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Inclusion properties of hypergeometric type functions and related integral transforms

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Abstract. In this work, conditions on the parameters a, b and c are given so that the normalized Gaussian hypergeometric function $zF(a, b; c; z)$, where

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1,$$

is in certain class of analytic functions. Using Taylor coefficients of functions in certain classes, inclusion properties of the Hohlov integral transform involving $zF(a, b; c; z)$ are obtained. Similar inclusion results of the Komatu integral operator related to the generalized polylogarithm are also obtained. Various results for the particular values of these parameters are deduced and compared with the existing literature.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that contains functions univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is called starlike, denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. The class of all convex functions, denoted by $\mathcal{C}$, consists of the functions $f \in \mathcal{A}$ such that zf' is starlike. A function $f \in \mathcal{A}$ is said to be *close-to-convex* with respect to a fixed starlike function $g \in S^*$ if and only if $\text{Re}\left(e^{i\lambda}\dfrac{zf'(z)}{g(z)}\right) > 0$ for $z \in \mathbb{D}$ and $\lambda \in \mathbb{R}$. . Let $\mathcal{K}$ denote the subclass of all such close-to-convex functions, where $\lambda = 0$. Various generalization of these classes and various other subclasses of S exist in the literature. For example the class of starlike functions of order σ, denoted by $S^*(\sigma)$, $0 \le \sigma < 1$, which has the analytic characterization $\mathrm{Re}\,\dfrac{zf'(z)}{f(z)} > \sigma$, is the generalization of the class $S^*(0) = S^*$. Note that $C(\sigma)$, the class of convex functions of order σ contains all functions $f \in S$ for which $zf' \in S^*(\sigma)$.

We introduce the class $R^{\tau}_{\gamma,\alpha}(\beta)$, with $0 \le \gamma < 1$, $0 \le \alpha \le 1$, $\tau \in \mathbb{C}\backslash\{0\}$ and $\beta < 1$ as

$$R^{\tau}_{\gamma,\alpha}(\beta) := \left\{ f \in \mathcal{A} : \left| \frac{(1-\alpha+2\gamma)\frac{f}{z} + (\alpha-2\gamma)f' + \gamma z f'' - 1}{2\tau(1-\beta) + (1-\alpha+2\gamma)\frac{f}{z} + (\alpha-2\gamma)f' + \gamma z f'' - 1} \right| < 1,\, z \in \mathbb{D} \right\}. \tag{1.2}$$

Note that few particular cases of this class discussed in the literature.

1. The class $R^{\tau}_{\gamma,\alpha}(\beta)$ for $\alpha = 2\gamma + 1$, was considered in [16], where references about other particular cases in this direction are provided.
2. The class $R^{\tau}_{\gamma,\,\alpha}(\beta)$ for $\tau = e^{i\eta}\cos\eta$, where $-\pi/2 < \eta < \pi/2$ is considered in [1] (see also [2, 3]), and the properties of certain integral transforms of the type
$$V_{\lambda}(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in R^{(e^{i\eta}\cos\eta)}_{0,\gamma}(\beta) \tag{1.3}$$
with $\beta < 1$, $\gamma < 1$ and $|\eta| < \pi/2$, under suitable restriction on $\lambda(t)$ was discussed using duality techniques for various values of γ in [1]. For other interesting cases, we refer to [3, 16] and references therein.
3. The class $R^{\tau}_{0,1}(0)$ with $\tau = e^{i\eta}\cos\eta$ was considered in [10] with reference to the univalency of partial sums.

It is clear that the geometric properties of certain integral transforms under duality techniques, which is one of recent research interest (for example, see [1, 3] and references therein), cannot be proved easily as the results involve certain multiple integrals and it is difficult to check the conditions given for the existence of the inclusion results for these integral transforms. For this purpose, the inclusion properties of certain special functions to be in the analytic subclasses like $R^{(e^{i\eta}\cos\eta)}_{\gamma,\,\alpha}(\beta)$ are studied using techniques other than duality methods which motivates this work.

Among various results related to the integral operator (1.3) available in the literature, an important and interesting result is application of the operator (1.3) when $\lambda(t)$ is related to the function $zF(a,b;c;z)$. Here by $F(a,b;c;z)$ we mean the well-known Gaussian hypergeometric function

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n \tag{1.4}$$

$z \in \mathbb{D}$, with $(\lambda)_n$ being the Pochhammer symbol given by $(\lambda)_n = \lambda(\lambda+1)_{n-1}$, $(\lambda)_0 = 1$. Also, there has been considerable interest to find conditions on the parameters a, b, and c such that the normalized hypergeometric functions $(c/ab)\,(F(a,b;c;z) - 1)$ or $zF(a,b;c;z)$ belong to one of the known subclasses of $\mathcal{S}$. For more details on the basic ideas of Gaussian hypergeometric functions, we refer to [11] and on the applications related to geometric function theory, we refer to [1, 14, 15, 16] and references therein.

Related to $F(a,b;c;z)$ is the Hohlov operator $H_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z)$, where $*$ denotes the well-known Hadamard product or convolution. This operator is particular case of a general integral transform studied in [5]. To be more specific, the properties of certain integral transforms of the type

$$V_\lambda(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in R_{\gamma,\alpha}^{(e^{i\eta}\cos\eta)}(\beta) \tag{1.5}$$

under suitable restriction on $\lambda(t)$ was discussed by many authors [1, 3, 5]. In particular, if

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}t^{b-1}(1-t)^{c-b-1}$$

then $V_\lambda(f) = \mathcal{L}(b,c)(f)(z)$ which is the well-known Carlson-Schaffer operator. Note that $H_{1,b,c}(f)(z) = \mathcal{L}(b,c)(f)(z)$. The following lemma exhibits the relation between the integral operator in discussion with the Hohlov operator.

Lemma 1.1. *If $f \in \mathcal{A}$ and $c-a+1 > b > 0$, then*

$$V_\lambda(f)(z) = H_{a,b,c}(f)(z)$$

where

$$H_{a,b,c}(f)(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_0^1 \frac{(1-t)^{c-a-b}}{\Gamma(c-a-b+1)}t^{b-2}F(c-a,1-a;c-a-b+1;1-t)f(tz)dt.$$

The Komatu operator $K_a^p : \mathcal{A} \to \mathcal{A}$ [9] is defined as

$$K_a^p[f](z) = \frac{(1+a)^p}{\Gamma(p)}\int_0^1 \left(\log(\frac{1}{t})\right)^{p-1}t^{a-1}f(tz)dt,$$

where $a > -1$ and $p \geq 0$. It has a series representation as

$$K_a^p[f](z) = z + \sum_{n=2}^{\infty}\frac{(1+a)^p}{(n+a)^p}a_n z^n$$

and in terms of convolution, we can write

$$K_a^p[f](z) = \mathcal{K}_a^p(z) * f(z),$$

where $\mathcal{K}_a^p(z) = z + \sum_{n=2}^{\infty}\frac{(1+a)^p}{(n+a)^p}z^n$.

In this paper we study the operators $H_{a,b,c}(f)(z)$ and $K_a^p[f](z)$ for various choices of the function f.

The paper is organized as follows. In Section 2, some preliminary results about the Gaussian hypergeometric function $F(a,b;c;z)$ and conditions on the Taylor coefficients of $f \in R_{\gamma,\alpha}^{\tau}(\beta)$ are given which are used in the subsequent sections. Conditions on the triplets a, b, c are obtained so that in Section 3 inclusion properties of $F(a,b;c;z)$ and its normalized case to be in the class $R_{\gamma,\alpha}^{\tau}(\beta)$ are discussed and in Section 4, inclusion properties of $zF(a,b;c;z) * f(z)$ for f in various subclasses of S are discussed. Similar type of inclusion results for the Komatu operator is discussed in Section 5. In the last section, certain remarks are given to provide motivation for further research in this direction.

2. Preliminary results

The following result is available in [16], which can also be easily verified by simple computation.

Lemma 2.1. *Let $F(a,b;c;z)$ be the Gaussian hypergeometric function as given in* (1.4)*. Then we have the following*

(i) *For* $\operatorname{Re}(c-a-b)>0$ *and* $c\neq 0,-1,-2,\ldots,$

$$F(a,b;c;1)=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.1}$$

(ii) *For* $a,b>0,\ c>a+b+1,$

$$\sum_{n=0}^{\infty}\frac{(n+1)(a)_n(b)_n}{(c)_n(1)_n}=F(a,b;c;1)\left[\frac{ab}{c-1-a-b}+1\right]. \tag{2.2}$$

(iii) *For* $a\neq 1,\ b\neq 1$ *and* $c\neq 1$ *with* $c>\max\{0,a+b-1\},$

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}=\frac{(c-1)}{(a-1)(b-1)}\left[F(a-1,b-1;c-1;1)-1\right]. \tag{2.3}$$

(iv) *For* $a\neq 1$ *and* $c\neq 1$ *with* $c>\max\{0,2\operatorname{Re}a-1\},$

$$\sum_{n=0}^{\infty}\frac{|(a)_n|^2}{(c)_n(1)_{n+1}}=\frac{(c-1)}{|a-1|^2}\left[F(a-1,\bar{a}-1;c-1;1)-1\right]. \tag{2.4}$$

Proof. Part (i) is the well-known Gauss summation formula. Part (ii) follows from splitting the left hand side into two parts and applying (2.1). For part (iii), using the fact that $\lambda(\lambda+1)_m=(\lambda)_{m+1}$, in place of a, b and c, the required result follows. Part (iv) is nothing but Part (iii) with $b=\bar{a}$. □

In order to obtain the objective, we need conditions on the Taylor coefficients of $R^{\tau}_{\gamma,\alpha}(\beta)$ which is given in the following results.

Lemma 2.2. *Let $f(z)\in\mathcal{S}$ and is of the form* (1.1)*. If $f(z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$, then*

$$|a_n|\le\frac{2|\tau|(1-\beta)}{1+(n-1)(\alpha-2\gamma+\gamma n)},\quad n=2,3,\ldots. \tag{2.5}$$

Equality holds for the function

$$f(z)=\frac{1}{z^{(1/\nu)-1}}\frac{1}{\mu\nu}\int_0^z\frac{1}{t^{\frac{1}{\mu}-\frac{1}{\nu}+1}}\int_0^t w^{\frac{1}{1-\frac{1}{\mu}}}\left(1+\frac{2(1-\beta)\tau w^{n-1}}{1-w^{n-1}}\right)dw, \tag{2.6}$$

where $\mu+\nu=\alpha-\gamma$ and $\mu\nu=\gamma$.

Proof. Clearly $f \in R^{\tau}_{\gamma,\alpha}(\beta)$ is equivalent to

$$1+\frac{1}{\tau}\left((1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma z f''-1\right)=\frac{1+(1-2\beta)w(z)}{1-w(z)},$$

where $w(z)$ is analytic in $\mathbb{D}$ and satisfies the condition $w(0)=0$, $|w(z)|<1$ for $z\in\mathbb{D}$. Hence we have

$$\frac{1}{\tau}\left((1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma z f''-1\right)$$
$$=w(z)\left(2(1-\beta)+\frac{1}{\tau}\left((1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma z f''-1\right)\right).$$

Using (1.1) and $w(z)=\sum_{n=1}^{\infty}b_n z^n$ we have

$$\left[2(1-\beta)+\frac{1}{\tau}\left(\sum_{n=2}^{\infty}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}\right)\right]\left[\sum_{n=1}^{\infty}b_n z^n\right]$$
$$=\frac{1}{\tau}\sum_{n=2}^{\infty}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}.$$

Equating the coefficients of the powers of z^{n-1} on both sides of the above equation, it is easy to observe that the coefficient a_n in right hand side of the above expression depends only on $a_2,\dots,a_{n-1}$ and the left hand side of the above expression. Hence, for $n\geq 2$ this gives

$$\left[2(1-\beta)+\frac{1}{\tau}\left(\sum_{n=2}^{k-1}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}\right)\right]w(z)$$
$$=\frac{1}{\tau}\sum_{n=2}^{k}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}+\sum_{n=k+1}^{\infty}d_n z^{n-1}.$$

Using $|w(z)|<1$, this reduces to the inequality

$$\left|2(1-\beta)+\frac{1}{\tau}\left(\sum_{n=2}^{k-1}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}\right)\right|$$
$$>\left|\frac{1}{\tau}\sum_{n=2}^{k}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}+\sum_{n=k+1}^{\infty}d_n z^{n-1}\right|.$$

Squaring the above inequality and integrating around $|z|=r$, $0<r<1$, we get

$$4(1-\beta)^2+\frac{1}{|\tau|^2}\left(\sum_{n=2}^{k-1}[1+(\alpha-2\gamma+\gamma n)(n-1)]^2|a_n|^2r^{2(n-1)}\right)$$
$$>\frac{1}{|\tau|^2}\sum_{n=2}^{k}[1+(\alpha-2\gamma+\gamma n)(n-1)]^2|a_n|^2r^{2(n-1)}+\sum_{n=k+1}^{\infty}|d_n|^2r^{2(n-1)}.$$

and letting $r \to 1$ we obtain

$$4(1-\beta)^2 \geq \frac{1}{|\tau|^2}[1+(\alpha-2\gamma+\gamma n)(n-1)]^2|a_n|^2$$

which gives the desired result. For sharpness, consider the function

$$(1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma z f'' = 1+\frac{2(1-\beta)\tau z^{n-1}}{1-z^{n-1}} := p(z).$$

Simplifying and using the fact $\mu+\nu=\alpha-\gamma$ and $\mu\nu=\gamma$ gives (2.6). ☐

Remark 2.3. The condition given in (2.5) is equivalent to the condition

$$|a_n| \leq \frac{2|\tau|(1-\beta)}{1+\alpha(n-1)+\gamma(n-1)(n-2)}, \quad n=2,3,\ldots, \tag{2.7}$$

which will be used in the sequel.

Lemma 2.4. *Let $f(z)$ be of the of the form* (1.1). *Then a sufficient condition for $f(z)$ to be in $R^{\tau}_{\gamma,\alpha}(\beta)$ is*

$$\sum_{n=2}^{\infty}[1+(n-1)(\alpha-2\gamma+\gamma n)]|a_n| \leq |\tau|(1-\beta). \tag{2.8}$$

This condition is also necessary if $\eta=0$ in (1.2) *and $a_n<0$ in* (1.1).

Proof. Using (1.1) it is easy to see that

$$\begin{aligned}
&\operatorname{Re} e^{i\eta}\left((1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma z f''-\beta\right)\\
&= (1-\beta)\cos\eta+\operatorname{Re} e^{i\eta}\sum_{n=2}^{\infty}\Big(1+(\alpha-2\gamma+\gamma n)(n-1)\Big)a_n z^{n-1}\\
&\geq (1-\beta)\cos\eta-\sum_{n=2}^{\infty}\Big|\Big(1+(\alpha-2\gamma+\gamma n)(n-1)\Big)\Big|\,|a_n| \geq 0,
\end{aligned}$$

using (2.8). The resultant obtained above is equivalent to the analytic characterization of $f \in R^{\tau}_{\gamma,\alpha}(\beta)$ and the proof is complete. ☐

3. Inclusion results for $zF(a,b;c;z)$

Theorem 3.1. *Let a,b,c and γ satisfy any one of the following conditions such that $T_i(a,b,c,\gamma) \leq |\tau|(1-\beta)$ for $i=1,2,3$.*

(i) *$a,b>0$, $c>a+b+2$ and*

$$T_1(a,b,c,\gamma) = F(a,b;c;1)+\alpha\frac{a\,b}{c}F(a+1,b+1;c+1;1)+\gamma\frac{(a)_2\,(b)_2}{(c)_2}F(a+2,b+2;c+2;1)-1.$$

(ii) *$a,b \in \mathbb{C}\backslash\{0\}$, $|a|\neq 1, |b|\neq 1$, $c>|a|+|b|+2$ and*
$T_2(a,b,c,\gamma)$

$$= F(|a|+1,|b|+1;c+1;1)\left(\alpha\frac{|ab|}{c}+\gamma\frac{(|a|)_2(|b|)_2}{(c)(c-|a|-|b|-2)}+\frac{c-|a|-|b|-1}{c}\right)-1.$$

(iii) $-1 < a < 0$, $-1 < b < 0$, $c > 0$ *and*
$T_3(a,b,c,\gamma)$

$$= F(a+1,b+1;c+1;1)\left(\alpha\frac{ab}{c}+\gamma\frac{(a)_2(b)_2}{(c)(c-a-b-2)}+\frac{c-a-b-1}{c}\right)-1.$$

Then $zF(a,b;c;z)$ *is in* $R^{\tau}_{\gamma,\alpha}(\beta)$.

Proof. Clearly $zF(a,b;c;z)$ has the series representation of the form (1.1) where

$$a_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \tag{3.1}$$

Using Lemma 2.4, it suffices to prove that

$$\sum_{n=2}^{\infty}[1+(n-1)(\alpha-2\gamma+\gamma n)]|a_n| \le |\tau|(1-\beta),$$

which is equivalent in writing

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)+\gamma(n-1)(n-2)]|a_n| \le |\tau|(1-\beta) \Longrightarrow f\in R^{\tau}_{\gamma,\alpha}(\beta). \tag{3.2}$$

Case (i): Let $a,b>0$ and $c>a+b+2$. Then the series in the left hand side of (3.2) can be written as

$$\begin{aligned} S :&= \sum_{n=2}^{\infty}\left(1+\alpha(n-1)+\gamma(n-1)(n-2)\right)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= \sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}+\alpha\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}}+\gamma\frac{(a)_2(b)_2}{(c)_2}\sum_{n=3}^{\infty}\frac{(a+2)_{n-3}(b+2)_{n-3}}{(c+2)_{n-3}(1)_{n-3}}. \end{aligned}$$

An easy computation by using the hypothesis of the theorem and applying (2.1), we get the required result.

Case (ii): Let $a,b\in\mathbb{C}\backslash\{0\}$, $c>|a|+|b|+2$. Since $|(a)_n|\le(|a|)_n$, we have from (3.2),

$$\begin{aligned} S :&= \sum_{n=2}^{\infty}\Big(1+\alpha(n-1)+\gamma(n-1)(n-2)\Big)|a_n| \\ &\le \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_{n+1}}+\alpha\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n}+\gamma\sum_{n=1}^{\infty}(n-1)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}}. \end{aligned} \tag{3.3}$$

Note that the third sum in the right hand side of (3.3) is equivalent to

$$\begin{aligned} &\sum_{n=0}^{\infty} n\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n} \\ &\quad= \sum_{n=0}^{\infty}(n+1)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n}-\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n} \\ &\quad= \frac{|ab|}{c}\sum_{n=0}^{\infty}(n+1)\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}-\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}. \end{aligned}$$

Using the above value in (3.3) we get that the inequality (3.3) is equivalent to

$$S \leq \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_{n+1}}$$

$$+(\alpha-\gamma)\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}+\gamma\frac{|ab|}{c}\sum_{n=0}^{\infty}(n+1)\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}. \quad (3.4)$$

Now applying (2.3) and the hypothesis of the theorem in the first sum of (3.4) gives

$$\left(\frac{c-|a|-|b|-1}{c}F(|a|+1,|b|+1;c+1;1)-1\right). \quad (3.5)$$

Similarly applying (2.2) and the hypothesis of the theorem in the third sum of (3.4) gives

$$\frac{|ab|}{c}\left(F(|a|+1,|b|+1;c+1,1)\left(\frac{(|a|+1)(|b|+1)}{c-|a|-|b|-2}+1\right)\right). \quad (3.6)$$

Clearly the second sum of (3.4) is related to (2.1) which gives

$$\frac{|ab|}{c}F(|a|+1,|b|+1;c+1;1).$$

Now substituting this resultant and (3.5) and (3.6) in (3.4) gives the required result. Case (iii): Let $-1<a<0$, $-1<b<0$ and $c>0$. The result follows by proceeding in a similar way to the previous case. □

Since the substitution $a=\bar{b}$ in Theorem 3.1 is useful in characterizing polynomials with positive coefficients when b is some negative integer, we give the corresponding result independently, wherein only the second case can be applied.

Corollary 3.2. *Let $c>2\operatorname{Re}b+2$ and $T_4(b,c,\gamma)\leq|\tau|(1-\beta)$ where*

$$T_4(b,c,\gamma)=F(\overline{b+1},b+1;c+1;1)\left(\alpha\frac{|b|^2}{c}+\gamma\frac{(|b|)_2{}^2}{c(c-2\operatorname{Re}b-2)}+\frac{c-2\operatorname{Re}b-1}{c}\right)-1.$$

Then $zF(\bar{b},b;c;z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

Note that the results in Corollary 3.2 can also be obtained directly by using (2.4) instead of (2.3), as used in Theorem 3.1.

Further, if we set $\alpha=1$ and $\gamma=0$, then by choosing $\beta=0$ and $\tau=e^{i\eta}\cos\eta$ with $-\pi/2<\eta<\pi/2$, we get the functions in the class $R^{\tau}_{\gamma,\alpha}(\beta)$ satisfying the analytic criterion $\operatorname{Re}f'>0$ which implies that $f(z)$ is close-to-convex with respect to the starlike function $g(z)=z$. Hence the following result is immediate.

Corollary 3.3. *Let $c>2|b-1|+3$ and*

$$F(\bar{b},b;c;1)\leq\frac{2(c-1)}{|b-2|^2+c-3}, \quad (3.7)$$

then $zF(\bar{b},b;c;z)$ is close-to-convex with respect to the starlike function $g(z)=z$.

Remark 3.4. Corollary 3.3, with the absence of α, β, γ and τ, is much useful, in particular, for extracting polynomials with positive coefficients, which is the main idea behind choosing $a=\bar{b}$. Moreover, if we take $b=-m$, then (3.7) gives

$$F(-m,-m;c;1)\left(\frac{m^2+4m+c+1}{2(c-1)}\right)\leq 1.$$

But, when m is sufficiently large, c has to be chosen so large to have the value in the left side bounded by 1. This is given by the condition that $c>2m+5$. In the case of $m=2$, c need to be larger than 9 and should satisfy $c^3-18c^2-75c-104\geq 0$ so that the corresponding polynomial $1+\frac{4}{c}z+\frac{2}{c(c+1)}z^2$ is close-to-convex. It is easy to see that the condition is satisfied for c more than $21.68057259\ldots$, which is obtained using mathematical software. Hence if m is chosen as a larger negative integer then this result is true for polynomials having their coefficients very small, which is not interesting.

Instead, if we consider, Theorem 3.1, with either $a=-m$ or $b=-m$ we can still extract polynomials that can have smaller values of c, with coefficients having alternate signs, that satisfy the hypothesis given in Theorem 3.1.

In Theorem 3.1, if we take $a=1$, we get the result for the incomplete beta function $zF(1,b;c;z)$. Since the incomplete beta function plays an important role in geometric function theory (for example, see [15]), we give the result for the incomplete beta function independently as

Theorem 3.5. *Let b,c and γ satisfy any one of the following conditions such that $T_i(b,c,\gamma)\leq|\tau|(1-\beta)$ for $i=1,2$.*

(i) $b>0$, $c>b+3$ *and*

$$T_1(b,c,\gamma)=F(1,b;c;1)+\alpha\frac{b}{c}F(2,b+1;c+1;1)+\gamma\frac{2\,(b)_2}{(c)_2}F(3,b+2;c+2;1)-1.$$

(ii) $b\in\mathbb{C}\backslash\{0\}$, $c>|b|+3$ *and*

$$T_2(b,c,\gamma)=F(2,|b|+1;c+1;1)\left(\alpha\frac{|b|}{c}+\gamma\frac{2(|b|)_2}{(c)(c-|b|-3)}+\frac{c-|b|-2}{c}\right)-1.$$

Then the incomplete beta function $\phi(b;c;z):=zF(1,b;c;z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

Remark 3.6. Note that at $\alpha=1$, $\gamma=0$, $\beta=0$ and $\tau=e^{i\eta}\cos\eta$ with $-\pi/2<\eta<\pi/2$ the above result reduces to $c>b+3$, $b>0$. Under these conditions, the normalized incomplete beta function $zF(1,b;c;z)$ is close-to-convex with respect to the starlike function $g(z)=z$.

Consider the operator of the form $G(a,b;c;z):=\int_0^z F(a,b;c;t)dt$. Then we have

$$G(a,b;c;z):=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}z^n=z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n,$$

where a_n is given as in (3.1). This is the normalized form of the hypergeometric function $F(a,b;c;z)$ which has many interesting properties. Note that a function may fail to inherit its geometric properties under such normalization. For example, $1+z$ is convex univalent in $\mathbb{D}$, whereas its normalized form $z(1+z)$ is not even univalent.

Theorem 3.7. *Let $a, b \in \mathbb{C}\backslash\{0\}$ with $|a| \neq 1$, $|b| \neq 1$ and $|c| > |a|+|b|+1$ such that $T(a,b,c,\gamma) \leq |\tau|(1-\beta)$ where*
$T(a,b,c,\gamma)$

$$= F(a,b;c;1)\left(\frac{\gamma\, ab}{c-a-b-1}+\alpha+\frac{(1-\alpha+2\gamma)(c-a-b)}{(a-1)(b-1)}\right)-\frac{(1-\alpha+2\gamma)(c-1)}{(a-1)(b-1)}.$$

Then $G(a,b;c;z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

Proof. We have $G(a,b;c;z) = z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}z^n$. So it is sufficient to prove that

$$\sum_{n=2}^{\infty}\left[1+\alpha(n-1)+\gamma(n-1)(n-2)\right]|a_n| \leq |\tau|(1-\beta).$$

The left hand side of the above inequality can be expressed as

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}+\alpha\sum_{n=1}^{\infty}n\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}+\gamma\sum_{n=1}^{\infty}n(n-1)\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}. \tag{3.8}$$

For the third part (3.8), writing $n(n-1) = n(n+1)-2(n+1)+2$ and adding with the second part of (3.8) gives

$$(1-\alpha+2\gamma)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}+(\alpha-2\gamma)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}+\frac{\gamma ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}. \tag{3.9}$$

Now, using the hypothesis and comparing the first part of (3.9) with (2.3), second and third part of (3.9) with (2.1) gives the required result upon simplification. □

Check, if at $a = \overline{b}$ in the above result gives the following Corollary.

Corollary 3.8. *Let $a = \overline{b}$, $0 < b \neq 1$ and $c > 2\mathrm{Re\,b}+1$ such that $T(\overline{b},b,c,\gamma) \leq |\tau|(1-\beta)$ where*
$T(\overline{b},b,c,\gamma)$

$$= F(b,\overline{b};c;1)\left(\frac{\gamma\,|b|^2(\alpha-2\gamma)}{c-2\mathrm{Re\,b}-1}+\frac{(1-\alpha+2\gamma)(c-2\mathrm{Reb})}{|b-1|^2}\right)-\left(\frac{(1-\alpha+2\gamma)(c-1)}{|b-1|^2}\right)$$

Then $G(\overline{b},b;c;z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

4. Inclusion properties of $H_{a,\,b\,c}(f)(z)$

Our next interest is to find the inclusion properties of

$$H_{a,\,b\,c}(f)(z) = zF(a,b;c;z) * f(z),$$

where $f(z)$ is in certain subclass of S. For this, we recall certain subclasses that are necessary for further discussion. We begin with the following definition.

Definition 4.1. [4] Let $f \in \mathcal{A}$, $0 \le k < \infty$, and $0 \le \sigma < 1$. Then $f \in k - UCV(\sigma)$ if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \ge k\left|\frac{zf''(z)}{f'(z)}\right| + \sigma. \tag{4.1}$$

This class generalizes various other classes which are worthy to mention here. The class $k-UCV(0)$, called as k-uniformly convex is due to [8], and has the geometric characterization that for $0 \le k < \infty$, the function $f \in \mathcal{A}$ is said to be k-uniformly convex in $\mathbb{D}$, if f is convex in $\mathbb{D}$, and the image of every circular arc γ contained in $\mathbb{D}$, with center ζ, where $|\zeta| \le k$, is convex.

The class $1 - UCV(0) = UCV$ [7] (see also [12]) describes geometrically the domain of values of the expression $p(z) = 1 + \dfrac{zf''(z)}{f'(z)}$, $z \in \mathbb{D}$, as $f \in UCV$ if and only if p is in the conic region

$$\Omega = \{\omega \in \mathbb{C} : (\mathrm{Im}\omega)^2 < 2\ \mathrm{Re}\,\omega - 1\}.$$

Using Alexander transform a related class $k - \mathcal{S}_p(\sigma)$ is obtained as $f \in k - UCV(\sigma) \iff zf' \in k - \mathcal{S}_p(\sigma)$. Results for the condition on the Taylor coefficients of functions in these classes are available in the literature. Among them, we mention the results that serve our purpose.

Lemma 4.2. [4] *A function $f \in \mathcal{A}$ is in $k - UCV(\sigma)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\,[n(1+k) - (k+\sigma)]\,|a_n| \le 1 - \sigma. \tag{4.2}$$

It was also found that the condition (4.2) is necessary if $f \in \mathcal{A}$ given by (1.1) has $a_n < 0$. Further that the condition

$$\sum_{n=2}^{\infty} [n(1+k) - (k+\sigma)]\,|a_n| \le 1 - \sigma. \tag{4.3}$$

is sufficient for f to be in $k - \mathcal{S}_p(\sigma)$ and turns out to be also necessary if $f \in \mathcal{A}$ given by (1.1) has $a_n < 0$.

Theorem 4.3. *Let $f \in \mathcal{A}$ be defined as in (1.1). Suppose that $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a| + |b| + 1$ be such that, for $k \ge 0$, $0 \le \sigma < 1$,*
$F(|a|+1, |b|+1; c+1; 1)$

$$(|ab|(1+k) + (1-\sigma)(c - |a| - |b| - 1) \le c(1-\sigma)\left(1 + \frac{\alpha - 3\gamma}{2|\tau|(1-\beta)}\right). \tag{4.4}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ and $0 \le \beta < 1$, $H_{a,b\,c}(f)(z) \in k - UCV(\sigma)$.

Proof. Let $f \in \mathcal{A}$ be defined as in Theorem 4.3. Considering (4.2), from Lemma 2.2, we need to prove that if $f \in \mathcal{A}$ satisfies (2.5), then

$$\sum_{n=2}^{\infty} n\Big(\, n(1+k) - (k+\sigma)\Big)\,|A_n| \le 1 - \sigma, \tag{4.5}$$

where

$$A_n = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}a_n, \qquad n \geq 2.$$

Since $1+\alpha(n-1)+\gamma(n-1)(n-2) \geq n(\alpha-3\gamma)$ for $0 \leq \gamma \leq 1$ and $n \geq 2$, using $|(a,n)| \leq (|a|,n)$ it is enough if we prove that

$$T := \sum_{n=2}^{\infty} n\frac{(n)(1+k)-(k+\sigma)}{n}\frac{(|a|,n-1)(|b|,n-1)}{(|c|,n-1)(1,n-1)} \leq \frac{(1-\sigma)(\alpha-3\gamma)}{2|\tau|(1-\beta)}.$$

Using $(n+2)(1+k)-(k+\sigma) = (n+1)(1+k)+(1-\sigma)$ and

$$F(a,b;c;1) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathrm{Re}\,(c-a-b) > 0,$$

we get

$$\begin{aligned} T &= (1+k)\sum_{n=0}^{\infty}(n+1)\frac{(|a|,n+1)(|b|,n+1)}{(c,n+1)(1,n+1)} + (1-\sigma)\sum_{n=0}^{\infty}\frac{(|a|,n+1)(|b|,n+1)}{(c,n+1)(1,n+1)} \\ &= (1+k)\frac{ab}{c}\left(\frac{\Gamma(c-|a|-|b|-1)\Gamma(c+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right) + (1-\sigma)\left(\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right) \\ &= \left(\frac{\Gamma(c-|a|-|b|-1)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right)\Big(|ab|(1+k)+(1-\sigma)(c-|a|-|b|-1)\Big) - (1-\sigma), \end{aligned}$$

which by using the hypothesis, gives the required result. □

Another sufficient condition for the class $k-UCV$ is also given in [8] by the following result.

Lemma 4.4. [8] *Let $f \in \mathcal{S}$ and has the form (1.1). If for some k, $0 \leq k < \infty$, the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2}, \tag{4.6}$$

holds, then $f \in k-UCV$. The number $1/(k+2)$ cannot be increased.

It is interesting to observe that, even though σ is not involved in this sufficient condition, this condition holds for $f \in k-UCV(\sigma)$, by the method of proof given for Lemma 4.4 in [8]. Also that, using the Alexander transform, a result for $f \in k-S_p(\sigma)$ analogous to (4.6) cannot be obtained by replacing a_n by a_n/n as in many other situations.

To compare the results we are interested in giving a theorem equivalent to Theorem 4.3, by using (4.6) instead of (4.2). Since σ is not involved in (4.6), we present this result for the case $\sigma = 0$ only. The proof of this theorem is similar to Theorem 4.3 and we omit details.

Theorem 4.5. *Let $f \in \mathcal{A}$ be defined as in (1.1). Suppose that $a,b \in \mathbb{C}\backslash\{0\}$, $c > |a|+|b|+1$ be such that, for $k \geq 0$, $0 \leq \alpha < 1$,*

$$F(|a|+1,|b|+1;c+1;1)\frac{|ab|}{c} \leq \frac{(\alpha-3\gamma)}{2|\tau|(1-\beta)(k+2)}. \tag{4.7}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$,$0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $H_{a,\,b,\,c}(f)(z) \in k{-}UCV$.

If we let $a = \bar{b}$ in $F(a, b; c; z)$ we get polynomials with positive coefficients when b is some negative integer. Hence the above Theorems are useful in characterizing convex polynomials and we give the corresponding results independently.

Corollary 4.6. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > 2\mathrm{Re}b + 1$ *and* b, c *satisfy*

$$F(b+1, \bar{b}+1; c+1; 1)(|b|^2(1+k) + (1-\sigma)(c - 2\mathrm{Re}b - 1) \le c(1-\sigma)\left(1 + \frac{\alpha - 3\gamma}{2|\tau|(1-\beta)}\right). \tag{4.8}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $H_{\bar{b},\,b\,c}(f)(z) \in k - UCV(\sigma)$.

Corollary 4.7. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > 2\mathrm{Re}b + 1$ *be such that*

$$F(b+1, \bar{b}+1; c+1; 1)\frac{|b|^2}{c} \le \frac{(\alpha - 3\gamma)}{2|\tau|(1-\beta)(k+2)}. \tag{4.9}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$,$0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $H_{\bar{b},\,b,\,c}(f)(z) \in k{-}UCV$ *where* $k \ge 0$.

The Hohlov operator $H_{a,\,b,\,c}(f)(z)$ reduces to the Carlson-Shaffer operator $\mathcal{L}(b, c)(f)(z)$ if $a = 1$. Hence we give the statement of the following results.

Corollary 4.8. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > b+2$ *are such that, for* $k \ge 0$, $0 \le \sigma < 1$ *and*

$$\frac{(c-1)}{(c-b-1)(c-b-2)}(|b|(1+k) + (1-\sigma)(c-b-2) \le (1-\sigma)\left(1 + \frac{\alpha - 3\gamma}{2|\tau|(1-\beta)}\right). \tag{4.10}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $\mathcal{L}(b, c)(f)(z) \in k - UCV(\sigma)$.

Corollary 4.9. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > b+2$ *are such that, for* $k \ge 0$, $0 \le \sigma < 1$ *and*

$$\frac{(\alpha - 3\gamma)}{2|\tau|(1-\beta)(k+2)}\Big((c-1)^2 + (2b+1)(c-1) + b(b+1)\Big) - b(c-1) \ge 0. \tag{4.11}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $\mathcal{L}(b, c)(f)(z) \in k{-}UCV$.

Let $\mathcal{S}^*_\lambda$ $(\lambda > 0)$, denotes the class of functions in $\mathcal{S}$ such that $\left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda$.

A sufficient condition for $f \in \mathcal{A}$ of the form (1.1) to be in $\mathcal{S}^*_1 \subset \mathcal{S}^*$, is given by $\sum_{n=2}^{\infty} n|a_n| \le 1$, and is proved by many authors. For example, see [6]. A particular extension of this, due to [13], is

$$\sum_{n=2}^{\infty}(n+\lambda-1)|a_n| \le \lambda \Longrightarrow f \in \mathcal{S}^*_\lambda. \tag{4.12}$$

Theorem 4.10. *Let $a, b > 0$ or $a \in \mathbb{C}\backslash\{0\}$ with $a = \bar{b}$. Further, let $|a| \neq 1$, $|b| \neq 1$, and $0 \neq c > a+b$ be such that*

$$F(a,b;c;1)\left(1+\frac{(\lambda-1)(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right) \le \frac{(\lambda-1)(c-1)}{(|a|-1)(|b|-1)} + \lambda\left(1+\frac{(\alpha-3\gamma)}{2|\tau|(1-\beta)}\right). \tag{4.13}$$

Suppose that $f \in \mathcal{A}$ be defined as in (1.1). Then, for $f \in R^\tau_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$, $0 \le \beta < 1$, and $\lambda > 0$, $H_{a,\,b,\,c}(f)(z) \in \mathcal{S}^*_\lambda$.

Proof. Let $f(z)$ be of the form (1.1). In view of (4.12), it suffices to prove that

$$\sum_{n=2}^{\infty}(n+\lambda-1)|A_n| \le \lambda, \tag{4.14}$$

where

$$A_n = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}\, a_n, \qquad n \ge 2.$$

Since $f \in R^\tau_{\gamma,\alpha}(\beta)$, using (2.5) and $1+\alpha(n-1)+\gamma(n-1)(n-2) \ge n(\alpha-3\gamma)$, we need only to show that

$$T := \sum_{n=2}^{\infty}\frac{(|a|,n-1)(|b|,n-1)}{(c,n-1)(1,n-1)} + (\lambda-1)\sum_{n=2}^{\infty}\frac{(|a|,n-1)(|b|,n-1)}{(c,n-1)(1,n)} \le \frac{\lambda\,(\alpha-3\gamma)}{2|\tau|(1-\beta)}.$$

But this last inequality is true by the hypothesis of the theorem and (2.3). □

5. Inclusion properties of $K^p_a[f](z)$

Theorem 5.1. *Let $f \in \mathcal{A}$ be as in (1.1). Suppose $a > -1, p \ge 0$ and*

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\sigma)]B_n(a,p) \le \frac{(1-\sigma)(\alpha-3\gamma)}{2|\tau|(1-\beta)}, \tag{5.1}$$

where $B_n(a,p) = \dfrac{(1+a)^p}{(n+a)^p}$. Then for $f \in \mathcal{R}^\tau_{\gamma,\alpha}(\beta), 0 \le \gamma \le 1, 0 \le \alpha \le 1$ and $0 \le \beta < 1$, we have $K^p_a[f](z) \in k-UCV(\sigma)$.

Proof. Since $f \in \mathcal{R}^\tau_{\gamma,\alpha}(\beta)$, we have from Lemma 2.2 and the fact

$$1+\alpha(n-1)+\gamma(n-1)(n-2) \ge n(\alpha-3\gamma), n \ge 2$$

that

$$|a_n| \le \frac{2|\tau|(1-\beta)}{n(\alpha-3\gamma)}.$$

Now using Lemma 4.2, it is enough to show that

$$\sum_{n=2}^{\infty} n\left[n(1+k)-(k+\sigma)\right]|A_n| \le 1-\sigma,$$

where $A_n = B_n(a,p)a_n$. Clearly, the above inequality is true if (5.1) holds. □

It is easy to see that, for all $n \ge 2$,

$$B_n(a,p) = \frac{(1+a)^p}{(n+a)^p} < 1, \qquad a > -1, \quad p \ge 0$$

which leads to

Corollary 5.2. *Let $f \in \mathcal{A}$ be as in (1.1). Suppose $a > -1, p \ge 0$ and*

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\sigma)] \le \frac{(1-\sigma)(\alpha-3\gamma)}{2|\tau|(1-\beta)}.$$

Then for $f \in \mathcal{R}^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ and $0 \le \beta < 1$, we have $K^p_a[f](z) \in k - UCV(\sigma)$.

Theorem 5.3. *Let $p \ge 0$, $a > -1$ and $f \in \mathcal{A}$ be as in (1.1). Suppose that*

$$\sum_{n=2}^{\infty}[n+\lambda-1]\frac{B_n(a,p)}{n} \le \frac{\lambda(\alpha-3\gamma)}{2|\tau|(1-\beta)}, \tag{5.2}$$

where $B_n(a,p) = \dfrac{(1+a)^p}{(n+a)^p}$. Then for $f \in \mathcal{R}^{\tau}_{\gamma,\alpha}(\beta), 0 \le \gamma \le 1, 0 \le \alpha \le 1$ and $0 \le \beta < 1$, we have $K^p_a[f](z) \in \mathcal{S}^*_\lambda$.

Proof. Since $f \in \mathcal{R}^{\tau}_{\gamma,\alpha}(\beta)$, Lemma 2.2 gives

$$|a_n| \le \frac{2|\tau|(1-\beta)}{1+\alpha(n-1)+\gamma(n-1)(n-2)}.$$

Using the fact that $1+\alpha(n-1)+\gamma(n-1)(n-2) \ge n(\alpha-3\gamma), n \ge 2$, we obtain

$$|a_n| \le \frac{2|\tau|(1-\beta)}{n(\alpha-3\gamma)}. \tag{5.3}$$

Now $K^p_a[f](z) \in \mathcal{S}^*_\lambda$ if

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n+\lambda-1]\left|\frac{(1+a)^p}{(n+a)^p}a_n\right| \le \lambda \\
\Longrightarrow\quad &\sum_{n=2}^{\infty}[n+\lambda-1]\frac{(1+a)^p}{(n+a)^p}|a_n| \le \lambda \\
\Longrightarrow\quad &\sum_{n=2}^{\infty}[n+\lambda-1]\frac{(1+a)^p}{(n+a)^p}\frac{2|\tau|(1-\beta)}{n(\alpha-3\gamma)} \le \lambda, \qquad \text{using (5.3)} \\
\Longrightarrow\quad &\sum_{n=2}^{\infty}[n+\lambda-1]\frac{(1+a)^p}{(n+a)^p}\frac{1}{n} \le \frac{\lambda(\alpha-3\gamma)}{2|\tau|(1-\beta)},
\end{aligned}$$

which is the hypothesis and the proof is complete. □

6. Concluding remarks

Remark 6.1. If $k = 0$ then it is clear from the analytic characterization that $k - UCV(\sigma)$ reduces to the class of Convex functions of order σ, denoted by $\mathcal{C}(\sigma)$. Similarly, (using Alexander transform), $k - \mathcal{S}_p(\sigma)$ reduces to the class of Starlike functions of order σ, $(\mathcal{S}^*(\sigma))$. For results regarding to these classes we refer to [6]. Further results on the restriction $k = 0$ can be found in the literature, e.g. see [8].

Remark 6.2. We note that Theorem 4.3 and Theorem 4.5 are not sharp. In particular, for a, b real with $\eta = 0$, $k = 0$ and $\sigma = 0$, we get from (2.5),

$$F(|a|+1, |b|+1; c+1; 1)\frac{|ab|}{c} + (c - |a| - |b| - 1) \leq 1 + \frac{\alpha - 3\gamma}{2(1-\beta)}. \tag{6.1}$$

This inequality for $\alpha = 1$ and $\gamma = 0$ further reduces to

$$F(|a|+1, |b|+1; c+1; 1)\frac{|ab|}{c} + (c - |a| - |b| - 1) \leq 1 + \frac{1}{2(1-\beta)}. \tag{6.2}$$

Similarly, (4.7) reduces to

$$F(|a|+1, |b|+1; c+1; 1)\frac{|ab|}{c} \leq \frac{1}{4(1-\beta)}. \tag{6.3}$$

From (6.2) and (6.3), it is easy to see that Theorem 4.5 is better for all c lying between $|a| + |b| + 1$ and $|a| + |b| + \frac{3}{2}$ and for all other values of c satisfying $c > |a| + |b| + \frac{3}{2}$, Theorem 4.3 is better.

Note that, in Theorem 4.10, $|a| \neq 1$ and $|b| \neq 1$. Hence Theorem 4.10 cannot be reduced to the important transforms such as Carlson-Schaffer integral operator, which leads to the following.

Problem 6.3. To find conditions on b and c such that the Carlson-Schaffer operator $\mathcal{L}(b, c)(f)(z)$ maps the class $R^{\tau}_{\gamma,\alpha}(\beta)$ onto S^*_{λ}.

Note that, for $p = 1$, the results given in Section 5 for the Komatu operator $K^p_a[f](z)$ reduce to the results for the Bernardi integral operator and coincide with the results of Section 4 for particular values of a, b and c. However, for no values of p or a, the Komatu operator $K^p_a[f](z)$ can be reduced to the Carlson-Schaffer operator $\mathcal{L}(b, c)(f)(z)$. Hence Problem 6.3 gains further significance.

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