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Korovkin type theorem in the space $\tilde{C}_b[0,\infty)$

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. A Korovkin type theorem is established in the space $\tilde{C}_b[0,\infty)$ of all uniformly continuous and bounded functions on $[0,\infty)$ for a sequence of positive linear operators, the approximation error being estimated with the aid of the usual modulus of continuity. As applications we obtain quantitative results for q-Baskakov operators.

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1. Introduction

The well-known Korovkin's theorem ensures the convergence of sequences of positive linear operators to the identity operator in the strong operator topology. For $C[0,1]$ the Banach space of all continuous functions f on $[0,1]$ equipped with the norm $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$, and for the test-functions $e_i(x) = x^i$, $x \in [0,1]$, $i \in \{0,1,2\}$, Korovkin's theorem is the following (see [5, p. 8]): *let* $(L_n)_{n\geq 1}$ *be a sequence of positive linear operators such that* $L_n : C[0,1] \to C[0,1]$. *Then* $\|L_n f - f\| \to 0$ *as* $n \to \infty$ *for all* $f \in C[0,1]$ *if and only if* $\|L_n e_i - e_i\| \to 0$ *as* $n \to \infty$ *for* $i \in \{0,1,2\}$. Specifically we recover Weierstrass' approximation theorem if we choose for L_n the Bernstein operators $B_n : C[0,1] \to C[0,1]$ defined by

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{1.1}$$

The so-called q-Bernstein operators were introduced by Phillips [12], and they are generalization of (1.1) based on q-integers. To present these operators we recall some notions of the q-calculus (see e.g. [11]). Let $q > 0$. For each non-negative integer n, the q-integers $[n]_q$ and the q-factorials $[n]_q!$ are defined by

$$[n]_q = \begin{cases} 1+q+\ldots+q^{n-1}, & \text{if} \quad n \geq 1 \\ 0, & \text{if} \quad n=0 \end{cases}$$

and

$$[n]_q! = \begin{cases} [1]_q[2]_q\ldots[n]_q, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n=0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Then the q-Bernstein operators $B_{n,q} : C[0,1] \to C[0,1]$ are introduced as

$$(B_{n,q}f)(x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k(1-x)(1-qx)\ldots(1-q^{n-k-1}x) f\left(\frac{[k]_q}{[n]_q}\right). \tag{1.2}$$

For $q=1$, we recover the operators (1.1). If $0<q<1$, then $B_{n,q}$ are positive linear operators. However, they do not satisfy the conditions of Korovkin's theorem, because $(B_{n,q}e_0)(x) = 1$, $(B_{n,q}e_1)(x) = x$ and

$$(B_{n,q}e_2)(x) = x^2 + \frac{1}{[n]_q}x(1-x) \to x^2 + (1-q)x(1-x) \neq x^2,$$

as $n \to \infty$ (see [12, pp. 513-514]). The investigation of convergence of operators (1.2) for $0<q<1$ fixed has resulted in the discovery of a Korovkin type theorem in $C[0,1]$ due to Wang [14]. Applying Wang's result to (1.2), there exists a limit operator $B_{\infty,q}$ on $C[0,1]$ such that $(B_{n,q}f)_{n\geq 1}$ converges to $B_{\infty,q}f$ uniformly on $[0,1]$ as $n \to \infty$. The operator $B_{\infty,q}$ was introduced by Il'inskii and Ostrovska [10], and it is called the limit q-Bernstein operator. Furthermore, in [6] and [7], we established new Korovkin type theorems for parameter depending sequences of operators defined on $C[0,1]$; these results are different from Wang's result.

On the other hand, in [8] and [9], Korovkin type theorems are studied in weighted spaces, showing that the direct analogue of Korovkin's theorem is not valid in spaces of functions defined on the semi-axis $[0,\infty)$ or on the whole real line, but under additional conditions can be obtained analogous theorem to Korovkin's theorem. Let φ be a strictly increasing continuous function on $[0,\infty)$ such that $\lim_{x\to\infty}\varphi(x) = +\infty$ and $\rho(x) = (1+\varphi^2(x))^{-1}$, $x \geq 0$. Further, let $B_\rho[0,\infty)$ be the set of all functions f satisfying the condition $\rho(x)|f(x)| \leq M_f$ for $x \geq 0$, where M_f is a positive constant depending only on f. We denote by $C_\rho[0,\infty)$ the space $C[0,\infty) \cap B_\rho[0,\infty)$ with the norm $\|f\|_\rho = \sup\{\rho(x)|f(x)| : x \geq 0\}$, and $C_\rho^*[0,\infty) = \{f \in C_\rho[0,\infty) : \lim_{x\to\infty}\rho(x)|f(x)| < \infty\}$. Gadjiev was the first in noticing the relevance of the spaces $C_\rho^*[0,\infty)$ in proving Korovkin type theorems. We have the following result [8]: *let* $(A_n)_{n\geq 1}$ *be a sequence of positive linear operators acting from* $C_\rho[0,\infty)$ *to* $B_\rho[0,\infty)$ *satisfying the conditions* $\lim_{n\to\infty}\|A_n\varphi^i - \varphi^i\|_\rho = 0$ *for* $i \in \{0,1,2\}$. *Then* $\lim_{n\to\infty}\|A_nf - f\|_\rho = 0$ *for any* $f \in C_\rho^*[0,\infty)$.

In what follows, let $C_b[0,\infty)$ be the space of all continuous and bounded functions f on $[0,\infty)$, equipped with the norm $\|f\| = \sup\{|f(x)| : x \geq 0\}$. Further, we set $\tilde{C}_b[0,\infty) = \{f \in C_b[0,\infty) : f$ is uniformly continuous on $[0,\infty)\}$. We consider the function $\rho \in C_b[0,\infty)$ such that $\rho(x) > 0$ for all $x \geq 0$, and the space $C_\rho[0,\infty) = \{f \in C[0,\infty) : \rho f$ is bounded on $[0,\infty)\}$ equipped with the norm $\|f\|_\rho = \sup\{\rho(x)|f(x)| : x \geq 0\}$. Obviously $C_\rho[0,\infty)$ is a Banach space, and for $\rho(x) = 1$, $x \geq 0$, we have $C_\rho[0,\infty) = C_b[0,\infty)$. The goal of the paper is to establish a Korovkin type theorem for a sequence of positive linear operators $(L_n)_{n\geq 1}$, where $L_n : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ and $(L_n)_{n\geq 1}$ converges to its limit operator $L_\infty : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$, which is not necessarily the identity operator. The approximation error $\|L_n f - L_\infty f\|_\rho$ will be estimated with the aid of the usual modulus of continuity of $f \in \tilde{C}_b[0,\infty)$ defined by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,\infty), |x-y| \leq \delta\}, \quad \delta > 0. \tag{1.3}$$

As applications we obtain quantitative estimates for some q-Baskakov operators.

2. Main result

For $W = \{g \in C_b[0,\infty) : g' \in C_b[0,\infty)\}$, $f \in C_b[0,\infty)$ and $\delta > 0$, the K-functional defined by $K(f;\delta) = \inf\{\|f-g\| + \delta\|g'\| : g \in W\}$ and the modulus of continuity (1.3) are equivalent (see [5, p. 177, Theorem 2.4]), i.e. there exists $C > 0$ such that

$$C^{-1}\omega(f;\delta) \leq K(f;\delta) \leq C\omega(f;\delta). \tag{2.1}$$

Throughout this paper C denotes positive constant independent of n and x, but not necessarily the same in different cases.

The next theorem is our Korovkin type theorem.

Theorem 2.1. *Let $(L_n)_{n\geq 1}$, $L_n : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ be a sequence of positive linear operators, and let $(\alpha_n)_{n\geq 1}$ be a positive sequence with $\alpha_n \to 0$ as $n \to \infty$. If the sequence $(\beta_n)_{n\geq 1}$ satisfies the conditions*

(i) $\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1} \leq C\alpha_n$ for all $n, p \geq 1$,
(ii) $\|L_n g - L_{n+1} g\|_\rho \leq C\beta_n \|g'\|$ for all $g \in W$ and $n \geq 1$,

then there exists a positive linear operator $L_\infty : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ such that $\|L_n f - L_\infty f\|_\rho \to 0$ as $n \to \infty$, where $f \in \tilde{C}_b[0,\infty)$ is arbitrary. Moreover

$$\|L_n f - L_\infty f\|_\rho \leq c\,\omega(f;\alpha_n) \tag{2.2}$$

for all $f \in \tilde{C}_b[0,\infty)$ and $n \geq 1$; c is a constant depending only on $\|L_1 e_0\|_\rho$.

Proof. By (i) and (ii), we have

$$\begin{aligned} \|L_n g - L_{n+p} g\|_\rho &\leq \|L_n g - L_{n+1} g\|_\rho + \|L_{n+1} g - L_{n+2} g\|_\rho + \ldots \\ &\quad + \|L_{n+p-1} g - L_{n+p} g\|_\rho \\ &\leq C(\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1})\|g'\| \\ &\leq C\alpha_n \|g'\| \end{aligned} \tag{2.3}$$

for all $g \in W$ and $n, p \geq 1$. Because $e_0 \in W$, we find, in view of (2.3), that $L_n e_0 = L_{n+p} e_0$ for $n, p \geq 1$. Hence

$$L_n e_0 = L_1 e_0 \tag{2.4}$$

for all $n \geq 1$. Further, $e_0 \in \tilde{C}_b[0,\infty)$ implies that $L_1 e_0 \in C_\rho[0,\infty)$, i.e.

$$\|L_1 e_0\|_\rho < \infty. \tag{2.5}$$

Taking into account that L_n are positive linear operators and (2.4) is satisfied, we obtain

$$\begin{aligned} \rho(x)|(L_n f)(x)| &\equiv \rho(x)|L_n(f,x)| \leq \rho(x)L_n(|f|,x) \leq \rho(x)L_n(\|f\|e_0,x) \\ &= \rho(x)\|f\|L_n(e_0,x) = \rho(x)\|f\|(L_n e_0)(x) \\ &= \rho(x)\|f\|(L_1 e_0)(x), \end{aligned}$$

where $f \in \tilde{C}_b[0,\infty)$ and $x \in [0,\infty)$. Hence, by (2.5),

$$\|L_n f\|_\rho \leq \|L_1 e_0\|_\rho \|f\| \tag{2.6}$$

for every $f \in \tilde{C}_b[0,\infty)$. Using (2.3) and (2.6), we find for arbitrary $g \in W$ that

$$\begin{aligned} \|L_n f - L_{n+p} f\|_\rho &\leq \|L_n f - L_n g\|_\rho + \|L_n g - L_{n+p} g\|_\rho \\ &\quad + \|L_{n+p} g - L_{n+p} f\|_\rho \\ &\leq 2\|L_1 e_0\|_\rho \|f-g\| + C\alpha_n \|g'\| \\ &\leq \max\{2\|L_1 e_0\|_\rho, C\}\{\|f-g\| + \alpha_n \|g'\|\}. \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W$, we get

$$\|L_n f - L_{n+p} f\|_\rho \leq \max\{2\|L_1 e_0\|_\rho, C\} K(f;\alpha_n).$$

Hence, by (2.1),

$$\|L_n f - L_{n+p} f\|_\rho \leq c\,\omega(f;\alpha_n), \tag{2.7}$$

where c depends on $\|L_1 e_0\|_\rho$. Further, for $f \in \tilde{C}_b[0,\infty)$ and $\alpha_n \to 0$ as $n \to \infty$, we have $\omega(f;\alpha_n) \to 0$ as $n \to \infty$. Thus, by (2.7), we obtain that $(L_n f)_{n\geq 1}$ is a Cauchy sequence in the Banach space $C_\rho[0,\infty)$. Therefore there exists an operator L_∞ on $\tilde{C}_b[0,\infty)$ such that $\|L_n f - L_\infty f\|_\rho \to 0$ for every $f \in \tilde{C}_b[0,\infty)$. This also implies that L_∞ is a positive linear operator on $\tilde{C}_b[0,\infty)$, because $L_n : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ are positive linear operators, $n \geq 1$. Now let $p \to \infty$ in (2.7), then we obtain the estimation (2.2), which completes the proof of the theorem. □

3. Applications

In what follows we shall use the following notation:

$$(z;q)_n = (1-z)(1-qz)\dots(1-q^{n-1}z),$$

where z is a real number, $0 < q < 1$ and $n = 1, 2, \dots$ Then

$$\left(\frac{q^2 x}{1+x}; q\right)_n = \left(1 - \frac{q^2 x}{1+x}\right)\left(1 - \frac{q^3 x}{1+x}\right)\dots\left(1 - \frac{q^{n+1} x}{1+x}\right)$$

and

$$(-qx;q)_{n+k} = (1+qx)(1+q^2 x)\dots(1+q^{n+k} x)$$

for $x \geq 0$ and $k = 0, 1, 2, \ldots$

In [2], Aral and Gupta introduced the operators $B^*_{n,q} : C_b[0,\infty) \to C[0,\infty)$, where $n = 1, 2, \ldots$ and $0 < q < 1$, given by

$$(B^*_{n,q}f)(x) = \left(\frac{q^2x}{1+x}; q\right)_n \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k. \tag{3.1}$$

In [13], C. Radu defined the operators $V^*_{n,q} : C_b[0,\infty) \to C[0,\infty)$,

$$(V^*_{n,q}f)(x) = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{(qx)^k}{(-qx;q)_{n+k}} f\left(\frac{[k]_q}{[n]_q q^{k-1}}\right), \tag{3.2}$$

where $n = 1, 2, \ldots$ and $0 < q < 1$ (see also [3, (2.1)]). When $q = 1$, the operators $B^*_{n,q}$ and $V^*_{n,q}$ become the classical Baskakov operator [4].

For (3.1) we compute the difference $(B^*_{n,q}g)(x) - (B^*_{n+1,q}g)(x)$, where $g \in W$ and $x \geq 0$. We have

$$\begin{aligned}
&(B^*_{n,q}g)(x) - (B^*_{n+1,q}g)(x) \\
&= \left(\frac{q^2x}{1+x}; q\right)_n \sum_{k=0}^{\infty} g\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k \\
&\quad - \left(\frac{q^2x}{1+x}; q\right)_{n+1} \sum_{k=0}^{\infty} g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k \\
&= \left(\frac{q^2x}{1+x}; q\right)_n \sum_{k=0}^{\infty} \left\{ g\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \right. \\
&\quad \left. - \frac{1+x(1-q^{n+2})}{1+x} g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \right\} \left(\frac{q^2x}{1+x}\right)^k \\
&= \left(\frac{q^2x}{1+x}; q\right)_n \sum_{k=1}^{\infty} \left\{ g\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \right. \\
&\quad \left. - g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \right\} \left(\frac{q^2x}{1+x}\right)^k + \left(\frac{q^2x}{1+x}; q\right)_n \\
&\quad \times \sum_{k=0}^{\infty} \left(1 - \frac{1+x(1-q^{n+2})}{1+x}\right) g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{q^2x}{1+x}\right)^k \\
&= \left(\frac{q^2x}{1+x}; q\right)_n \sum_{k=0}^{\infty} \left\{ g\left(\frac{[k+1]_q}{q^{k+2}[n]_q}\right) \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q - g\left(\frac{[k+1]_q}{q^{k+2}[n+1]_q}\right) \right. \\
&\quad \left. \times \begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q + g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^n \right\} \left(\frac{q^2x}{1+x}\right)^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left\{ \left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q \left(g\left(\frac{[k+1]_q}{q^{k+2}[n]_q}\right) - g\left(\frac{[k+1]_q}{q^{k+2}[n+1]_q}\right)\right)\right. \\
&\quad \left. + q^n \left[\begin{array}{c} n+k \\ k \end{array}\right]_q \left(g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) - g\left(\frac{[k+1]_q}{q^{k+2}[n+1]_q}\right)\right)\right\} \left(\frac{q^2x}{1+x}\right)^{k+1} \\
&= \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left\{ \left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q \int_{[k+1]_q/q^{k+2}[n+1]_q}^{[k+1]_q/q^{k+2}[n]_q} g'(u)\,du \right. \\
&\quad \left. + q^n \left[\begin{array}{c} n+k \\ k \end{array}\right]_q \int_{[k+1]_q/q^{k+2}[n+1]_q}^{[k]_q/q^{k+1}[n+1]_q} g'(u)\,du \right\} \left(\frac{q^2x}{1+x}\right)^{k+1},
\end{aligned}
$$

where we have used

$$
\left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q + q^n \left[\begin{array}{c} n+k \\ k \end{array}\right]_q = \left[\begin{array}{c} n+k+1 \\ k+1 \end{array}\right]_q .
$$

Hence

$$
\begin{aligned}
&|(B^*_{n,q}g)(x) - (B^*_{n+1,q}g)(x)| \\
&\leq \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left\{ \left[\begin{array}{c} n+k \\ k+1 \end{array}\right]_q \left| \frac{[k+1]_q}{q^{k+2}[n]_q} - \frac{[k+1]_q}{q^{k+2}[n+1]_q} \right| \right. \\
&\quad \left. + q^n \left[\begin{array}{c} n+k \\ k \end{array}\right]_q \left| \frac{[k]_q}{q^{k+1}[n+1]_q} - \frac{[k+1]_q}{q^{k+2}[n+1]_q} \right| \right\} \left(\frac{q^2x}{1+x}\right)^{k+1} \|g'\| \\
&= 2\|g'\| \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k \\ k \end{array}\right]_q \frac{q^n}{[n+1]_q} \frac{1}{q^{k+2}} \left(\frac{q^2x}{1+x}\right)^{k+1} \\
&= \frac{2q^{n-1}}{[n+1]_q} \|g'\| \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k \\ k \end{array}\right]_q \left(\frac{qx}{1+x}\right)^{k+1}. \qquad (3.3)
\end{aligned}
$$

Because (see [1, p. 420])

$$
\sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q z^k = (1-z)^{-1}(1-qz)^{-1}\dots(1-q^{n-1}z)^{-1}, \quad |z|<1,
$$

we have, by (3.3),

$$
\begin{aligned}
&|(B^*_{n,q}g)(x) - (B^*_{n+1,q}g)(x)| \\
&\leq \frac{2q^{n-1}}{[n+1]_q} \|g'\| \left(1-\frac{q^2x}{1+x}\right)\left(1-\frac{q^3x}{1+x}\right)\dots\left(1-\frac{q^{n+1}x}{1+x}\right) \\
&\quad \times \frac{qx}{1+x}\left(1-\frac{qx}{1+x}\right)^{-1}\left(1-\frac{q^2x}{1+x}\right)^{-1}\dots\left(1-\frac{q^{n+1}x}{1+x}\right)^{-1} \\
&= \frac{2q^{n-1}}{[n+1]_q} \|g'\| \frac{qx}{1+x} \frac{1+x}{1+x(1-q)} \\
&\leq \frac{2q^{n-1}}{[n+1]_q} \|g'\| \frac{q}{1-q} = \frac{2q^n}{1-q^{n+1}} \|g'\|. \qquad (3.4)
\end{aligned}
$$

We set $\beta_n = q^n/(1-q^{n+1})$, $n = 1,2,\dots$ Then

$$\begin{aligned}\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1} &= \frac{q^n}{1-q^{n+1}} + \frac{q^{n+1}}{1-q^{n+2}} + \ldots + \frac{q^{n+p-1}}{1-q^{n+p}} \\ &\le \frac{q^n}{1-q^{n+1}}(1+q+\ldots+q^{p-1}) \\ &\le \frac{q^n}{(1-q)(1-q^{n+1})} \qquad (3.5)\end{aligned}$$

for all $n, p = 1,2,\dots$ Due to (3.4) and (3.5), we can apply Theorem 2.1 (case $\rho(x) = 1$, $x \ge 0$) with $\alpha_n = q^n/(1-q)(1-q^{n+1})$, $n = 1,2,\dots$ Thus we obtain the following

Theorem 3.1. *For the operators $B^*_{n,q}$ defined by (3.1) and $q \in (0,1)$ given, there exists a positive linear operator $B^*_{\infty,q} : \tilde{C}_b[0,\infty) \to C_b[0,\infty)$ such that*

$$\|B^*_{n,q} f - B^*_{\infty,q} f\| \le C\,\omega(f; q^n/(1-q)(1-q^{n+1}))$$

for all $f \in \tilde{C}_b[0,\infty)$ and $n = 1,2,\dots$

Here C is independent of $\|B^*_{1,q} e_0\|$, because $B^*_{n,q} e_0 = e_0$ (see [2, Lemma 2]) implies that $\|B^*_{n,q} f\| \le \|f\|$, $f \in \tilde{C}_b[0,\infty)$. This justifies that $B^*_{n,q} f \in C_b[0,\infty)$ for $f \in \tilde{C}_b[0,\infty)$.

Now we shall study the sequence $(V^*_{n,q})_{n\ge 1}$ defined by (3.2). In the same way as above, we obtain the following representation for $(V^*_{n,q} g)(x) - (V^*_{n+1,q} g)(x)$, where $g \in W$ and $x \ge 0$:

$$\begin{aligned}&(V^*_{n,q} g)(x) - (V^*_{n+1,q} g)(x) \\ &\quad = \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \Bigg\{ q^n g\left(\frac{[k]_q}{[n+1]_q q^{k-1}}\right) \\ &\qquad - \frac{[n+k+1]_q}{[k+1]_q}\, g\left(\frac{[k+1]_q}{[n+1]_q q^k}\right) + \frac{[n]_q}{[k+1]_q}\, g\left(\frac{[k+1]_q}{[n]_q q^k}\right)\Bigg\}\end{aligned}$$

(see also [3, Theorem 6]). Hence, by $[n+k+1]_q = [n]_q + q^n[k+1]_q$, we get

$$\begin{aligned}&(V^*_{n,q} g)(x) - (V^*_{n+1,q} g)(x) \\ &= \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \Bigg\{ q^n \Bigg(g\left(\frac{[k]_q}{[n+1]_q q^{k-1}}\right) \\ &\quad - g\left(\frac{[k+1]_q}{[n+1]_q q^k}\right)\Bigg) + \frac{[n]_q}{[k+1]_q}\left(g\left(\frac{[k+1]_q}{[n]_q q^k}\right) - g\left(\frac{[k+1]_q}{[n+1]_q q^k}\right)\right)\Bigg\} \\ &= \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \Bigg\{ q^n \int_{[k+1]_q/[n+1]_q q^k}^{[k]_q/[n+1]_q q^{k-1}} g'(u)\,du \\ &\quad + \frac{[n]_q}{[k+1]_q} \int_{[k+1]_q/[n+1]_q q^k}^{[k+1]_q/[n]_q q^k} g'(u)\,du \Bigg\}.\end{aligned}$$

Then

$$
\begin{aligned}
&|(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x)| \\
&\le \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \left[\begin{array}{c} n+k \\ k \end{array} \right]_q \left\{ q^n \left| \frac{[k]_q}{[n+1]_q q^{k-1}} - \frac{[k+1]_q}{[n+1]_q q^k} \right| \right. \\
&\quad \left. + \frac{[n]_q}{[k+1]_q} \left| \frac{[k+1]_q}{[n]_q q^k} - \frac{[k+1]_q}{[n+1]_q q^k} \right| \right\} \|g'\| \\
&= \frac{2q^n}{[n+1]_q} \|g'\| \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \left[\begin{array}{c} n+k \\ k \end{array} \right]_q . \qquad (3.6)
\end{aligned}
$$

Because of [13, Remark 4], we have

$$
(V^*_{n+1,q}e_0)(x) = \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k \\ k \end{array} \right]_q q^{k(k-1)/2} \frac{(qx)^k}{(-qx;q)_{n+k+1}} = 1.
$$

Therefore, by (3.6), we obtain

$$
|(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x)| \le \frac{2q^{n+1}x}{[n+1]_q} \|g'\|
$$

or

$$
\frac{1}{1+qx} |(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x)| \le \frac{2q^n}{[n+1]_q} \|g'\|.
$$

With the notation $\rho(x) = 1/(1+qx)$, $x \ge 0$, we have

$$
\|V^*_{n,q}g - V^*_{n+1,q}g\|_\rho \le \frac{2q^n}{[n+1]_q} \|g'\|. \qquad (3.7)
$$

Now we set $\beta_n = q^n/[n+1]_q$, $n = 1, 2, \ldots$ Then

$$
\begin{aligned}
\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1} &\le \frac{q^n}{[n+1]_q}(1 + q + \ldots + q^{p-1}) \\
&\le \frac{q^n}{1 - q^{n+1}} \qquad (3.8)
\end{aligned}
$$

for all $n, p = 1, 2, \ldots$ Due to (3.7) and (3.8), we can apply Theorem 2.1 with $\alpha_n = q^n/(1 - q^{n+1})$, $n = 1, 2, \ldots$ In conclusion we obtain the following

Theorem 3.2. *For the operators $V^*_{n,q}$ defined by (3.2), $q \in (0,1)$ given and $\rho(x) = 1/(1+qx)$, $x \ge 0$, there exists a positive linear operator $V^*_{\infty,q} : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ such that*

$$
\|V^*_{n,q}f - V^*_{\infty,q}f\|_\rho \le C\,\omega(f; q^n/(1 - q^{n+1}))
$$

for all $f \in \tilde{C}_b[0,\infty)$ and $n = 1, 2, \ldots$

The constant C is independent of $\|V^*_{1,q}e_0\|_\rho$, because

$$
\begin{aligned}
\|V^*_{n,q}f\|_\rho &= \sup\{\rho(x)|(V^*_{n,q}f)(x)| : x \ge 0\} \le \sup\{|(V^*_{n,q}f)(x)| : x \ge 0\} \\
&\le \|f\| \sup\{(V^*_{n,q}e_0)(x) : x \ge 0\} = \|f\| \sup\{e_0(x) : x \ge 0\} = \|f\|,
\end{aligned}
$$

where $f \in \tilde{C}_b[0,\infty)$ (see [13, Remark 4]).

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