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Korovkin type approximation on an infinite interval via generalized matrix summability method using ideal

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Abstract. Following the notion of $A^{\mathcal{I}}$-summability method for real sequences [24] we establish a Korovkin type approximation theorem for positive linear operators on $UC_*[0,\infty)$, the Banach space of all real valued uniform continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0,\infty)$. In the last section, we extend the Korovkin type approximation theorem for positive linear operators on $UC_*([0,\infty)\times[0,\infty))$. We then construct an example which shows that our new result is stronger than its classical version.

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1. Introduction and background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. For a sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [17] first established the necessary and sufficient conditions for the uniform convergence of $\{L_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_1 = 1,\ e_2 = x,\ e_3 = x^2$ [1]. The study of the Korovkin type approximation theory has a long history and is a well-established area of research. In recent years, using the concept of uniform statistical convergence various statistical approximation results have been proved ([9]). Erkuş and Duman [13] studied a Korovkin type approximation theorem via A-statistical convergence in the space $H_w(I^2)$ where $I^2 = [0,\infty)\times[0,\infty)$ which was extended for double sequences of positive linear operators of two variables in A-statistical sense by Demirci and Dirik in [6, 8]. Further it was extended for double sequences of positive linear operators of two variables in $A_2^{\mathcal{I}}$-statistical sense and in the sense of $A_2^{\mathcal{I}}$-summability method, by Dutta et. al. [11, 10].

Our primary interest, in this paper, is to obtain general Korovkin type approximation theorem for positive linear operators on the space $UC_*(D)$, the Banach space of all real valued uniform continuous functions on $D := [0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and finite, endowed with the supremum norm $\|f\|_* = \sup_{x\in D} \mid f(x) \mid$ for $f \in UC_*(D)$, using the concept of $A^{\mathcal{I}}$-summability method for real sequences and test functions $1,\ e^{-x},\ e^{-y}$. In the last section, we extend the Korovkin-type approximation theorem for double sequence of positive linear operators on $UC_*([0,\infty)\times[0,\infty))$. We also construct an example which shows that our new result is stronger than its classical version.

The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [14]. Further investigations started in this area after the pioneering works of Šalát [22] and Fridy [15]. The notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et. al. [18] as a generalization of statistical convergence using the notion of ideals. On the other hand statistical convergence was generalized to A-statistical convergence by Kolk ([16]). Later a lot of works have been done on matrix summability and A-statistical convergence (see [2, 3, 5, 12, 16, 19, 23]). In particular, in [25, 24] the very general notion of $A^{\mathcal{I}}$-statistical convergence and $A^{\mathcal{I}}$-summability was introduced and studied.

Recall that a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$ there exists $N(\varepsilon)\in\mathbb{N}$ such that $|x_{mn}-L| < \varepsilon$ for all $m,n > N(\varepsilon)$ and denoted by $\lim_{m,n} x_{mn} = L$. A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \le M$ for all $(m,n)\in\mathbb{N}\times\mathbb{N}$. A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{j,k} \frac{|\{m \le j, n \le k : |x_{mn} - L| \ge \varepsilon\}|}{jk} = 0 \text{ [20]}.$$

Recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if $(i) A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; $(ii) A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e. $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$) then the family of sets $F(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Let $\mathcal{I}_0 = \{A \subset \mathbb{N}\times\mathbb{N} :$ there is $m(A) \in \mathbb{N}$ such that $i, j \ge m(A) \Longrightarrow (i,j) \notin A\}$. Then $\mathcal{I}_0$ is a non-trivial strongly admissible ideal [4].

2. A Korovkin type approximation for a sequence of positive linear operators of single variable

Throughout this section $\mathcal{I}$ denotes the non-trivial admissible ideal on $\mathbb{N}$. If $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $A = (a_{nk})_{n,k=1}^{\infty}$ is an infinite matrix, then Ax is the sequence whose n-th term is given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

A matrix A is called regular if $A \in (c, c)$ and

$$\lim_{k\to\infty} A_k\,(x) = \lim_{k\to\infty} x_k \text{ for all } x = \{x_k\}_{k\in\mathbb{N}} \in c$$

when c, as usual, stands for the set of all convergent sequences. It is well-known that the necessary and sufficient conditions for A to be regular are

$$R1) \quad ||A|| = \sup_n \sum_k |a_{nk}| < \infty;$$

$$R2) \quad \lim_n a_{nk} = 0, \text{ for each } k;$$

$$R3) \quad \lim_n \sum_k a_{nk} = 1.$$

We first recall the following definition

Definition 2.1. [25] Let $A = (a_{nk})$ be a non-negative regular summability matrix. Then a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-summable to a number L if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : |A_n(x) - L| \geq \varepsilon\} \in \mathcal{I}$ where $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$.

Thus $x = \{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-summable to a number L if and only if $\{A_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to L. In this case, we write $\mathcal{I} - \lim_n \sum_{k\in\mathbb{N}} a_{nk} x_k = L$.

It should be noted that for $\mathcal{I} = \mathcal{I}_d$, the set of all subsets of $\mathbb{N}$ with natural density zero, $A^{\mathcal{I}}$-summability reduces to statistical A-summability [12].

We now establish a Korovkin type approximation theorem for positive linear operators on $UC_*[0, \infty)$, the Banach space of all real valued uniform continuous functions on $[0, \infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0, \infty)$. If L be a positive linear operator then $L(f) \geq 0$ for any positive function f. Also we denote the value of $L(f)$ at a point $x \in [0, \infty)$ by $L(f; x)$.

Theorem 2.2. *Let $\{L_n\}$ be a sequence of positive linear operators from $UC_*[0, \infty)$ into itself and let, $A = (a_{jn})$ be a non-negative regular summability matrix then for all $f \in UC_*[0, \infty)$*

$$\mathcal{I} - \lim_n \left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\|_* = 0$$

if and only if the following statements hold

$$\mathcal{I} - \lim_n \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-pt}) - e^{-px} \right\|_* = 0, \ p = 0, 1, 2.$$

Proof. Since the necessity is clear, then it is enough to proof sufficiency. Our objective is to show that for given $\varepsilon > 0$ there exist constants C_0 , C_1 , C_2 (depending on $\varepsilon > 0$) such that

$$\left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\|_* \leq \varepsilon + C_2 \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x} \right\|_* + C_1 \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x} \right\|_* + C_0 \left\| \sum_{k=1}^{\infty} a_{nk} L_k(1) - 1 \right\|_* .$$

If this is done then our hypothesis implies that for any $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\| \geq \varepsilon \right\} \in \mathcal{I}.$$

Let $f \in UC_*[0,\infty)$ then $\exists$ a constant M such that $| f(x) | \leq M$ for each $x \in [0,\infty)$. Let ε be an arbitrary positive number. By hypothesis we may find $\delta := \delta(\varepsilon) > 0$ such that for every $t, x \in [0,\infty)$, $| e^{-t} - e^{-x} | < \delta$ implies $| f(t) - f(x) | < \varepsilon$. We can write $| f(t) - f(x) | < 2M \ \forall\ t, x \in [0,\infty)$. Also if $| e^{-t} - e^{-x} | \geq \delta$ then

$$| f(t) - f(x) | < \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Then for all $t, x \in [0,\infty)$,

$$| f(t) - f(x) | < \varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Then for $n \in \mathbb{N}$, using the linearity and the positivity of the operators L_n,

$$\begin{aligned}
\left| \sum_{k=1}^{\infty} a_{nk} L_k(f(t); x) - f(x) \right| &\leq \sum_{k=1}^{\infty} a_{nk} L_k(| f(t) - f(x) |; x) \\
&\quad + | f(x) | \left| \sum_{k=1}^{\infty} a_{nk} L_k(1; x) - 1 \right| \\
&\leq \sum_{k=1}^{\infty} a_{nk} L_k(\varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2; x) + | f(x) | \left| \sum_{k=1}^{\infty} a_{nk} L_k(1; x) - 1 \right| \\
&\leq \varepsilon + (\varepsilon + M) \left| \sum_{k=1}^{\infty} a_{nk} L_k(1; x) - 1 \right| + \frac{2M}{\delta^2} \sum_{k=1}^{\infty} a_{nk} L_k((e^{-t} - e^{-x})^2; x) \\
&\leq \varepsilon + (\varepsilon + M) \left| \sum_{k=1}^{\infty} a_{nk} L_k(1; x) - 1 \right| + \frac{2M}{\delta^2} | e^{-2x} | \left| \sum_{k=1}^{\infty} a_{nk} L_k(1; x) - 1 \right| \\
&\quad + \frac{2M}{\delta^2} \left| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}; x) - e^{-2x} \right| + \frac{4M}{\delta^2} | e^{-x} | \left| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}; x) - e^{-x} \right|
\end{aligned}$$

where $| e^{-kt} | \leq 1 \ \forall \ t \in [0,\infty)$ and $k \in \mathbb{N}$.
Then taking supremum over $x \in [0,\infty)$ we have

$$\left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\|_* \leq \varepsilon + K \left\{ \left\| \sum_{k=1}^{\infty} a_{nk} L_k(1) - 1 \right\|_* + \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x} \right\|_* + \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x} \right\|_* \right\}$$

where

$$K = \max\left\{ \varepsilon + M + \frac{2M}{\delta^2}, \frac{2M}{\delta^2}, \frac{4M}{\delta^2} \right\}.$$

For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$ let us define the following sets

$$D = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\|_* \geq r \right\}$$

$$D_1 = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(1) - 1 \right\|_* \geq \frac{r-\varepsilon}{3K} \right\}$$

$$D_2 = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x} \right\|_* \geq \frac{r-\varepsilon}{3K} \right\}$$

$$D_3 = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x} \right\|_* \geq \frac{r-\varepsilon}{3K} \right\}.$$

It follows that $D \subset D_1 \cup D_2 \cup D_3$. Since from hypotheses $D_1,\ D_2,\ D_3$ are belong to $\mathcal{I}$ so $D \in \mathcal{I}$ i.e.

$$\left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\| \geq \varepsilon \right\} \in \mathcal{I}$$

and this completes the proof. □

3. A Korovkin type approximation for a sequence of positive linear operators of two variables

Throughout this section $\mathcal{I}$ denotes the non-trivial strongly admissible ideal on $\mathbb{N}\times\mathbb{N}$. Let $A = (a_{jkmn})$ be a four dimensional summability matrix. For a given double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$, the A-transform of x, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} x_{mn}$$

provided the double series converges in Pringsheim sense for every $(j,k) \in \mathbb{N}^2$. In 1926, Robison [21] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a convergent double sequence is not necessarily bounded.

Recall that a four dimensional matrix $A = (a_{jkmn})$ is said to be RH-regular if it maps every bounded convergent double sequence into a convergent double sequence with the same limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = (a_{jkmn})$ is RH-regular if and only if

(i) $\lim_{j,k} a_{jkmn} = 0$ for each $(m, n) \in \mathbb{N}^2$,

(ii) $\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} = 1$,

(iii) $\lim_{j,k} \sum_{m\in\mathbb{N}} |a_{jkmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $\lim_{j,k} \sum_{n\in\mathbb{N}} |a_{jkmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{jkmn}|$ is convergent for each $(j, k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers M_0 and N_0 such that $\sum_{m,n>N_0} |a_{jkmn}| < M_0$ holds for every $(j, k) \in \mathbb{N}^2$.

Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix and let $K \subset \mathbb{N}^2$. Then the A-density of K is given by

$$\delta_A^{(2)}\{K\} = \lim_{j,k} \sum_{(m,n)\in K} a_{jkmn}.$$

Recall the following definition

Definition 3.1. [10] Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix. Then a real double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-summable to a number L if for every $\varepsilon > 0$, $\left\{(j, k) \in \mathbb{N}^2 : |(Ax)_{j,k} - L| \geq \varepsilon\right\} \in \mathcal{I}$.

Thus $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to a number L if and only if $(Ax)_{j,k}$ is $\mathcal{I}$-convergent to L. In this case, we write $\mathcal{I}_2 - \lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}x_{mn} = L$.

It should be noted that, if we take $\mathcal{I} = \mathcal{I}_d$, the set of all subsets of $\mathbb{N} \times \mathbb{N}$ with natural density zero, then $A_2^{\mathcal{I}}$-summability reduces to the notion of statistical A-summability for double sequence [2].

We now establish the Korovkin type approximation theorem for a double sequence of positive linear operators on $UC_* ([0, \infty) \times [0, \infty))$, the Banach space of all real valued uniform continuous functions defined on $[0, \infty) \times [0, \infty)$ with the property that $\lim_{(x,y)\to(\infty,\infty)} f(x, y)$ exists finitely for any $f \in UC_* ([0, \infty) \times [0, \infty))$ endowed with the supremum norm $||f||_* = \sup_{x,y\in[0,\infty)} |f(x, y)|$, in $A_2^{\mathcal{I}}$-summability method. If L be a positive linear operator then $L(f) \geq 0$ for any positive function f. Also we denote the value of $L(f)$ at a point $(x, y) \in [0, \infty) \times [0, \infty)$ by $L(f; x, y)$.

Theorem 3.2. *Assume $\mathcal{K} := [0, \infty) \times [0, \infty)$ and let $\{L_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators on $UC_* (\mathcal{K})$, the Banach space of all real valued uniform continuous functions defined on $\mathcal{K}$ with the property that $\lim_{(x,y)\to(\infty,\infty)} f(x, y)$ exists* *finitely for any $f \in UC_*(\mathcal{K})$ and let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix. Then for any $f \in UC_*(\mathcal{K})$,*

$$\mathcal{I}_2 - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* = 0$$

is satisfied if the following hold

$$\mathcal{I}_2 - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_* = 0, \ i = 0,1,2,3 \tag{3.1}$$

where $f_0 = 1, \ f_1 = e^{-x}, \ f_2 = e^{-y}, f_3 = e^{-2x} + e^{-2y}$.

Proof. Assume that (3.1) holds. Let $f \in UC_*(\mathcal{K})$. Our objective is to show that for given $\varepsilon > 0$ there exist constants $C_0, \ C_1, \ C_2, \ C_3$ (depending on $\varepsilon > 0$) such that

$$\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \le \varepsilon + \sum_{i=0}^{3} C_i \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_* .$$

If this is done then our hypothesis implies that for any $\varepsilon > 0$,

$$\{(j,k) \in \mathbb{N}^2 : \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \|_* \ge \varepsilon\} \in \mathcal{I}.$$

To this end, start by observing that for each $(u,v) \in \mathcal{K}$ the function $0 \le g_{uv} \in UC_*(\mathcal{K})$ defined by

$$g_{uv}(s,t) = (e^{-s} - e^{-u})^2 + ((e^{-t} - (e^{-v})^2$$

satisfies

$$g_{uv} = (e^{-x})^2 + (e^{-y})^2 - 2e^{-u}e^{-x} - 2e^{-v}e^{-y} + (e^{-u})^2 + (e^{-v})^2.$$

Since each L_{mn} is a positive operator, $L_{mn} g_{uv}$ is a positive function. In particular, we have for each $(u,v) \in \mathcal{K}$,

$$0 \le \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})(u,v)$$

$$= \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(\left(e^{-x}\right)^2 + \left(e^{-y}\right)^2 - 2e^{-u}e^{-x} - 2e^{-v}e^{-y} + \left(e^{-u}\right)^2 + \left(e^{-v}\right)^2 ; u,v \right) \right]$$

$$= \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(\left(e^{-x}\right)^2 + \left(e^{-y}\right)^2 ; u,v \right) - \left(e^{-u}\right)^2 - \left(e^{-v}\right)^2 \right]$$

$$-2e^{-u} \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(e^{-x}; u,v\right) - e^{-u} \right]$$

$$-2e^{-v} \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(e^{-y}; u,v\right) - e^{-v} \right]$$

$$+\left\{\left(e^{-u}\right)^2+\left(e^{-v}\right)^2\right\}\left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)-f_0\right]$$

$$\leq \left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_3)-f_3\right\|_* + 2e^{-u}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_1)-f_1\right\|_*$$

$$+2e^{-v}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_2)-f_2\right\|_*$$

$$+\left\{\left(e^{-u}\right)^2+\left(e^{-v}\right)^2\right\}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)-f_0\right\|_*.$$

Let $f\in UC_*(\mathcal{K})$. Then there exists a constant M such that $|f(x,y)|\leq M$ for each $(x,y)\in\mathcal{K}$. Let $\varepsilon>0$ be arbitrary. Then by the uniform continuity of f on $\mathcal{K}$ there exists a $\delta=\delta(\varepsilon)>0$ such that if $\mid e^{-x}-e^{-u}\mid<\delta$ and $\mid e^{-y}-e^{-v}\mid<\delta$ then

$$\mid f(x,y)-f(u,v)\mid<\varepsilon+\frac{2M}{\delta^2}\left[\left(e^{-x}-e^{-u}\right)^2+\left(e^{-y}-e^{-v}\right)^2\right]$$

for all $(x,y),(u,v)\in\mathcal{K}$.
Since each L_{mn} is positive and linear it follows that

$$-\varepsilon\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)-\frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv})$$

$$\leq \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f)-f(u,v)L_{mn}(f_0)$$

$$\leq\varepsilon\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)+\frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv}).$$

Therefore

$$\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f;u,v)-f(u,v)L_{mn}(f_0;u,v)\right|$$

$$\leq\varepsilon+\varepsilon\left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0;u,v)-f_0(u,v)\right]+\frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv})$$

$$\leq\varepsilon+\varepsilon\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)-f_0\right\|+\frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv}).$$

In particular, note that

$$\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f;u,v)-f(u,v)\right|$$

$$\leq \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f;u,v) - f(u,v) \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;u,v) \right|$$

$$+ \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} f(u,v) \right| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;u,v) - f_0(u,v) \right|$$

$$\leq \varepsilon + (M+\varepsilon) \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0 \right\|_* + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})$$

which implies

$$\begin{aligned}\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \leq \varepsilon &+ C_3 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_3) - f_3 \right\|_* \\ &+ C_2 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_2) - f_2 \right\|_* \\ &+ C_1 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_1) - f_1 \right\|_* \\ &+ C_0 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0 \right\|_*\end{aligned}$$

where there exist such A and B such that

$$C_0 = \left[\frac{2M}{\delta^2} \{(e^{-A})^2 + (e^{-B})^2\} + M + \varepsilon \right],\ C_1 = \frac{4M}{\delta^2} e^{-A},$$

$$C_2 = \frac{4M}{\delta^2} e^{-B} \text{ and } C_3 = \frac{2M}{\delta^2}.$$

i.e.

$$\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \leq \varepsilon + C \sum_{i=0}^{3} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_*,\ i = 0,1,2,3$$

where $C = \max\{C_0,\ C_1,\ C_2,\ C_3\}$.

For a given $\gamma > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \gamma$. Now let

$$U = \left\{ (j,k) \in \mathbb{N}^2 : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \geq \gamma \right\}$$

and

$$U_i = \left\{ (j,k) \in \mathbb{N}^2 : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_* \geq \frac{\gamma - \varepsilon}{4C} \right\},\ i = 0,1,2,3.$$

It follows that $U \subset \bigcup_{i=0}^{3} U_i$. By hypotheses each $U_i \in \mathcal{I}$, $i = 0, 1, 2, 3$ and consequently $U \in \mathcal{I}$ i.e.

$$\left\{ (j,k) \in \mathbb{N}^2 : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \geq \gamma \right\} \in \mathcal{I}.$$

This completes the proof of the theorem. □

Remark 3.3. We now show that our theorem is stronger than the statistical A-summable version [7] (and so the classical version). Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$. Choose an infinite subset $C = \{(p_i, q_i) : i \in \mathbb{N}\}$ (where $p_i \neq q_i$, $p_1 < p_2 < ...$, and $q_1 < q_2 < ...$) from $\mathcal{I} \setminus \mathcal{I}_d$ where $\mathcal{I}_d$ denotes the set of all subsets of $\mathbb{N} \times \mathbb{N}$ with natural density zero. Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ be given by

$$u_{mn} = \begin{cases} 1 & \text{if } m, n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{jkmn})$ be given by

$$a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i, m = 2p_i, n = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i, m = 2j+1, n = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$y_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} u_{mn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i \in \mathbb{N}. \end{cases}$$

Let $\varepsilon > 0$ be given. Then $\{(j,k) \in \mathbb{N}^2 : |y_{j,k} - 0| \geq \varepsilon\} = C \in \mathcal{I}$. Then the sequence $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to 0. Evidently this sequence is not statistically A-summable to 0.

Let $\mathcal{K} = [0, \infty) \times [0, \infty)$. We consider the following Baskakov operators

$$B_{mn} : UC_*(\mathcal{K}) \to UC_*(\mathcal{K})$$

defined by

$$B_{mn}(f; x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f\left(\frac{j}{n}, \frac{k}{n}\right) \binom{m-1+j}{j} \binom{n-1+k}{k} (1+x)^{-m-j} (1+y)^{-n-k} x^j y^k.$$

We now consider the double sequence $\{L_{mn}\}_{m,n\in\mathbb{N}}$ of positive linear operators defined by

$$L_{mn}(f; x, y) = (1 + u_{mn}) B_{mn}(f; x, y).$$

Then observe that

$$L_{mn}(f_0;x,y) = (1+u_{mn})f_0(x,y),$$

$$L_{mn}(f_1;x,y) = (1+u_{mn})\Big(1+x-xe^{-\frac{1}{m}}\Big)^{-m},$$

$$L_{mn}(f_2;x,y) = (1+u_{mn})\Big(1+y-ye^{-\frac{1}{n}}\Big)^{-n},$$

$$L_{mn}(f_3;x,y) = (1+u_{mn})\left[\Big(1+x-xe^{-\frac{1}{m}}\Big)^{-m}+\Big(1+y-ye^{-\frac{1}{n}}\Big)^{-n}\right].$$

Now as A is a nonnegative RH-regular summability matrix and $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to 0 then for any $\varepsilon > 0$,

$$\left\{(j,k)\in\mathbb{N}^2 : ||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_i)-f_i||_* \geq \varepsilon\right\}\in\mathcal{I},\ i=0,1,2,3.$$

Therefore by previous theorem

$$\left\{(j,k)\in\mathbb{N}^2 : ||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f)-f||_* \geq \varepsilon\right\}\in\mathcal{I}.$$

But since $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is not usual convergent and statistical A-summable so we can say that the classical version and statistical A-summable version of the previous theorem do not work for the operator defined above.

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