6\alpha^2+10\alpha+5)n+2(2\alpha+1)(\beta+1)^3-2(3\alpha^2+3\alpha+1)(\beta+1)^2}{(n+\beta+1)^4}x(1-x)$$

$$-\frac{2(2\alpha+1)(\beta+1)^3-2(3\alpha^2+3\alpha+1)(\beta+1)^2+(4\alpha^3+6\alpha^2+4\alpha+1)(\beta+1)}{(n+\beta+1)^4}x$$

$$+\frac{5\alpha^4+10\alpha^3+10\alpha^2+5\alpha+1}{5(n+\beta+1)^4},$$

$$K_n^{(\alpha,\beta)}((t-x)^5;x)=\frac{[(30\beta+70)x^2(1-x)^3+(30\alpha-30\beta-35)x^2(1-x)^2]}{2(n+\beta+1)^5}n^2$$

$$+\Bigg[-\frac{30(\beta+1)^3+60(\beta+1)^2+90(\beta+1)+72}{3(n+\beta+1)^5}x^2(1-x)^3$$

$$+\frac{60(\beta+1)^3-45(2\alpha-1)(\beta+1)^2-30(4\alpha-1)(\beta+1)-9(10\alpha+1)}{3(n+\beta+1)^5}x^2(1-x)^2$$

$$-\frac{30(\beta+1)^3-15(6\alpha+1)(\beta+1)^2+15(6\alpha^2+2\alpha+1)(\beta+1)+2(30\alpha^2+30\alpha+13)}{3(n+\beta+1)^5}x^2(1-x)$$

$$+\frac{30\alpha^3+75\alpha^2+75\alpha+28}{3(n+\beta+1)^5}x(1-x)\Bigg]n+O\left(\frac{1}{n^5}\right),$$

$$K_n^{(\alpha,\beta)}((t-x)^6;x)=\frac{15x^3(1-x)^3}{(n+\beta+1)^6}n^3+\Bigg[\frac{45(\beta+1)^2x^4(1-x)^2}{(n+\beta+1)^6}$$

$$-\frac{(120x^3(1-x)^3+15(6\alpha-1)x^3(1-x)^2)(\beta+1)}{(n+\beta+1)^6}$$

$$+\frac{130x^2(1-x)^4+10(12\alpha-7)x^2(1-x)^3+5(9\alpha^2-3\alpha+2)x^2(1-x)^2}{(n+\beta+1)^6}\Bigg]n^2+O\left(\frac{1}{n^5}\right).$$

3. Main results

Here, we will extend the results from [4] for modified operators $\overline{\overline{K}}_n^{(\alpha,\beta)}$ defined by (1.6).

It is easy to see that

$$\overline{\overline{K}}_n^{(\alpha,\beta)}(f;x)=b(x;n)K_{n-2}^{(\alpha,\beta+2)}(f;x)+d(x;n)K_{n-2}^{(\alpha+1,\beta+2)}(f;x)$$
$$+b(1-x;n)K_{n-2}^{(\alpha+2,\beta+2)}(f;x). \tag{3.1}$$

Lemma 3.1. *For $i = 0, 1, 2$, the moments of $\overline{\overline{K}}_n(t^i; x)$ are given by:*

$$\overline{\overline{K}}_n^{(\alpha,\beta)}(1;x) = (2b_2(n) - d_0(n))x^2 - (2b_2(n) - d_0(n))x + b_2(n) + b_1(n) + 2b_0(n),$$

$$\overline{\overline{K}}_n^{(\alpha,\beta)}(t;x) = \frac{(2(n-2)x + 2\alpha + 3)(2b_2(n) - d_0(n))}{2(n+\beta+1)} x(x-1) + \frac{(n-4)(b_2(n) + b_1(n)) + 2(n-2)b_0(n)}{n+\beta+1} x + \frac{(2\alpha+5)(b_2(n) + b_1(n)) + 2(2\alpha+3)b_0(n)}{2(n+\beta+1)},$$

$$\begin{aligned}\overline{\overline{K}}_n^{(\alpha,\beta)}(t^2;x) &= [(2b_2(n) - d_0(n))x(x-1) + b_2(n) + b_1(n) + 2b_0(n)] \\ &\times \left[\frac{(n-2)(n-3)}{(n+\beta+1)^2} x^2 + \frac{2(\alpha+1)(n-2)}{(n+\beta+1)^2} x + \frac{3\alpha^2 + 3\alpha + 1}{3(n+\beta+1)^2}\right] \\ &+ \left[\frac{2(2b_2(n) - d_0(n))x^3}{(n+\beta+1)^2} - \frac{(2(2b_2(n) - d_0(n)) + 4(b_2(n) + b_1(n)))x^2}{(n+\beta+1)^2}\right. \\ &\left.+ \frac{4(b_2(n) + b_1(n) + b_0(n))}{(n+\beta+1)^2} x\right](n-2) \\ &+ \frac{2(\alpha+1)(2b_2(n) - d_0(n)) + 2b_2(n)}{(n+\beta+1)^2} x^2 \\ &- \left[\frac{2(\alpha+1)(2b_2(n) - d_0(n))}{(n+\beta+1)^2} + \frac{4(\alpha+2)b_2(n) + 2(2\alpha+3)b_1(n)}{(n+\beta+1)^2}\right] x \\ &+ \frac{2(2\alpha+3)(b_2(n) + b_1(n) + b_0(n))}{(n+\beta+1)^2}.\end{aligned}$$

We want to demonstrate the uniform convergence of the sequence $(\overline{\overline{K}}_n^{(\alpha,\beta)} f)_{n\geq 2}$. For this purpose, we will consider that

$$\overline{\overline{K}}_n^{(\alpha,\beta)}(1;x) = 1$$

$$\Leftrightarrow 2b_2(n) - d_0(n) = 0 \text{ and } b_2(n) + b_1(n) + 2b_0(n) = 1. \tag{3.2}$$

Using these, we obtain that

$$\overline{\overline{K}}_n^{(\alpha,\beta)}(t;x) = x + \frac{4b_0(n) - \beta - 5}{n+\beta+1} x + \frac{2\alpha + 5 - 4b_0(n)}{2(n+\beta+1)}$$

$$\overline{\overline{K}}_n^{(\alpha,\beta)}(t^2;x) = \frac{n^2 - (9 - 8b_0(n))n + 16 - 2b_1(n) - 20b_0(n)}{(n+\beta+1)^2} x^2 + \frac{2(\alpha + 3 - 2b_0(n))n + 2b_1(n) + 8(\alpha+3)b_0(n) - 4(2\alpha+5)}{(n+\beta+1)^2} x + \frac{3\alpha^2 + 15\alpha + 19 - 6(2\alpha+3)b_0(n)}{3(n+\beta+1)^2}.$$

Assume that $\beta = 2\alpha$, for $b_0(n) = \dfrac{\beta+5}{4}$ the above expressions become

$$\widetilde{K}_n^{(\alpha,\beta)}(t;x) = x$$

$$\widetilde{K}_n^{(\alpha,\beta)}(t^2;x) = x^2 + \frac{n+2b_1(n)+(\beta+2)(\beta+5)}{(n+\beta+1)^2}x(1-x) - \frac{3\beta^2+18\beta+14}{12(n+\beta+1)^2}.$$

Taking $b_1(n) = -\dfrac{n+(\beta+2)(\beta+5)}{2}$ we have that

$$\widetilde{K}_n^{(\alpha,\beta)}(t^2;x) = x^2 - \frac{3\beta^2+18\beta+14}{12(n+\beta+1)^2}.$$

By (3.2) we obtain

$$b_2(n) = \frac{n+2+(\beta+1)(\beta+5)}{2}$$

and

$$d_0(n) = n+2+(\beta+1)(\beta+5).$$

In this situation, we can give other expressions for the first six central moments.

Lemma 3.2.

$$\widetilde{K}_n^{(\alpha,\beta)}(t-x;x) = 0,$$
$$\widetilde{K}_n^{(\alpha,\beta)}((t-x)^2;x) = -\frac{3\beta^2+18\beta+14}{12(n+\beta+1)^2},$$
$$\widetilde{K}_n^{(\alpha,\beta)}((t-x)^3;x) = -\frac{1-2x}{4(n+\beta+1)^3}[2(3\beta+7)nx(1-x)$$
$$+2(\beta^3+9\beta^2+21\beta+13)x(1-x)+\beta^3+9\beta^2+23\beta+15],$$
$$\widetilde{K}_n^{(\alpha,\beta)}((t-x)^4;x) = \frac{-3n^2}{(n+\beta+1)^4}x^2(1-x)^2 + O\left(\frac{1}{n^3}\right),$$
$$\widetilde{K}_n^{(\alpha,\beta)}((t-x)^5;x) = \frac{15(\beta+3)n^2}{(n+\beta+1)^5}x^2(1-x)^2(2x-1) + O\left(\frac{1}{n^4}\right),$$
$$\widetilde{K}_n^{(\alpha,\beta)}((t-x)^6;x) = \frac{-30n^3}{(n+\beta+1)^6}x^3(1-x)^3 + O\left(\frac{1}{n^4}\right).$$

Using this, we will prove the following result:

Theorem 3.3. *For $x\in[0,1]$, if $f\in C^{(6)}([0,1])$, we have*

$$\widetilde{K}_n^{(\alpha,\beta)}(f;x) - f(x) = O\left(\frac{1}{n^2}\right), \tag{3.3}$$

for sufficient large n.

Proof. Applying the Taylor's formula to the operators $\widetilde{K}_n^{(\alpha,\beta)}$ we have

$$\widetilde{K}_n^{(\alpha,\beta)}(f;x) = f(x) + \sum_{k=1}^{6}\frac{1}{k!}\widetilde{K}_n^{(\alpha,\beta)}((t-x)^k;x)f^{(k)}(x)$$
$$+\widetilde{K}_n^{(\alpha,\beta)}(\rho(t;x)(t-x)^6;x),$$

where ρ is a continuous function.

It is sufficient to prove that

$$|\widetilde{K}_n^{(\alpha,\beta)}(\rho(t;x)(t-x)^6;x)| = O\left(\frac{1}{n^2}\right). \tag{3.4}$$

We know that operators are not positive, so we rewrite them like this

$$\widetilde{K}_n^{(\alpha,\beta)}(f;x) = \widetilde{K}_{n,1}^{(\alpha,\beta)}(f;x) - \widetilde{K}_{n,2}^{(\alpha,\beta)}(f;x)$$

where

$$\begin{aligned}\widetilde{K}_{n,1}^{(\alpha,\beta)}(f;x) = (b_2(n)x^2 + b_0(n)) \cdot K_{n-2}^{(\alpha,\beta+2)}(f;x) + d_0(n)x \cdot K_{n-2}^{(\alpha+1,\beta+2)}(f;x)\\ + b_2(n)x^2 \cdot K_{n-2}^{(\alpha+2,\beta+2)}(f;x)\end{aligned}$$

and

$$\begin{aligned}\widetilde{K}_{n,2}^{(\alpha,\beta)}(f;x) = -b_1(n)x \cdot K_{n-2}^{(\alpha,\beta+2)}(f;x) + d_0(n)x^2 \cdot K_{n-2}^{(\alpha+1,\beta+2)}(f;x)\\ +((2b_2(n) + b_1(n))x - (b_2(n) + b_1(n) + b_0(n))) \cdot K_{n-2}^{(\alpha+2,\beta+2)}(f;x).\end{aligned}$$

We note that $\widetilde{K}_{n,1}^{(\alpha,\beta)}$ and $\widetilde{K}_{n,2}^{(\alpha,\beta)}$ are linear and positive operators.

$$\begin{aligned}|\widetilde{K}_n^{(\alpha,\beta)}(\rho(t;x)(t-x)^6;x)| \le |\widetilde{K}_{n,1}^{(\alpha,\beta)}(\rho(t;x)(t-x)^6;x)|\\ + |\widetilde{K}_{n,2}^{(\alpha,\beta)}(\rho(t;x)(t-x)^6;x)|.\end{aligned} \tag{3.5}$$

Computing $\widetilde{K}_{n,i}^{(\alpha,\beta)}((t-x)^6;x)$, $i=1,2$, we obtain the following expressions

$$\widetilde{K}_{n,i}^{(\alpha,\beta)}((t-x)^6;x) = \frac{15x^4(1-x)^3(1+x)n^4}{(n+\beta+1)^6} + A_i(\alpha,\beta,x)\frac{n^3}{(n+\beta+1)^6}, \ i=1,2$$

where

$$\begin{aligned}A_1(\alpha,\beta,x) = 15(\beta^2+6\beta+7)x^4(1-x)^3(1+x)\\ -[120x^4(1-x)^3(1+x) + 15(6\alpha+5)x^4(1-x)^2(1+x)](\beta+3)\\ +45x^5(1-x)^2(1+x)(\beta+3)^2 + 130x^3(1-x)^4(1+x)\\ +10(12\alpha+5)x^3(1-x)^3(1+x) + 5(9\alpha^2+15\alpha+8)x^3(1-x)^2(1+x)\\ +45x^4(1-x)^2 + \frac{15(\beta+5)x^3(1-x)^3}{4}\end{aligned}$$

and

$$\begin{aligned}A_2(\alpha,\beta,x) = 15(\beta^2+6\beta+7)x^4(1-x)^3(1+x)\\ -[120x^4(1-x)^3(1+x) + 15(6\alpha+5)x^4(1-x)^2(1+x)](\beta+3)\\ +45x^5(1-x)^2(1+x)(\beta+3)^2 + 130x^3(1-x)^4(1+x)\\ +10(12\alpha+5)x^3(1-x)^3(1+x) + 5(9\alpha^2+15\alpha+8)x^3(1-x)^2(1+x)\\ +45x^3(1-x)^2 - \frac{15(\beta+1)x^3(1-x)^3}{4}.\end{aligned}$$

Because ρ is a continuous function, there exists an $M>0$ such that $|\rho(t;x)| < M$, $\forall\, x,t \in [0,1]$. Using the above results for $\widetilde{K}_{n,i}^{(\alpha,\beta)}((t-x)^6;x)$, we obtain

$$\begin{aligned}|\widetilde{K}_{n,i}^{(\alpha,\beta)}(\rho(t;x)(t-x)^6;x)| \le M\left|\frac{15x^4(1-x)^3(1+x)n^4}{(n+\beta+1)^6} + O\left(\frac{1}{n^3}\right)\right|\\ = O\left(\frac{1}{n^2}\right), \ i=1,2.\end{aligned}$$

So, (3.4) is proved.
Combining this with Lemma 3.2, we complete the proof of theorem.

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