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Approximation by max-product operators of Kantorovich type

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Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. We associate to various linear Kantorovich type approximation operators, nonlinear max-product operators for which we obtain quantitative approximation results in the uniform norm, shape preserving properties and localization results.

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1. Introduction

The general form of a linear and positive discrete operator attached to $f : I \to [0, +\infty)$ can be defined by

$$D_n(f)(x) = \sum_{k\in I_n} p_{n,k}(x) f(x_{n,k}), x \in I, n \in \mathbb{N},$$

where $p_{n,k}(x)$ are various kinds of function basis on I with $\sum_{k\in I_n} p_{n,k}(x) = 1$, I_n are finite or infinite families of indices and $\{x_{n,k}; k \in I_n\}$ represents a division of I.

Based on the Open Problem 5.5.4, pp. 324-326 in [7], to each $D_n(f)(x)$, can be attached the max-product type operator defined by

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k\in I_n} p_{n,k}(x) \cdot f(x_{n,k})}{\bigvee_{k\in I_n} p_{n,k}(x)}, x \in I, n \in \mathbb{N}. \tag{1.1}$$

Here $\bigvee_{k\in A} a_k = \sup_{k\in A} a_k$.

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Thus, in a series of papers we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated case), Baskakov operators (truncated and nontruncated case), Meyer-König and Zeller operators and Bleimann-Butzer-Hahn operators. All these results were collected in the very recent research monograph [2].

Remark 1.1. The max-product operators can also be naturally called as **possibilistic operators**, since they can be obtained by analogy with the Feller probabilistic scheme used to generate positive and linear operators, by replacing the probability (σ-additive), with a maxitive set function and the classical integral with the possibilistic integral (see, e.g. [2], Chapter 10, Section 10.2). If, for example, $p_{n,k}(x)$, $n \in \mathbb{N}$, $k = 0, \dots, n$ is a polynomial basis, then the operators $L_n^{(M)}(f)(x)$ become piecewise rational functions.

Now, to each max-product operator $L_n^{(M)}$, we can formally attach its Kantorovich variant, defined by

$$LK_n^{(M)}(f)(x) = \frac{\bigvee_{k \in I_n} p_{n,k}(x) \cdot (1/(x_{n,k+1} - x_{n,k})) \cdot \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt}{\bigvee_{k \in I_n} p_{n,k}(x)}, \tag{1.2}$$

with $\{x_{n,k}; k \in I_n\}$ a division of the finite or infinite interval I.

The goal of this paper is to study these Kantorovich-type versions for various max-product operators. Firstly, we prove that these operators are subadditive, positively homogeneous and monotone. For continuous functions we prove quantitative estimates, in most of the cases very good Jackson type estimates, shape preserving properties and localization results.

2. Uniform and pointwise approximation

Keeping the notations in the formulas (1.1) and (1.2), let us denote

$$C_+(I) = \{f : I \to \mathbb{R}_+; f \text{ is continuous on } I\},$$

where I is a bounded or unbounded interval and suppose that all $p_{n,k}(x)$ are continuous functions on I, satisfying $p_{n,k}(x) \geq 0$, for all $x \in I, n \in \mathbb{N}, k \in I_n$ and $\sum_{k \in I_n} p_{n,k}(x) = 1$, for all $x \in I, n \in \mathbb{N}$.

In many cases, for the Kantorovich max-product operator $K_n^{(M)}$ we could deduce quantitative estimates in approximation, by using the elaborated methods we used for the Bernstein-type max-product in the book [2]. However, here we will use a more simple method, which will be based on the already obtained estimates for the original type max-product operators denoted by $L_n^{(M)}$.

Firstly, we present the following result.

Lemma 2.1. *(i) For any $f \in C_+(I)$, $LK_n^{(M)}(f)$ is continuous on I.*

(ii) If $f \leq g$ then $LK_n^{(M)}(f) \leq LK_n^{(M)}(g)$.

(iii) $LK_n^{(M)}(f + g) \leq LK_n^{(M)}(f) + LK_n^{(M)}(g)$.

(iv) If $f \in C_+(I)$ and $\lambda \geq 0$ then $LK_n^{(M)}(\lambda f) = \lambda LK_n^{(M)}(f)$.

(v) If $LK_n^{(M)}(e_0) = e_0$, where $e_0(x) = 1$, for all $x \in I$, then for any $f \in C_+(I)$, we have

$$\left|LK_n^{(M)}(f)(x) - f(x)\right| \leq \left[1 + \frac{1}{\delta} LK_n^{(M)}(\varphi_x)(x)\right] \omega_1(f;\delta),$$

for any $x \in I$ and $\delta > 0$. Here, $\varphi_x(t) = |t - x|$, $t \in I$ and

$$\omega_1(f;\delta) = \sup\{|f(x) - f(y)|;\ x, y \in I,\ |x - y| \leq \delta\}.$$

(vi) $\left|LK_n^{(M)}(f) - LK_n^{(M)}(g)\right| \leq LK_n^{(M)}(|f - g|)$.

Proof. The proofs of (i)-(iv) are immediate from the definition of $K_n^{(M)}$. As for the proof of (v) and (vi), we exactly follow the proof of e.g., Theorem 1.1.2, pp. 16-17 in [2]. □

Lemma 2.2. *With the notations in (1.1) and (1.2), suppose that, in addition,*

$$|x_{n,k+1} - x_{n,k}| \leq \frac{C}{n+1}$$

for all $k \in I_n$, with $C > 0$ an absolute constant. Then, for all $x \in I$ and $n \in \mathbb{N}$, we have

$$LK_n^{(M)}(\varphi_x)(x) \leq L_n^{(M)}(\varphi_x)(x) + \frac{C}{n+1}.$$

Proof. If $f \in C_+(I)$, then by the integral mean value theorem, there exists $\xi_{n,k} \in (x_{n,k}, x_{n,k+1})$, such that

$$\int_{x_{n,k}}^{x_{n,k+1}} f(t)dt = (x_{n,k+1} - x_{n,k}) \cdot f(\xi_{n,k}),$$

which immediately leads to

$$LK_n^{(M)}(f)(x) = \frac{\bigvee_{k\in I_n} p_{n,k}(x) \cdot f(\xi_{n,k})}{\bigvee_{k\in I_n} p_{n,k}(x)}. \tag{2.1}$$

Applying this form for $f(t) = \varphi_x(t)$, we get

$$LK_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k\in I_n} p_{n,k}(x) \cdot |\xi_{n,k} - x|}{\bigvee_{k\in I_n} p_{n,k}(x)}$$

$$\leq \frac{\bigvee_{k\in I_n} p_{n,k}(x) \cdot |\xi_{n,k} - x_{n,k}|}{\bigvee_{k\in I_n} p_{n,k}(x)} + L_n^{(M)}(\varphi_x)(x) \leq \frac{C}{n+1} + L_n^{(M)}(\varphi_x)(x),$$

which proves the lemma. □

Corollary 2.3. *With the notations in (1.1) and (1.2) and supposing that, in addition,*

$$|x_{n,k+1} - x_{n,k}| \leq \frac{C}{n+1}$$

for all $k \in I_n$, for any $f \in C_+(I)$, we have

$$\left|LK_n^{(M)}(f)(x) - f(x)\right| \leq 2\left[\omega_1(f; L_n^{(M)}(\varphi_x)(x)) + \omega_1(f; C/(n+1))\right] \tag{2.2}$$

for any $x \in I$ and $n \in \mathbb{N}$.

Proof. By using Lemma 2.2, from the estimate in Lemma 2.1, (v), we immediately get

$$\left|LK_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1(f; L_n^{(M)}(\varphi_x)(x) + C/(n+1))$$
$$\leq 2\left[\omega_1(f; L_n^{(M)}(\varphi_x)(x)) + \omega_1(f; C/(n+1))\right],$$

which proves the corollary. □

This corollary shows that knowing quantitative estimates in approximation by a given max-product operator, we can deduce a quantitative estimate for its Kantorovich variant. Also, this method does not worsen the orders of approximation of the original operators. Let us exemplify below for several known max-product operators.

Firstly, let us choose $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, $I = [0,1]$, $I_n = \{0, \ldots, n-1\}$ and $x_{n,k} = \frac{k}{n+1}$. In this case, $L_n^{(M)}$ in (1.1) become the Bernstein max-product operators. Let us denote by $BK_n^{(M)}$ their Kantorovich variant, given by the formula

$$BK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n}{k}x^k(1-x)^{n-k} \cdot (n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{n} \binom{n}{k}x^k(1-x)^{n-k}}. \tag{2.3}$$

We can state the following result.

Theorem 2.4. *(i) If $f \in C_+([0,1])$, then we have*

$$|BK_n^{(M)}(f)(x) - f(x)| \leq 24\omega_1(f; 1/\sqrt{n+1}) + 2\omega_1(f; 1/(n+1)), x \in [0,1], n \in \mathbb{N}.$$

(ii) If $f \in C_+([0,1])$ is concave on $[0,1]$, then we have

$$|BK_n^{(M)}(f)(x) - f(x)| \leq 6\omega_1(f; 1/n), x \in [0,1], n \in \mathbb{N}.$$

(iii) If $f \in C_+([0,1])$ is strictly positive on $[0,1]$, then we have

$$|BK_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1(f; 1/n) \cdot \left(\frac{n\omega_1(f; 1/n)}{m_f} + 4\right) + 2\omega_1(f; 1/n),$$

for all $x \in [0,1], n \in \mathbb{N}$, where $m_f = \min\{f(x); x \in [0,1]\}$.

Proof. (i) is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 2.1.5, p. 30, in [2].

(ii) is immediate from Corollary 2.3 (with $C = 1$) and from Corollary 2.1.10, p. 36 in [2].

(iii) is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 2.2.18, p. 63 in [2]. □

Now, let us choose $p_{n,k}(x) = \frac{(nx)^k}{k!}$, $I = [0, +\infty)$, $I_n = \{0, \ldots, n, \ldots,\}$ and $x_{n,k} = \frac{k}{n+1}$. In this case, $L_n^{(M)}$ in (1.1) become the non-truncated Favard-Szász-Mirakjan max-product operators. Let us denote by $FK_n^{(M)}$ their Kantorovich variant defined by

$$FK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}. \tag{2.4}$$

We can state the following result.

Theorem 2.5. *(i) If $f : [0, +\infty) \to [0, +\infty)$ is bounded and continuous on $[0, +\infty)$, then we have*

$$|FK_n^{(M)}(f)(x) - f(x)| \leq 16\omega_1(f; \sqrt{x}/\sqrt{n}) + 2\omega_1(f; 1/n), x \in [0, +\infty), n \in \mathbb{N}.$$

(ii) If $f : [0, +\infty) \to [0, +\infty)$ is continuous, bounded, non-decreasing, concave function on $[0, +\infty)$, then we have

$$|FK_n^{(M)}(f)(x) - f(x)| \leq 4\omega_1(f; 1/n), x \in [0, +\infty), n \in \mathbb{N}.$$

Proof. (i) is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 3.1.4, p. 162, in [2].

(ii) is immediate from Corollary 2.3 (with $C = 1$) and from Corollary 3.1.8, p. 168 in [2]. □

If we choose $p_{n,k}(x) = \frac{(nx)^k}{k!}$, $I = [0, 1]$, $I_n = \{0, \ldots, n\}$ and $x_{n,k} = \frac{k}{n+1}$. In this case, $L_n^{(M)}$ in (1.1) become the truncated Favard-Szász-Mirakjan max-product operators. Let us denote by $TK_n^{(M)}$ their Kantorovich variant given by the formula

$$TK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \frac{(nx)^k}{k!} \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{n} \frac{(nx)^k}{k!}}. \tag{2.5}$$

We can state the following result.

Theorem 2.6. *(i) If $f \in C_+([0, 1])$, then we have*

$$|TK_n^{(M)}(f)(x) - f(x)| \leq 12\omega_1(f; 1/\sqrt{n}) + 2\omega_1(f; 1/n), x \in [0, 1], n \in \mathbb{N}.$$

(ii) If $f \in C_+([0, 1])$ is non-decreasing, concave function on $[0, 1]$, then we have

$$|TK_n^{(M)}(f)(x) - f(x)| \leq 4\omega_1(f; 1/n), x \in [0, 1], n \in \mathbb{N}.$$

Proof. (i) is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 3.2.5, p. 178, in [2].

(ii) is immediate from Corollary 2.3 (with $C = 1$) and from Corollary 3.2.7, p. 182 in [2]. □

Now, let us choose $p_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$, $I = [0, +\infty)$, $I_n = \{0, \ldots, n, \ldots, \}$ and $x_{n,k} = \frac{k}{n+1}$. In this case, $L_n^{(M)}$ in (1.1) become the non-truncated Baskakov max-product operators. Let us denote by $VK_n^{(M)}$ their Kantorovich variant defined by

$$VK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}}. \tag{2.6}$$

We can state the following result.

Theorem 2.7. *(i) If $f : [0,+\infty) \to [0,+\infty)$ is bounded and continuous on $[0,+\infty)$, then for all $x \in [0,+\infty)$ and $n \geq 3$, we have*

$$|VK_n^{(M)}(f)(x) - f(x)| \leq 24\omega_1(f; \sqrt{x(x+1)}/\sqrt{n-1}) + 2\omega_1(f; 1/(n+1)).$$

(ii) If $f : [0,+\infty) \to [0,+\infty)$ is continuous, bounded, non-decreasing, concave function on $[0,+\infty)$, then for $x \in [0,+\infty)$ and $n \geq 3$ we have

$$|VK_n^{(M)}(f)(x) - f(x)| \leq 4\omega_1(f; 1/n).$$

Proof. (i) is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 4.1.6, p. 196, in [2].

(ii) is immediate from Corollary 2.3 (with $C = 1$) and from Corollary 4.1.9, p. 206 in [2]. □

If we choose $p_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$, $I = [0,1]$, $I_n = \{0, \ldots, n\}$ and $x_{n,k} = \frac{k}{n+1}$, then in this case, $L_n^{(M)}$ in (1.1) become the truncated Baskakov max-product operators. Let us denote by $UK_n^{(M)}$ their Kantorovich variant defined by

$$UK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}}. \tag{2.7}$$

We can state the following result.

Theorem 2.8. *(i) If $f \in C_+([0,1])$, then we have,*

$$|UK_n^{(M)}(f)(x) - f(x)| \leq 48\omega_1(f; 1/\sqrt{n+1}) + 2\omega_1(f; 1/(n+1)), x \in [0,1], n \geq 2.$$

(ii) If $f \in C_+([0,1])$ is non-decreasing, concave function on $[0,1]$, then we have

$$|UK_n^{(M)}(f)(x) - f(x)| \leq 6\omega_1(f; 1/n), x \in [0,1], n \in \mathbb{N}.$$

Proof. (i) is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 4.2.6, p. 217, in [2].

(ii) is immediate from Corollary 2.3 (with $C = 1$) and from Corollary 4.2.9, p. 223 in [2]. □

Now, let us choose $p_{n,k}(x) = \binom{n+k}{k} x^k$, $I = [0,1]$, $I_n = \{0, \ldots, n, \ldots\}$ and $x_{n,k} = \frac{k}{n+1+k}$. In this case, $L_n^{(M)}$ in (1.1) become the Meyer-König and Zeller max-product operators. Also, it is easy to see that $|x_{n,k+1} - x_{n,k}| \leq \frac{1}{n+1}$, for all $k \in I_n$. Let us denote by $ZK_n^{(M)}$ their Kantorovich variant defined by

$$ZK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \binom{n+k}{k} x^k \cdot \frac{(n+k+1)(n+k+2)}{n+1} \int_{k/(n+1+k)}^{(k+1)/(n+k+2)} f(t)dt}{\bigvee_{k=0}^{\infty} \binom{n+k}{k} x^k}. \tag{2.8}$$

The following result holds.

Theorem 2.9. *(i) If $f \in C_+([0,1])$, then for $n \geq 4$, $x \in [0,1]$, we have*

$$|ZK_n^{(M)}(f)(x) - f(x)| \leq 36\omega_1(f; \sqrt{x}(1-x)/\sqrt{n}) + 2\omega_1(f; 1/n).$$

(ii) If $f \in C_+([0,1])$ is non-decreasing concave function on $[0,1]$, then for $x \in [0,1]$ and $n \geq 2x$ we have

$$|ZK_n^{(M)}(f)(x) - f(x)| \leq 4\omega_1(f; 1/n).$$

Proof. (i) is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 6.1.4, p. 248, in [2].

(ii) is immediate from Corollary 2.3 (with $C = 1$) and from Corollary 6.1.7, p. 256 in [2]. □

In what follows, let us choose $p_{n,k}(x) = h_{n,k}(x)$-the fundamental Hermite-Fejér interpolation polynomials based on the Chebyshev knots of first kind

$$x_{n,k} = \cos\left(\frac{2(n-k)+1}{2(n+1)}\pi\right),$$

$I = [-1,1]$, and $I_n = \{0, \ldots, n\}$. In this case, $L_n^{(M)}$ in (1.1) become the Hermite-Fejér max-product operators. Also, applying he mean value theorem to cos, it is easy to see that $|x_{n,k+1} - x_{n,k}| \leq \frac{4}{n+1}$, for all $k \in I_n$. Let us denote by $HK_n^{(M)}$ their Kantorovich variant defined by

$$HK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} h_{n,k}(x) \cdot \frac{1}{x_{n,k}-x_{n,k+1}} \cdot \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt}{\bigvee_{k=0}^{\infty} h_{n,k}(x)}, \tag{2.9}$$

where $x_{n,k} = \cos\left(\frac{2(n-k)+1}{2(n+1)}\pi\right)$.

The following result holds.

Theorem 2.10. *If $f \in C_+([-1,1])$, then for $n \in \mathbb{N}$, $x \in [-1,1]$, we have*

$$|HK_n^{(M)}(f)(x) - f(x)| \leq 30\omega_1(f; 1/n).$$

Proof. It is immediate from Corollary 2.3 (with $C = 4$) and from Theorem 7.1.5, p. 286, in [2]. □

Now, let us consider choose $p_{n,k}(x) = e^{-|x-k/(n+1)|}$, $I = (-\infty, +\infty)$, $I_n = \mathbb{Z}$-the set of integers and $x_{n,k} = \frac{k}{n+1}$. In this case, $L_n^{(M)}$ in (1.1) become the Picard max-product operators. Let us denote by $\mathcal{P}K_n^{(M)}$ their Kantorovich variant defined by

$$\mathcal{P}K_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} e^{-|x-k/(n+1)|} \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{\infty} e^{-|x-k/(n+1)|}}. \tag{2.10}$$

We can state the following result.

Theorem 2.11. *If $f : \mathbb{R} \to [0, +\infty)$ is bounded and uniformly continuous on $\mathbb{R}$, then we have*

$$|\mathcal{P}K_n^{(M)}(f)(x) - f(x)| \leq 6\omega_1(f; 1/n), x \in \mathbb{R}, n \in \mathbb{N}.$$

Proof. It is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 10.3.1, p. 423, in [2]. □

In what follows, let us choose $p_{n,k}(x) = e^{-(x-k/(n+1))^2}$, $I = (-\infty, +\infty)$, $I_n = \mathbb{Z}$-the set of integers and $x_{n,k} = \frac{k}{n+1}$. In this case, $L_n^{(M)}$ in (1.1) become the Weierstrass max-product operators. Let us denote by $\mathcal{W}K_n^{(M)}$ their Kantorovich variant defined by

$$\mathcal{W}K_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} e^{-(x-k/(n+1))^2} \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{\infty} e^{-(x-k/(n+1))^2}}. \tag{2.11}$$

We can state the following result.

Theorem 2.12. *If $f : \mathbb{R} \to [0, +\infty)$ is bounded and uniformly continuous on $\mathbb{R}$, then we have*

$$|\mathcal{W}K_n^{(M)}(f)(x) - f(x)| \leq 4\omega_1(f; 1/\sqrt{n}) + 2\omega_1(f; 1/n), x \in \mathbb{R}, n \in \mathbb{N}.$$

Proof. It is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 10.3.3, p. 425, in [2]. □

At the end of this section, let us choose $p_{n,k}(x) = \frac{1}{n^2(x-k/n)^2+1}$, $I = (-\infty, +\infty)$, $I_n = \mathbb{Z}$-the set of integers and $x_{n,k} = \frac{k}{n+1}$. In this case, $L_n^{(M)}$ in (1.1) become the Poisson-Cauchy max-product operators. Let us denote by $\mathcal{C}K_n^{(M)}$ their Kantorovich variant

$$\mathcal{C}K_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{1}{n^2(x-k/(n+1))^2+1} \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\bigvee_{k=0}^{\infty} \frac{1}{n^2(x-k/(n+1))^2+1}}. \tag{2.12}$$

We can state the following result.

Theorem 2.13. *If $f : \mathbb{R} \to [0, +\infty)$ is bounded and uniformly continuous on $\mathbb{R}$, then we have*

$$|\mathcal{C}K_n^{(M)}(f)(x) - f(x)| \leq 6\omega_1(f; 1/n), x \in \mathbb{R}, n \in \mathbb{N}.$$

Proof. It is immediate from Corollary 2.3 (with $C = 1$) and from Theorem 10.3.5, p. 426, in [2]. □

Remark 2.14. All the Kantorovich kind max-product operators $LK_n^{(M)}$ given by (1.2) are defined and used for approximation of positive valued functions. But, they can be used for approximation of lower bounded functions of variable sign too, by introducing the new operators

$$N_n^{(M)}(f)(x) = LK_n^{(M)}(f + c)(x) - c,$$

where $c > 0$ is such that $f(x) + c > 0$, for all x in the domain of definition of f.

It is easy to see that the operators $N_n^{(M)}$ give the same approximation orders as $LK_n^{(M)}$.

3. Shape preserving properties for the Bernstein-Kantorovich max-product operators

In this section we deal with the shape preserving properties of the Bernstein-Kantorovich max-product operators $BK_n^{(M)}$ given by (2.3).

We can prove the following.

Theorem 3.1. *Let $f \in C_+([0,1])$.*

(i) If f is non-decreasing (non-increasing) on $[0,1]$, then for all $n \in \mathbb{N}$, $BK_n^{(M)}(f)$ is non-decreasing (non-increasing, respectively) on $[0,1]$.

(ii) If f is quasi-convex on $[0,1]$ then for all $n \in \mathbb{N}$, $BK_n^{(M)}(f)$ is quasi-convex on $[0,1]$. Here quasi-convexity on $[0,1]$ means that $f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$, for all $x, y, \lambda \in [0,1]$.

Proof. (i) By using the formula (2.1) for $LK_n^{(M)}$, we can write $BK_n^{(M)}(f)$ under the form

$$BK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \cdot f(\xi_{n,k})}{\bigvee_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}},$$

where $\xi_{n,k} \in (x_{n,k}, x_{n,k+1})$, for all $k = 0, \dots, n$.

Then, by analogy with the proofs for the Bernstein max-product operators (see [2], pp. 39-41, the proofs for the Bernstein-Kantorovich max-product operators, will be based on the properties of the functions

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(\xi_{n,k}).$$

Now, analyzing the proofs of Lemma 2.1.13, Corollary 2.1.14, Theorem 2.1.15 and Corollary 2.1.16 in [2], pp. 39-41, it is easy to see that they work identically for the above $f_{k,n,j}$ too and we immediately obtain the required conclusions.

(ii) Since as in the case of the max-product Bernstein operators in Corollary 2.1.18, p. 41 in [2], this point is based on the properties from the above point (i) and on the properties in the above Lemma 2.1, (i)-(iv), we easily get the required conclusion for this point too. □

In what follows, we will prove that $BK_n^{(M)}$ preserves quasi-concavity too. This property holds in the case of the operator $B_n^{(M)}$ (By Theorem 5.1 in [5]). However, it is difficult to adapt the proof to our case. Instead, we can prove this property by finding a direct correspondence between the operators $B_n^{(M)}$ and $BK_n^{(M)}$.

Let us notice that the operator $BK_n^{(M)}$ can be obtained from the operator $B_n^{(M)}$. Suppose that f is arbitrary in $C_+([0,1])$. Let us consider

$$f_n(x) = (n+1) \int_{nx/(n+1)}^{(nx+1)/(n+1)} f(t)dt \tag{3.1}$$

It is readily seen that $B_n^{(M)}(f_n)(x) = BK_n^{(M)}(f)(x)$, for all $x \in [0,1]$. We also notice that $f_n \in C_+([0,1])$. What is more, if f is strictly positive then so is f_n.

A function $f : [a,b] \to \mathbb{R}$ is quasi-concave if $-f$ is quasi-convex. If f is continuous, quasi-concavity equivalently means that there exists $c \in [a,b]$ such that f is nondecreasing on $[a,c]$ and nonincreasing on $[c,b]$.

We are now in position to prove that $BK_n^{(M)}$ preserves quasi-concavity too.

Theorem 3.2. *Let* $f \in C_+([0,1])$. *If* f *is quasi-concave on* $[0,1]$ *then* $BK_n^{(M)}(f)$ *is quasi-concave on* $[0,1]$.

Proof. For some arbitrary $n \geq 1$ let us consider the function f_n given by (3.1). Moreover, let $c \in [0,1]$ such that f is nondecreasing on $[0,c]$ and nonincreasing on $[c,1]$. Then, let $j(c) \in \{0,\ldots,n\}$ such that

$$\frac{j(c)}{n+1} \leq c \leq \frac{j(c)+1}{n+1}.$$

Next, we consider the function g_n which interpolates f_n at all the knots $\frac{k}{n}$, $k = 0,1,\ldots,n$, and which is continuous on $[0,1]$ and affine on any interval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k = 0,1,\ldots,n-1$. It means that g_n is the continuous polygonal line which interpolates f_n at all the knots $\frac{k}{n}$, $k = 0,1,\ldots,n$. This easily implies that

$$B_n^{(M)}(f_n)(x) = B_n^{(M)}(g_n)(x),\ x \in [0,1],$$

hence,

$$BK_n^{(M)}(f)(x) = B_n^{(M)}(g_n)(x),\ x \in [0,1].$$

Let us now choose arbitrary $0 \leq k_1 < k_2 \leq j(c)-1$. We have

$$g_n\left(\frac{k_1}{n}\right) = (n+1)\int_{k_1/(n+1)}^{(k_1+1)/(n+1)} f(t)dt$$

and

$$g_n\left(\frac{k_2}{n}\right) = (n+1)\int_{k_2/(n+1)}^{(k_2+1)/(n+1)} f(t)dt.$$

As $\frac{k_1+1}{n+1} \leq \frac{k_2}{n+1}$ and f is nondecreasing on $[0, \frac{k_2+1}{n+1}]$, we easily obtain (after applying the mean value theorem) that $g_n\left(\frac{k_1}{n}\right) \leq g_n\left(\frac{k_2}{n}\right)$. The construction of g_n easily implies that g_n is nondecreasing on $\left[0, \frac{j(c)-1}{n}\right]$. By similar reasoning we get that g_n is nonincreasing on $\left[\frac{j(c)+1}{n}, 1\right]$. Now, suppose that $f\left(\frac{j(c)}{n+1}\right) \geq f\left(\frac{j(c)+1}{n+1}\right)$. The quasi-concavity of f implies that $f(x) \geq f\left(\frac{j(c)+1}{n+1}\right)$ for any $x \in \left[\frac{j(c)}{n+1}, \frac{j(c)+1}{n+1}\right]$. Since there exists $x_0 \in \left[\frac{j(c)}{n+1}, \frac{j(c)+1}{n+1}\right]$ such that

$$(n+1)\int_{j(c)/(n+1)}^{(j(c)+1)/(n+1)} f(t)dt = f(x_0) = g_n\left(\frac{j(c)}{n}\right),$$

and since $f\left(\frac{j(c)+1}{n+1}\right) \geq g_n\left(\frac{j(c)+1}{n}\right)$ (this is true indeed as f is nondecreasing on $\left[\frac{j(c)+1}{n+1}, 1\right]$), we get that $g_n\left(\frac{j(c)}{n}\right) \geq g_n\left(\frac{j(c)+1}{n}\right)$. Therefore, g_n is nonincreasing on $\left[\frac{j(c)}{n}, \frac{j(c)+1}{n}\right]$. This implies that g_n is nondecreasing on $\left[0, \frac{j(c)-1}{n}\right]$ and nonincreasing on $\left[\frac{j(c)}{n},1\right]$. But f is affine on $\left[\frac{j(c)-1}{n},\frac{j(c)}{n}\right]$ which means that it is monotone on this interval. Clearly this implies that g_n is either nondecreasing on $\left[0,\frac{j(c)-1}{n}\right]$ and nonincreasing on $\left[\frac{j(c)-1}{n},1\right]$ or, it is nondecreasing on $\left[0,\frac{j(c)}{n}\right]$ and nonincreasing on $\left[\frac{j(c)}{n},1\right]$. It means that g_n is quasi-concave on $[0,1]$. By similar reasonings we get to the same conclusion if $f\left(\frac{j(c)}{n+1}\right)\leq f\left(\frac{j(c)+1}{n+1}\right)$. The only difference is that now g_n is either nondecreasing on $\left[0,\frac{j(c)}{n}\right]$ and nonincreasing on $\left[\frac{j(c)}{n},1\right]$ or, it is nondecreasing on $\left[0,\frac{j(c)+1}{n}\right]$ and nonincreasing on $\left[\frac{j(c)+1}{n},1\right]$. Thus, we just proved that g_n is quasi-concave on $[0,1]$. By Theorem 5.1 in [5] (see also Theorem 2.2.22 in the book, it follows that $B_n^{(M)}(g_n)$ is quasi-concave on $[0,1]$. As $B_n^{(M)}(g_n)=BK_n^{(M)}(f)$, it follows that $BK_n^{(M)}(f)$ is quasi-concave on $[0,1]$. □

As an important side remark, let us note that in Theorem 5.1 of paper [5](see also the book [2]), it is proved that if f is quasi-concave and c is a maximum point of f then there exists a maximum point of $B_n^{(M)}(f)$ such that $|c-c\prime|\leq\frac{1}{n+1}$. By the construction of g_n it follows that one maximum point of g_n is between the values $\frac{j(c)-1}{n}$, $\frac{j(c)}{n}$ or $\frac{j(c)+1}{n}$. If we denote this value with c_n then one can easily check that $|c_n-c|\leq\frac{2}{n}$. Now, applying the afore mentioned property obtained in [5], let $c\prime$ be a maximum point of $B_n^{(M)}(g_n)=BK_n^{(M)}(f)$, such that $|c\prime-c_n|\leq\frac{1}{n+1}$. This easily implies that $|c\prime-c|\leq\frac{3}{n}$. So, we obtained a quite similar result for the operator $BK_n^{(M)}$ in comparison with the operator $B_n^{(M)}$.

4. Approximation of Lipschitz functions by Bernstein-Kantorovich max-product operators

Let us return to the functions f_n given in (3.1) and let us find now an upper bound for the approximation of f by f_n in terms of the uniform norm. For some $x\in[0,1]$, using the mean value theorem, there exists $\xi_x\in\left[\frac{nx}{n+1},\frac{nx+1}{n+1}\right]$ such that $f_n(x)=f(\xi_x)$. We also easily notice that $|\xi_x-x|\leq\frac{1}{n+1}$. It means that

$$|f(x)-f_n(x)|\leq\omega_1(f;1/(n+1)),x\in\mathbb{R},n\in\mathbb{N}. \tag{4.1}$$

In particular, if f is Lipschitz with constant C then f_n is Lipschitz continuous with constant $3C$. These estimation are useful to prove some inverse results in the case of the operator $BK_n^{(M)}$ by using analogue results already obtained for the operator $B_n^{(M)}$.

Below we present a result which gives for the class of Lipschitz function the order of approximation $1/n$ in the approximation by the operator $BK_n^{(M)}$, hence an analogue result which holds in the case of the operator $B_n^{(M)}$.

Theorem 4.1. *Suppose that f is Lipschitz on $[0,1]$ with Lipschitz constant C and suppose that the lower bound of f is $m_f > 0$. Then we have*

$$\left\| BK_n^{(M)}(f) - f \right\| \leq 2C \left(\frac{C}{m_f} + 5 \right) \cdot \frac{1}{n},\ n \geq 1.$$

Proof. The estimation is immediate using the estimation from Corollary 2.4, (iii), taking into account that $\omega_1(f; 1/n) \leq C/n$. □

5. Localization results for Bernstein-Kantorovich max-product operators

We firstly prove a very strong localization property of the operator $BK_n^{(M)}$.

Theorem 5.1. *Let $f, g : [0,1] \to [0,\infty)$be both bounded on $[0,1]$ with strictly positive lower bounds and suppose that there exist $a, b \in [0,1]$, $0 < a < b < 1$ such that $f(x) = g(x)$ for all $x \in [a,b]$. Then for all $c, d \in [a,b]$ satisfying $a < c < d < b$ there exists $\widetilde{n} \in \mathbb{N}$ depending only on f, g, a, b, c, d such that $BK_n^{(M)}(f)(x) = BK_n^{(M)}(g)(x)$ for all $x \in [c,d]$and $n \in \mathbb{N}$ with $n \geq \widetilde{n}$.*

Proof. Let us choose arbitrary $x \in [c,d]$ and for each $n \in \mathbb{N}$ let $j_x \in \{0, 1, \ldots, n\}$ be such that $x \in [j_x/(n+1), (j_x+1)/(n+1)]$. Then by relation (4.17) in [1] we have

$$BK_n^{(M)}(f)(x) = B_n^{(M)}(f_n)(x) = \bigvee_{k=0}^{n} (f_n)_{k,n,j_x}(x), \tag{5.1}$$

where for $k \in \{0, 1, \ldots, n\}$ we have

$$(f_n)_{k,n,j_x} = \frac{\binom{n}{k}}{\binom{n}{j_x}} \left(\frac{x}{1-x} \right)^{k-j_x} f_n\left(\frac{k}{n} \right). \tag{5.2}$$

and each f_n is given by (3.1). Let us denote with m_f, M_f and m_{f_n}, M_{f_n} respectively, the minimums and maximum values of the functions f and f_n, respectively. By the mean value theorem, one can easily notice that for any $x \in [0,1]$ there exists $\xi_{n,x} \in [0,1]$ such that $f_n(x) = f(\xi_{n,x})$. It means that $0 < m_f \leq m_{f_n} \leq M_{f_n} \leq M_f$. In what follows, the proof is very similar to the proof of Theorem 2.1 in [6] (see also Theorem 2.4.1 in [2]). However, as often we will use f_n instead of f, especially since the constants obtained in the proof of Theorem 2.1 in [6] depend on f, in our setting these constants would depend on f_n, hence, they would depend on n, if we would apply directly the results in [6]. Therefore, there are some differences in the two proofs as our intention is to obtain constants that do not depend on f_n.

We need the set $I_{n,x} = \{k \in \{0, 1, \ldots, n\} : j_x - a_n \leq k \leq j_x + a_n\}$, where $a_n = \left[\sqrt[3]{n^2} \right]$ (here $[a]$ denotes the integer part of a). Now, suppose that $k \notin I_{n,x}$, and let us discuss first the case when $k < j_x - a_n$. If we look over the proof of Theorem 2.1 in [6], we observe that this proof is split in cases i) and ii). Case i) corresponds to the case when $k < j_x - a_n$. Furthermore this case is divided in two subcases i_a) and i_b). In subcase i_a) the inequality

$$\frac{f_{j_x,n,,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(1+\frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{f(j_x/n)}{f(k/n)}$$

is obtained which then gives

$$\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(1+\frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{m_f}{M_f}.$$

Applying this reasoning but considering f_n instead of f, we get

$$\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)} \geq \left(1+\frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{f_n(j_x/n)}{f_n(k/n)}.$$

But since $m_f \leq m_{f_n} \leq M_{f_n} \leq M_f$, we get

$$\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)} \geq \left(1+\frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{m_f}{M_f}.$$

We get the same conclusion all cases and subcases, that is, any lower bound for $\frac{f_{j_x,n,,j_x}(x)}{f_{k,n,j_x}(x)}$ is also a lower bound for $\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)}$, for any k outside of $I_{n,x}$. Since in..., we proved that there exists $N_0 \in \mathbb{N}$ which may depend only on f, a, b, c, d, such that for any $n \geq N_0$, $k \in \{0, 1, \ldots, n\}$, with $k < j_x - a_n$ or $k > j_x + a_n$, we have $\frac{f_{j_x,n,,j_x}(x)}{f_{k,n,j_x}(x)} \geq 1$, it follows that $\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)} \geq 1$, for any $n \geq N_0$, $k \in \{0, 1, \ldots, n\}$, with $k < j_x - a_n$ or $k > j_x + a_n$. Combining this fact with relations (5.1)-(5.2), we get that

$$BK_n^{(M)}(f)(x) = \bigvee_{k \in I_{n,x}} (f_n)_{k,n,j_x}(x), x \in [c,d],\ n \geq N_0.$$

Using a similar reasoning as in the proof of Theorem 2.1 in [6], in what follows, we will prove that N_0 can be replaced if necessary with a larger value $\widetilde{N}_1$ such that $[\frac{k}{n+1}, \frac{k+1}{n+1}] \subseteq [a,b]$ for any $k \in I_{n,x}$. Let us choose arbitrary $x \in [c,d]$ and $n \in \mathbb{N}$ so that $n \geq N_0$. If there exists $k \in I_{n,x}$ such that $k/(n+1) \notin [c,d]$ then we distinguish two cases. Either $\frac{k}{n+1} < c$ or $\frac{k}{n+1} > d$. In the first case we observe that

$$0 < c - \frac{k}{n+1} \leq x - \frac{k}{n+1} \leq \frac{j_x+1}{n+1} - \frac{k}{n+1} \leq \frac{j_x+1}{n+1} - \frac{k}{n+1} \leq \frac{a_n+1}{n+1}.$$

Since $\lim_{n\to\infty} \frac{a_n+1}{n+1} = 0$, it results that for sufficiently large n we necessarily have $\frac{a_n+1}{n+1} < c - a$ which clearly implies that $\frac{k}{n+1} \in [a,c]$. In the same manner, when $\frac{k}{n+1} > d$, for sufficiently large n we necessarily have $\frac{k}{n+1} \in [d,b]$. By similar reasoning it results that for sufficiently large n we necessarily have $\frac{k}{n+1} \in [a,b]$.Summarizing, there exists a constant $\widetilde{N}_1 \in \mathbb{N}$ independent of any $x \in [c,d]$ such that

$$BK_n^{(M)}(f)(x) = \bigvee_{k \in I_{n,x}} (f_n)_{k,n,j_x}(x),\ x \in [c,d],\ n \geq \widetilde{N}_1$$

and in addition for any $x \in [c,d]$, $n \geq \widetilde{N}_1$ and $k \in I_{n,x}$, we have $[\frac{k}{n+1}, \frac{k+1}{n+1}] \subseteq [a,b]$. Also, it is easy to check that $\widetilde{N}_1$ depends only on a,b,c,d,f.

Now, for $k \in \{0,1,\ldots,n\}$ taking

$$(g_n)_{k,n,j_x} = \frac{\binom{n}{k}}{\binom{n}{j_x}} \left(\frac{x}{1-x}\right)^{k-j_x} g_n\left(\frac{k}{n}\right),$$

applying the same reasoning, there exists $\widetilde{N}_2 \in \mathbb{N}$ which may depend only on a,b,c,d,g, such that

$$BK_n^{(M)}(g)(x) = \bigvee_{k \in I_{n,x}} (g_n)_{k,n,j_x}(x),\ x \in [c,d],\ n \geq \widetilde{N}_2$$

and in addition for any $x \in [c,d]$, $n \geq \widetilde{N}_2$ and $k \in I_{n,x}$, we have $[\frac{k}{n+1}, \frac{k+1}{n+1}] \subseteq [a,b]$. Since $f(x) = g(x)$, $x \in [a,b]$, we get that for any $n \geq \widetilde{n} = \max\{\widetilde{N}_1, \widetilde{N}_2\}$, $k \in I_{n.x}$ and $x \in [c,d]$, it holds that $(f_n)_{k,n,j_x}(x) = (g_n)_{k,n,j_x}(x)$. Thus, for any $n \geq \widetilde{n}$ and $x \in [c,d]$, we have $BK_n^{(M)}(f)(x) = BK_n^{(M)}(g)(x)$. The proof is complete now. □

As in the case of the Bernstein max-product operator, we can present a local direct approximation result as an immediate consequence of the localization result in Theorem 5.1.

Corollary 5.2. *Let $f : [0,1] \to [0,\infty)$ be bounded on $[0,1]$ with the lower bound strictly positive and $0 < a < b < 1$ be such that $f|_{[a,b]} \in Lip\,[a,b]$ with Lipschitz constant $\overline{C}$. Then, for any $c,d \in [0,1]$ satisfying $a < c < d < b$, we have*

$$\left|BK_n^{(M)}(f)(x) - f(x)\right| \leq \frac{C}{n} \text{ for all } n \in \mathbb{N} \text{ and } x \in [c,d],$$

where the constant C depends only on f and a,b,c,d.

Proof. Let us define the function $F : [0,1] \to \mathbb{R}$,

$$F(x) = \begin{cases} f(a) & \text{if} \quad x \in [0,a], \\ f(x) & \text{if} \quad x \in [a,b], \\ f(b) & \text{if} \quad x \in [b,1]. \end{cases}$$

The hypothesis immediately imply that F is a strictly positive Lipschitz function on $[0,1]$. Then, according to Theorem 4.1 and noting that the minimum of F is above the minimum of f, m_f, it results that

$$\left|BK_n^{(M)}(F)(x) - F(x)\right| \leq 2\overline{C}\left(\frac{\overline{C}}{m_f} + 5\right) \cdot \frac{1}{n}, \text{for all } x \in [0,1], n \in \mathbb{N}.$$

Now, let us choose arbitrary $c,d \in [a,b]$ such that $a < c < d < b$. Then, by Theorem 5.1 it results the existence of $\widetilde{n} \in \mathbb{N}$ which depends only on a,b,c,d,f,F such that $BK_n^{(M)}(F)(x) = BK_n^{(M)}(f)(x)$ for all $x \in [c,d]$. But since actually the function F depends on the function f, by simple reasonings we get that in fact $\widetilde{n}$ depends only on a,b,c,d and f.

Therefore, for arbitrary $x \in [c,d]$ and $n \in \mathbb{N}$ with $n \geq \widetilde{n}$ we obtain

$$\left|BK_n^{(M)}(f)(x) - f(x)\right| = \left|BK_n^{(M)}(F)(x) - F(x)\right| \leq 2\overline{C}\left(\frac{\overline{C}}{m_f} + 5\right) \cdot \frac{1}{n},$$

where C_1 and $\widetilde{n}$ depend only on a, b, c, d and f.

Now, denoting

$$C_2 = \max_{1 \leq n < \widetilde{n}} \{n \cdot \|BK_n^{(M)}(f) - f\|_{[c,d]}\},$$

we finally obtain

$$|BK_n^{(M)}(f)(x) - f(x)| \leq \frac{C}{n}, \text{for all } n \in \mathbb{N},\ x \in [c,d],$$

with $C = \max\{2\overline{C}\left(\frac{\overline{C}}{m_f} + 5\right), C_2\}$ depending only on a, b, c, d and f. □

In a previous section we proved that $BK_n^{(M)}$ preserves monotonicity and more generally quasi-convexity. By the localization result in Theorem 5.1 and then applying a very similar reasoning to the one used in the proof of Corollary 5.2, we obtain local versions for these shape preserving properties. Indeed, in all cases it will suffice to consider the same F as in the proof of Corollary 5.2 as this function will be monotone or quasi-convex/quasi-concave, respectively, whenever f will be monotone or quasi-convex/quasi-concave, respectively. For this reason we omit the proofs of the following corollaries (see also the corresponding local shape preserving properties proved for the operator $B_n^{(M)}$ in [6]).

Corollary 5.3. *Let $f : [0,1] \to [0,\infty)$be bounded on $[0,1]$ with strictly positive lower bound and suppose that there exists $a, b \in [0,1]$, $0 < a < b < 1$, such that f is nondecreasing (nonincreasing) on $[a,b]$. Then for any $c, d \in [a,b]$ with $a < c < d < b$, there exists $\widetilde{n} \in \mathbb{N}$ depending only on a, b, c, d and f, such that $B_n^{(M)}(f)$ is nondecreasing (nonincreasing) on $[c,d]$ for all $n \in \mathbb{N}$ with $n \geq \widetilde{n}$.*

Corollary 5.4. *Let $f : [0,1] \to [0,\infty)$be a continuous and strictly positive function and suppose that there exists $a, b \in [0,1]$, $0 < a < b < 1$, such that f is quasi-convex on $[a,b]$. Then for any $c, d \in [a,b]$ with $a < c < d < b$, there exists $\widetilde{n} \in \mathbb{N}$ depending only on a, b, c, d and $f$such that $B_n^{(M)}(f)$ is quasi-convex on $[c,d]$ for all $n \in \mathbb{N}$ with $n \geq \widetilde{n}$.*

Corollary 5.5. *Let $f : [0,1] \to [0,\infty)$be a continuous and strictly positive function and suppose that there exists $a, b \in [0,1]$, $0 < a < b < 1$, such that f is quasi-concave on $[a,b]$. Then for any $c, d \in [a,b]$ with $a < c < d < b$, there exists $\widetilde{n} \in \mathbb{N}$ depending only on a, b, c, d and f, such that $B_n^{(M)}(f)$ is quasi-concave on $[c,d]$ for all $n \in \mathbb{N}$ with $n \geq \widetilde{n}$.*

Remark 5.6. As in the cases of Bernstein-type max-product operators studied in the research monograph [2], for the the max-product Kantorovich type operators we can find natural interpretation as possibilistic operators, which can be deduced from the Feller scheme written in terms of the possibilistic integral. These approaches also offer new proofs for the uniform convergence, based on a Chebyshev type inequality in the theory of possibility.

Remark 5.7. In the recently submitted paper [3], we have introduced the more general Kantorovich max-product operators based on a generalized (φ, ψ)-kernel, by the formula

$$K_n^{(M)}(f;\varphi,\psi)(x) = \frac{1}{b} \cdot \frac{\bigvee_{k=0}^{n} \frac{\varphi(nx-kb)}{\psi(nx-kb)} \cdot \left[(n+1)\int_{kb/(n+1)}^{(k+1)b/(n+1)} f(v)\,dv\right]}{\bigvee_{k=0}^{n} \frac{\varphi(nx-kb)}{\psi(nx-kb)}}, \qquad (5.3)$$

where $b > 0$, $f : [0, b] \to \mathbb{R}_+$, $f \in L^p[0, b]$, $1 \le p \le \infty$ and φ and ψ satisfy some properties specific to max-product operators and proved pointwise, uniform or L^p convergence quantitative approximation results. For particular choices of (φ, ψ), we have obtained approximation results for many other max-product Kantorovich operators, including for example the sampling operators based on sinc-type kernels.

Remark 5.8. In another recently in preparation paper [4], we have generalized the max-product Kantorovich operators from the above Remark 2), by replacing the classical linear integral $\int dv$, by the nonlinear Choquet integral $(C)\int d\mu(v)$ with respect to a monotone and submodular set function μ obtaining and studying the max-product Kantorovich-Choquet operators given by the formula

$$K_n^{(M)}(f;\varphi,\psi)(x)$$
$$= \frac{1}{b} \cdot \frac{\bigvee_{k=0}^{n} \frac{\varphi(nx-kb)}{\psi(nx-kb)} \cdot \left[(C)\int_{kb/(n+1)}^{(k+1)b/(n+1)} f(v)\,d\mu(v)/\mu\left(\left[\frac{kb}{n+1}, \frac{(k+1)b}{n+1}\right]\right)\right]}{\bigvee_{k=0}^{n} \frac{\varphi(nx-kb)}{\psi(nx-kb)}}, \qquad (5.4)$$

It is worth noting that the max-product Kantorovich-Choquet operators are **doubly nonlinear operators**: firstly due to max and secondly, due to the Choquet integral.

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