Stud. Univ. Babeş-Bolyai Math. 65(2020), No. 2, 199–210
DOI: 10.24193/subbmath.2020.2.03

Application of Ruscheweyh q-differential operator to analytic functions of reciprocal order

Shahid Mahmood, Saima Mustafa and Imran Khan

Abstract. The core object of this paper is to define and study new class of analytic function using Ruscheweyh q-differential operator. We also investigate a number of useful properties such as inclusion relation, coefficient estimates, subordination result,for this newly subclass of analytic functions.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic functions, Subordination, Functions with positive real part, Ruscheweyh q-differential operator, reciprocal order.

1. Introduction

Quantum calculus (q-calculus) is simply the study of classical calculus without the notion of limits. The study of q-calculus attracted the researcher due to its applications in various branches of mathematics and physics, see detail [8]. Jackson [10, 12] was the first to give some application of q-calculus and introduced the q-analogue of derivative and integral. Later on Aral and Gupta [5, 6, 7] defined the q-Baskakov Durrmeyer operator by using q-beta function while the author's in [2, 3, 4] discussed the q-generalization of complex operators known as q-Picard and q-Gauss-Weierstrass singular integral operators. Recently, Kanas and Răducanu [13] defined q-analogue of Ruscheweyh differential operator using the concepts of convolution and then studied some of its properties. The application of this differential operator was further studied by Mohammed and Darus [1] and Mahmood and Sokół [14]. The aim of the current paper is to define a new class of analytic functions of reciprocal order involving q-differetial operator.

Let $\mathcal{A}$ be the class of functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{M}(\alpha)$ denote a subclass of $\mathcal{A}$ consisting of functions which satisfy the inequality

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$. And let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality:

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} < \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$. These classes were studied by Owa et al. [16, 18]. Shams et al. [20] have introduced the k-uniformly starlike $\mathcal{SD}\,(k, \alpha)$ and k-uniformly convex $\mathcal{CD}\,(k, \alpha)$ of order α, for some $k\,(k \geq 0)$ and $\alpha\,(0 \leq \alpha < 1)$. Using these ideas in above defined classes, Junichi et al. [17] introduced the following classes.

Definition 1.1. Let $f \in \mathcal{A}$. Then f is said to be in class $\mathcal{MD}\,(k, \alpha)$ if it satisfies

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$ and $k\,(k \leq 0)$.

Definition 1.2. An analytic function f of the form (1.1) belongs to the class $\mathcal{ND}\,(k, \alpha)$, if and only if

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} < k\left|\frac{(zf'(z))'}{f'(z)} - 1\right| + \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$ and $k\,(k \leq 0)$.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence holds, see [11, 15].

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For two analytic functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{U}),$$

For $t \in \mathbb{R}$ and $q > 0$, $q \neq 1$, the number $[t, q]$ is defined in [14] as

$$[t, q] = \frac{1 - q^t}{1 - q}, \quad [0, q] = 0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n, q]! = [1, q]\,[2, q]\,[3, q] \cdots [n, q]\,, \quad ([0, q]! = 1)\,.$$

We have $\lim\limits_{q \to 1} [n, q] = n$. Throughout in this paper we will assume q to be fixed number between 0 and 1.

The q-derivative operator or q-difference operator for $f \in \mathcal{A}$ is defined as

$$\partial_q f(z) = \frac{f\,(qz) - f(z)}{z\,(q - 1)}, \quad z \in \mathbb{U}.$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $z \in \mathbb{U}$

$$\partial_q z^n = [n, q]\, z^{n-1}, \quad \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q]\, a_n z^{n-1}.$$

The q-generalized Pochhammer symbol for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ is defined as

$$[t, q]_n = [t, q]\, [t + 1, q]\, [t + 2, q] \cdots [t + n - 1, q]\,,$$

and for $t > 0$, let q-gamma function is defined as

$$\Gamma_q\,(t + 1) = [t, q]\, \Gamma_q\,(t) \;\text{ and } \Gamma_q\,(1) = 1.$$

Definition 1.3. [14] For a function $f(z) \in \mathcal{A}$, the Ruscheweyh q-differential operator is defined as

$$\mathfrak{D}_q^\mu f(z) = \phi\,(q, \mu + 1; z) * f(z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} a_n z^n, \qquad (z \in \mathbb{U} \text{ and } \mu > -1)\,, \quad (1.2)$$

where

$$\phi\,(q, \mu + 1; z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} z^n, \quad (1.3)$$

and

$$\Phi_{n-1} = \frac{\Gamma_q\,(\mu + n)}{[n - 1, q]!\Gamma_q\,(\mu + 1)} = \frac{[\mu + 1, q]_{n-1}}{[n - 1, q]!}. \quad (1.4)$$

From (1.2), it can be seen that

$$L_q^0 f(z) = f(z) \text{ and } L_q^1 f(z) = z\partial_q f(z),$$

and

$$L_q^m f(z) = \frac{z\partial_q^m \left(z^{m-1} f(z)\right)}{[m, q]!}, \qquad (m \in \mathbb{N})\,.$$

$$\lim_{q \to 1^-} \phi\,(q, \mu + 1; z) = \frac{z}{(1 - z)^{\mu+1}},$$

and

$$\lim_{q \to 1^-} \mathfrak{D}_q^\mu f(z) = f(z) * \frac{z}{(1 - z)^{\mu+1}}.$$

This shows that in case of $q \to 1^-$, the Ruscheweyh q-differential operator reduces to the Ruscheweyh differential operator $D^\delta\,(f(z))$ (see [19]). From (1.2) the following identity can easily be derived.

$$z\partial\mathfrak{D}_q^\mu f(z) = \left(1 + \frac{[\mu, q]}{q^\mu}\right) \mathfrak{D}_q^\mu f(z) - \frac{[\mu, q]}{q^\mu} \mathfrak{D}_q^\mu f(z). \quad (1.5)$$

If $q \to 1^-$, then

$$z\left(\mathfrak{D}_q^\mu f(z)\right)' = (1 + \mu)\, \mathfrak{D}_q^\mu f(z) - \mu\mathfrak{D}_q^\mu f(z).$$

Now using the Ruscheweyh q-differential operator, we define the following class.

Definition 1.4. Let $f \in \mathcal{A}$. Then f is in the class $\mathcal{KD}_q(k,\alpha,\gamma)$ if

$$\mathfrak{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right\}<k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right|+\alpha,$$

for some $k\,(k\leq 0)$, $\alpha\,(\alpha>1)$ and for some $\gamma\in\mathbb{C}\setminus\{0\}$.

We note that $\mathcal{LD}_2^0(1,1,\alpha)=\mathcal{M}(\alpha)$ and $\mathcal{LD}_1^0(1,1,\alpha)=\mathcal{N}(\alpha)$, the classes introduced by Owa et al. [16, 18]. When we take $\gamma=1,2$, $c=1$, and $a=1$ the class $\mathcal{KD}_q(k,\alpha,\gamma)$ reduces to the classes $\mathcal{MD}(k,\alpha)$ and $\mathcal{ND}(k,\alpha)$ (see [17]). For $1<\alpha<4/3$ the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were investigated by Uralegaddi et al. [21].

2. Preliminary results

Lemma 2.1. [9]*For a positive integer t, we have*

$$\sigma\sum_{j=1}^{t}\frac{(\sigma)_{j-1}}{(j-1)!}=\frac{(\sigma)_t}{(t-1)!}. \tag{2.1}$$

Proof. Consider

$$\begin{aligned}
&\sigma\sum_{j=1}^{t}\frac{(\sigma)_{j-1}}{(j-1)!}\\
=\;&\sigma\left(1+\frac{\sigma}{1}+\frac{(\sigma)_2}{2!}+\frac{(\sigma)_3}{3!}+\frac{(\sigma)_4}{4!}+\cdots+\frac{(\sigma)_{t-1}}{(t-1)!}\right)\\
=\;&\sigma(1+\sigma)\left(1+\frac{\sigma}{2}+\frac{\sigma(\sigma+2)}{2\times 3}+\cdots+\frac{\sigma(\sigma+2)\cdots(\sigma+t-2)}{2\times\cdots\times(t-1)}\right)\\
=\;&\sigma(1+\sigma)\frac{(\sigma+2)}{2}\left(1+\frac{\sigma}{3}+\cdots+\frac{\sigma(\sigma+3)\cdots(\sigma+t-2)}{3\times 4\times\cdots\times(t-1)}\right)\\
=\;&\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\left(1+\frac{\sigma}{4}+\cdots+\frac{\sigma(\sigma+4)\cdots(\sigma+t-2)}{4\times\cdots\times(t-1)}\right)\\
=\;&\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\left(1+\frac{\sigma}{5}+\cdots+\frac{\sigma\cdots(\sigma+t-2)}{5\times 6\times\cdots\times(t-1)}\right)\\
=\;&\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(1+\frac{\sigma}{t-1}\right)\\
=\;&\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(\frac{\sigma+(t-1)}{t-1}\right)\\
=\;&\frac{(\sigma)_t}{(t-1)!}.
\end{aligned}$$

□

3. Main results

With the help of the definition of $\mathcal{KD}_q(k,\alpha,\gamma)$, we prove the following results.

Theorem 3.1. *If $f(z)\in\mathcal{KD}_q(k,\alpha,\gamma)$, then*

$$f(z)\in\mathcal{KD}_q\left(0,\frac{\alpha-k}{1-k},\gamma\right).$$

Proof. Because $k\le 0$, we have

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right\} &< k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right|+\alpha,\\ &\le k\mathfrak{Re}\left(\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right)+\alpha-k,\end{aligned}$$

which implies that

$$(1-k)\,\mathfrak{Re}\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)<\alpha-k.$$

After simplification, we obtain

$$\mathfrak{Re}\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]<\frac{\alpha-k}{1-k},(k\le 0,\ \alpha>1\text{ and }). \tag{3.1}$$

This completes the proof. □

Theorem 3.2. *If $f(z)\in\mathcal{KD}_q(k,\alpha,\gamma)$ and if $f(z)$ has the form (1.1), then*

$$|a_n|\le\frac{(\sigma)_{n-1}}{(n-1)!\Phi_{n-1}}, \tag{3.2}$$

where

$$\sigma=\frac{2|\gamma|(\alpha-1)}{q(1-k)}. \tag{3.3}$$

Proof. Let us define a function

$$p(z)=\frac{(\alpha-k)-(1-k)\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]}{\alpha-1}. \tag{3.4}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and $\mathfrak{Re}\{p(z)\}>0$ for $z\in\mathbb{U}$. We can write

$$\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]=\frac{(\alpha-k)-(\alpha-1)p(z)}{1-k} \tag{3.5}$$

If we take $p(z)=1+\sum_{n=1}^{\infty}p_nz^n$, then (3.5) can be written as

$$z\partial_q\mathfrak{D}_q^\mu f(z)-\mathfrak{D}_q^\mu f(z)=-\frac{\gamma(\alpha-1)}{1-k}\left(\mathfrak{D}_q^\mu f(z)\right)\left(\sum_{n=1}^{\infty}p_nz^n\right).$$

this implies that

$$\left[\sum_{n=2}^{\infty}q\,[n-1]\,\Phi_{n-1}a_nz^n\right]=-\frac{\gamma(\alpha-1)}{1-k}\left(\sum_{n=1}^{\infty}\Phi_{n-1}a_nz^n\right)\left(\sum_{n=1}^{\infty}p_nz^n\right).$$

Using Cauchy product $\left(\sum\limits_{n=1}^{\infty} x_n\right) \cdot \left(\sum\limits_{n=1}^{\infty} y_n\right) = \sum\limits_{j=1}^{\infty} \sum\limits_{k=1}^{j} x_k y_{k-j}$, we obtain

$$q\,[n-1]\,\Phi_{n-1} a_n z^n = -\frac{\gamma(\alpha-1)}{1-k} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j} \right) z^n.$$

Comparing the coefficients of nth term on both sides, we obtain

$$a_n = \frac{-\gamma(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}\,(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j}.$$

By taking absolute value and applying triangle inequality, we get

$$|a_n| \leq \frac{|\gamma|\,(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}\,(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1}\,|a_j|\,|p_{n-j}|\,.$$

Applying the coefficient estimates $|p_n| \leq 2$ $(n \geq 1)$ for Caratheodory functions [11], we obtain

$$\begin{aligned} |a_n| \quad &\leq \quad \frac{2\,|\gamma|\,(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1}\,|a_j| \\ &= \quad \frac{\sigma}{[n-1]\,\Phi_{n-1}} \sum_{j=1}^{n-1} \psi_{j-1}\,|a_j|\,, \end{aligned} \tag{3.6}$$

where $\sigma = 2|\gamma|(\alpha-1)/q(1-k)$. To prove (3.2) we apply mathematical induction. So for $n = 2$, we have from (3.6)

$$|a_2| \leq \frac{\sigma}{\Phi_1} = \frac{(\sigma)_{2-1}}{[2-1]!\Phi_{2-1}}, \tag{3.7}$$

which shows that (3.2) holds for $n = 2$. For $n = 3$, we have from (3.6)

$$|a_3| \leq \frac{\sigma}{[3-1]\,\Phi_{3-1}} \left\{1 + \Phi_1\,|a_2|\right\},$$

using (3.7), we have

$$|a_3| \leq \frac{\sigma}{[2]\,\Phi_2}\,(1+\sigma) = \frac{(\sigma)_{3-1}}{[3-1]\,\Phi_{3-1}},$$

which shows that (3.2) holds for $n = 3$. Let us assume that (3.2) is true for $n \leq t$, that is,

$$|a_t| \leq \frac{(\sigma)_{t-1}}{[t-1]!\Phi_{t-1}} \quad j = 1, 2, \ldots, t. \tag{3.8}$$

Using (3.6) and (3.8), we have

$$\begin{aligned}|a_{t+1}| &\leq \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\Phi_{j-1}|a_j| \\ &\leq \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\psi_{j-1}\frac{(\sigma)_{j-1}}{[j-1]!\Phi_{j-1}} \\ &= \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\frac{(\sigma)_{j-1}}{[j-1]!}.\end{aligned}$$

Applying (2.1), we have

$$\begin{aligned}|a_{t+1}| &\leq \frac{1}{t\Phi_t}\frac{(\sigma)_t}{[t-1]!} \\ &= \frac{1}{\Phi_t}\frac{(\sigma)_t}{[t]!}.\end{aligned}$$

Consequently, using mathematical induction, we have proved that (3.2) holds true for all n, $n \geq 2$. This completes the proof. □

Theorem 3.3. *If a function $f \in \mathcal{KD}_q(k,\alpha,\gamma)$, then*

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} \prec 1 + 2(\alpha_1 - 1) - \frac{2(\alpha_1-1)}{1-z} \quad (z \in \mathbb{U}), \tag{3.9}$$

$$\alpha_1 = \frac{\alpha - k}{1-k}. \tag{3.10}$$

Proof. If $f(z) \in \mathcal{KD}_q(k,\alpha,\gamma)$, then by (3.1)

$$\mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} - 1\right)\right\} < \alpha_1. \tag{3.11}$$

Then there exists a Schwarz function $w(z)$ such that

$$\frac{\alpha_1 - \left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} - 1\right)\right\}}{\alpha_1 - 1} = \frac{1+w(z)}{1-w(z)}, \tag{3.12}$$

and

$$\mathfrak{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

Therefore, from (3.12), we obtain

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} = 1 + \gamma(\alpha_1 - 1)\left(1 - \frac{1+w(z)}{1-w(z)}\right).$$

This gives

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} = 1 + 2\gamma(\alpha_1 - 1) - \frac{2\gamma(\alpha_1-1)}{1-w(z)}$$

and hence

$$\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} \prec 1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma\left(\alpha_1-1\right)}{1-z} \quad (z\in\mathbb{U}).$$

which was required in (3.9). □

Theorem 3.4. *If function $f\in\mathcal{KD}_q\left(k,\alpha,\gamma\right)$, then we have*

$$\frac{1-[1+2\gamma(\alpha_1-1)]\,r}{1-r} \leq \mathfrak{Re}\left\{\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}\right\} \leq \frac{1+[1+2\gamma(\alpha_1-1)]\,r}{1+r}, \tag{3.13}$$

for $|z|=r<1$ and α_1 is defined by (3.10).

Proof. By the virtue of Theorem (3.3), let us take the function $\phi(z)$ defined by

$$\phi(z)=1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma(\alpha_1-1)}{1-z} \quad \left(z\in\mathbb{U}\right).$$

Letting $z=re^{i\theta}(0\leq r<1)$, we see that

$$\mathfrak{Re}\phi(z)=1+2\gamma\left(\alpha_1-1\right)+\frac{2\gamma\left(1-\alpha_1\right)\left(1-r\cos\theta\right)}{1+r^2-2r\cos\theta}.$$

Let us define

$$\psi(t)=\frac{1-rt}{1+r^2-2rt} \quad \left(t=\cos\theta\right).$$

Since $\psi'(t)=\dfrac{r\left(1-r^2\right)}{\left(1+r^2-2rt\right)^2}\geq 0$, because $r<1$. Therefore we get

$$1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma\left(\alpha_1-1\right)}{1-r} \leq \mathfrak{Re}\phi(z) \leq 1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma\left(\alpha_1-1\right)}{1+r}.$$

After simplification, we have

$$\frac{1-\left[1+2\gamma\left(\alpha_1-1\right)\right]r}{1-r} \leq \mathfrak{Re}\phi(z) \leq \frac{1+\left[1+2\gamma\left(\alpha_1-1\right)\right)]\,r}{1+r}.$$

Since we note that $\dfrac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}\prec\phi(z),(z\in\mathbb{U})$ by Theorem 3.3 and $\phi(z)$ is analytic in $\mathbb{U}$, we proved the inequality (3.13). □

Theorem 3.5. *If $f\in\mathcal{A}$ satisfies*

$$\left|\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right| < \frac{(\alpha-1)|\gamma|}{(1-k)} \quad z\in\mathbb{U}, \tag{3.14}$$

for some $k\,(k\leq 0)$, $\alpha\,(\alpha>1)$ and $\gamma\in\mathbb{C}\setminus\{0\}$. Then $f\in\mathcal{KD}_q(k,\alpha,\gamma)$.

Proof.

$$\begin{aligned}
& \left|\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right| < \frac{(\alpha-1)|\gamma|}{(1-k)} \\
\Rightarrow\quad & \left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right| < \frac{\alpha-1}{1-k} \\
\Rightarrow\quad & (1-k)\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right| + 1 < \alpha \\
\Rightarrow\quad & \left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right| + 1 < k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right| + \alpha \\
\Rightarrow\quad & \mathfrak{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right\} + 1 < k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right)\right| + \alpha \\
\Rightarrow\quad & f \in \mathcal{LD}_b^k(a,c,\beta)
\end{aligned}$$

□

Corollary 3.6. *Let $f \in \mathcal{A}$ be of the form (1.1) and satisfies*

$$\left|\frac{\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}}\right| < \frac{(\alpha-1)|\gamma|}{q(1-k)} \quad z \in \mathbb{U}, \tag{3.15}$$

for some $k\,(k \le 0)$, $\beta\,(\beta > 1)$ and for some $b \in \mathbb{C}\setminus\{0\}$. Then $f \in \mathcal{KD}_q(k,\alpha,\gamma)$..

Proof. We have

$$\mathfrak{D}_q^{\mu}f(z) = z + \sum_{n=2}^{\infty}\Phi_{n-1}a_nz^n$$

and by (1.5)

$$z\partial\mathfrak{D}_q^{\mu}f(z) = z + \sum_{n=2}^{\infty}[n]\,\Phi_{n-1}a_nz^n.$$

Therefore, (3.14) follows immediately (3.15). □

Theorem 3.7. *Let $f \in \mathcal{A}$ be of the form (1.1) and satisfies*

$$\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n| < y \quad z \in \mathbb{U}, \tag{3.16}$$

for some $k\,(k \le 0)$, $\beta\,(\beta > 1)$ and for some $b \in \mathbb{C}\setminus\{0\}$ and where

$$y = \frac{(\alpha-1)|\gamma|}{q(1-k)} > 0.$$

Then $f \in \mathcal{KD}_q(k,\alpha,\gamma)$.

Proof. We have

$$\begin{aligned}
&\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n|<y\\
\Rightarrow\quad &\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n|<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n|\\
\Rightarrow\quad &0<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n|\\
\Rightarrow\quad &0<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n||z^{n-1}|\\
\Rightarrow\quad &0<y\left|1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}\right| \qquad (3.17)
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n|<y\\
\Rightarrow\quad &\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n||z^{n-1}|<y\\
\Rightarrow\quad &\sum_{n=2}^{\infty}[n-1]\,|\Phi_{n-1}||a_n||z^{n-1}|<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n||z^{n-1}|\\
\Rightarrow\quad &\left|\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_nz^{n-1}\right|<y\left|1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}\right|\\
\Rightarrow\quad &\left|\frac{\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}}\right|<y,
\end{aligned}$$

because of (3.17). By (3.15) it follows $f\in\mathcal{LD}_b^k(a,c,\beta)$. □

References

[1] Aldweby, H., Darus, M., *Some subordination results on q-analogue of Ruscheweyh differential operator,* Abstr. Appl. Anal., Vol. 2014, Art. ID 958563, 6 pages.

[2] Anastassiu, G.A., Gal, S.G., *Geometric and approximation properties of some singular integrals in the unit disk,* J. Inequal. Appl., Vol. 2006, Art. ID 17231, 19 pages.

[3] Anastassiu, G.A., Gal, S.G., *Geometric and approximation properties of generalized singular integrals,* J. Korean Math. Soc., **23**(2006), no. 2, 425-443.

[4] Aral, A., *On the generalized Picard and Gauss Weierstrass singular integrals,* J. Comput. Anal. Appl., **8**(2006), no. 3, 249-261.

[5] Aral, A., Gupta, V., *On q-Baskakov type operators*, Demonstr. Math., **42**(2009), no. 1, 109-122.

[6] Aral, A., Gupta, V., *On the Durrmeyer type modification of the q-Baskakov type operators,* Nonlinear Anal. Theory, Methods and Appl., **72**(2010), no. 3-4, 1171-1180.

[7] Aral, A., Gupta, V., *Generalized q-Baskakov operators,* Math. Slovaca, **61**(2011), no. 4, 619-634.

[8] Aral, A., Gupta, V., Agarwal, R.P., *Applications of q-Calculus in Operator Theory,* Springer, New York, NY, USA, 2013.

[9] Arif, M., Mahmood, S., Sokół, J., Dziok, J., *New subclass of analytic functions in conical domain associated with a linear operator,* Acta Math. Sci., **36**B(2016), no. 3, 1-13.

[10] Jackson, F.H., *On q-functions and a certain difference operator,* Trans. Roy. Soc. Edinburgh, **46**(1909), no. 2, 253-281.

[11] Goodman, A.W., *Univalent Functions*, Vol. I, II, Polygonal Publishing House, Washington, New Jersey, 1983.

[12] Jackson, F.H., *On q-definite integrals,* Quart. J. Pure Appl. Math., **41**(1910), 193-203.

[13] Kanas, S., Raducanu, D., *Some class of analytic functions related to conic domains,* Math. Slovaca, **64**(2014), no. 5, 1183-1196.

[14] Mahmmod, S., Sokół, J., *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator,* J. Results Math., **71**(2017), 1345-1357.

[15] Miller, S.S., Mocanu, P.T., *Differential Subordinations Theory and Applications*, Marcel Decker Inc., New York, 2000.

[16] Nishiwaki, J., Owa, S., *Coefficient estimates for certain classes of analytic functions*, J. Inequal. Pure Appl. Math., **3**(2002), 1-5.

[17] Nishiwaki, J., Owa, S., *Certain classes of analytic functions concerned with uniformly starlike and convex functions*, Appl. Math. Comput., **187**(2007), 350-355.

[18] Owa, S., Srivastava, H.M., *Some generalized convolution properties associated with cerain subclasses of analytic functions*, J. Inequal. Pure Appl. Math., **3**(2002), no. 3, 1-13.

[19] Ruscheweyh,St., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[20] Shams, S., Kulkarni, S.R., Jahangiri, J.M., *Classes of uniformly starlike and convex functions*, Int. J. Math. Math. Sci., **55**(2004), 2959-2961.

[21] Uralegaddi, B.A., Ganigi, M.D., Sarangi, S.M., *Univalent functions with positive coefficients*, Tamkang J. Math., **25**(1994), 225-230.

Shahid Mahmood
Corresponding author
Department of Mechanical Engineering, Sarhad University of Science
and
I.T. Landi Akhun Ahmad, Hayatabad Link. Ring Road, Peshawar, Pakistan
e-mail: `shahidmahmood757@gmail.com`

Saima Mustafa
Department of Statistics & Mathematics
PMAS-Arid Agriculture University, Rawalpindi
e-mail: `saimamustafa28@gmail.com`

Imran Khan
Department of Basic Sciences and Islamyat University of Engineering and Technology
Peshawar, Pakistan
e-mail: ikhanqau1@gmail.com