Stud. Univ. Babeş-Bolyai Math. 65(2020), No. 2, 303–308
DOI: 10.24193/subbmath.2020.2.11

Ulam stability of Volterra integral equation on a generalized metric space

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Abstract. The aim of this paper is to give some Ulam-Hyers stability results for Volterra integral equations on a generalized metric space. In this case we consider the Volterra integral equation in the Krasnoselski-Krein and Naguno-Perron-Van Kampen conditions. Here we present only Ulam-Hyers stability for the Volterra integral equation.

Mathematics Subject Classification (2010): 45G10, 45M10, 47H10, 47J20.

Keywords: Volterra integral equations, Ulam-Hyers stability, generalized metric space, Krasnoselski-Krein conditions, Naguno-Perron-Van Kampen conditions.

1. Introduction

The Ulam stability is an important concept in the theory of Volterra integral equations. This problem has been studied by L.P. Castro and A. Ramos [1], N. Cădariu and V. Radu [2], S.M. Jung [3], I.A. Rus [9], [10], I.A. Rus and N. Lungu [11]. But, on a generalized metric spaces this problem has been studied in the papers [1] and [10]. In what follows we shall present Ulam-Hyers stability of a Volterra integral equation on a generalized metric space, N. Lungu [5]. Here, we consider a Volterra integral equation in the Krasnoselski-Krein and Naguno-Perron-Van Kampen conditions. In the present work we consider a generalized metric space (X, d), where $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$ is a generalized metric on X. For these we need some notions and results from the generalized metric spaces theory.

Let (X, d) be a generalized metric space. On X we have the following equivalence relation:

$$x \sim y \Leftrightarrow d(x, y) < \infty, \ \forall\, x, y \in X.$$

Let $X = \bigcup\limits_{\lambda \in \Lambda} X_\lambda$ be the canonical decomposition of X after this equivalence relation. We denote

$$d_\lambda(x, y) = d(x, y)\Big|_{X_\lambda \times X_\lambda}$$

and we have that (X_λ, d_λ) is a metric space ([7]).

In this paper we need the following two theorems (see W.A.J. Luxemburg [6], I.A. Rus [7], [8]):

Theorem 1.1. *Let (X, d) be a generalized complete metric space and $A : X \to X$ an operator with the property:*

$$\exists\ \alpha \in [0, 1] \text{ such that } d(x, y) < \infty \Rightarrow d(A(x), A(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$.

If there exists $x_0 \in X$ such that $d(x_0, A(x_0)) < +\infty$, then the operator A has at least one fixed point.

Theorem 1.2. (Luxemburg-Jung). *Let (X, d) be a generalized complete metric space and his canonical decomposition $X = \bigcup X_\lambda$. If $A : X \to X$ ia a contraction, then the operator A have in every X_λ, for which exists u_λ such that*

$$d(u_\lambda, A(u_\lambda)) < +\infty,$$

a unique fixed point.

2. Ulam-Hyers stability in the generalized Krasnoselski-Krein conditions

In what follows we shall consider the following integral equation

$$u(x, y) = h(x, y) + \int_0^x \int_0^y f(s, t, u(s, t))dsdt \tag{2.1}$$

$$f : [0, a) \times [0, b) \times \mathbb{R} \to \mathbb{R},\ \ h : [0, a) \times [0, b) \to \mathbb{R},$$
$$f \in C([0, a) \times [0, b) \times \mathbb{R}, \mathbb{R}),$$
$$h \in C([0, a) \times [0, b), \mathbb{R}),\ \ u \in C([0, a) \times [0, b), \mathbb{R}),$$
$$(x, y) \in [0, a) \times [0, b),\ \ D = [0, a) \times [0, b).$$

Let X be the set:

$$X = C(D) \tag{2.2}$$

and the generalized metrics:

$$d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$$
$$d(u_1, u_2) := \sup_D \frac{|u_1(x, y) - u_2(x, y)|}{(xy)^{p\sqrt{k}}} \tag{2.3}$$

for all $u_1, u_2 \in X$, $p > 1$, $k > 0$.

It is known that the space (X, d) is a generalized complete metric space.

Let $a, b \in (0, \infty]$ and $\varepsilon > 0$. In what follows we denote by A the operator

$$A : X \to X$$

$$A(u)(x, y) := \text{the second part of (2.1)}.$$

Then the equation (2.1) becomes

$$u(x, y) = A(u)(x, y). \tag{2.4}$$

For the fixed point equation (2.4) we have:

Definition 2.1. ([10]) The equation (2.4) is Ulam-Hyers stable if there exists the positive real number $C_f > 0$ such that, for each $\varepsilon \in \mathbb{R}_+^*$ and each solution v of the inequation

$$d(v, Av) \leq \varepsilon \tag{2.5}$$

there exists a solution $u \in X$ of (2.4) such that

$$d(u, v) \leq C_f \cdot \varepsilon.$$

In this case we have

Theorem 2.2. *We suppose that:*
(i) $f : E \to \mathbb{R}$ is continuous and bounded on E, $E = D \times \mathbb{R}$;
(ii) $a < \infty$, $b < \infty$;
(iii) f verifies the generalized Krasnoselski-Krein conditions ([4])*:*

$$|f(x, y, u_1) - f(x, y, u_2)| \leq \frac{k}{xy}|u_1 - u_2|, \ k > 0 \tag{2.6}$$

$$|f(x, y, u_1) - f(x, y, u_2)| \leq \frac{c}{(xy)^\beta}|u_1 - u_2|^\alpha, \ c > 0 \tag{2.7}$$

$$\alpha \in (0, 1), \ \beta < \alpha, \ k(1-\alpha)^2 < (1-\beta)^2, \ \beta < p\sqrt{k}, \ xy \neq 0,$$
$$p^2 k(1-\alpha)^2 < (1-\beta)^2, \ \textit{for all } (x, y, u) \in E.$$

Then the equation (2.4) is Ulam-Hyers stable.

Proof. We consider $X = C(D)$ and $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Let v be a solution of the inequation (2.5) and there exists $\lambda \in \Lambda$ such that $v \in X_\lambda$. By Luxemburg-Jung theorem (Theorem 1.2), the equation (2.4) has a unique solution u in X_λ.

From (2.1), (2.5), (2.6) and (2.7), we have:

$$|v(x, y) - u(x, y)| \leq \left| v(x, y) - h(x, y) - \int_0^x \int_0^y f(s, t, v(s, t)) ds dt \right|$$
$$+ \int_0^x \int_0^y |f(s, t, v(s, t)) - f(s, t, u(s, t))| ds dt. \tag{2.8}$$

Hence, from (2.4), (2.6) and (2.7), we have

$$|v(x, y) - u(x, y)| \leq |v(x, y) - A(v)(x, y)| + \int_0^x \int_0^y \frac{k}{st} |v(s, t) - u(s, t)| ds dt,$$

or

$$|v(x, y) - u(x, y)| \leq |v(x, y) - A(v)(x, y)| + \int_0^x \int_0^y k d(u, v)(st)^{p\sqrt{k}-1} ds dt,$$

and

$$|v(x, y) - u(x, y)| \leq |v(x, y) - A(v)(x, y)| + k d(u, v) \frac{(xy)^{p\sqrt{k}}}{p^2 k},$$

from where we have

$$d(u, v) \leq \varepsilon + \frac{1}{p^2} d(u, v)$$

and

$$d(u,v) \leq \frac{p^2}{p^2-1}\varepsilon \tag{2.9}$$

then

$$d(u,v) \leq C_f \cdot \varepsilon$$

where

$$C_f = \frac{p^2}{p^2-1}.$$

So, from Definition 2.1, the equation (2.4) is Ulam-Hyers stable.

Example 2.3. Let us consider the equation (2.1) in the Krasnoselski-Krein conditions (2.6)+(2.7) and

$$f(x,y,u) = u(x,y)xye^{-x^2y^2}, \ h(x,y) = x^2y^2,$$

then $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $k = 1$, $p = 2$.

In this case we have $c_f = \frac{p^2}{p^2-1}$ and for $p = 2$, $c_f = \frac{4}{3}$, hence the equation (2.1) is Ulam-Hyers stable.

3. Ulam-Hyers stability in the generalized Naguno-Perron-Van Kampen conditions

In this case we consider the integral equation (2.1) in the same conditions. Let $X = C(D)$ and the generalized metrics

$$d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$$

$$d(u_1,u_2) = \sup_D \frac{|u_1(x,y) - u_2(x,y)|}{(xy)^{p+1}} \tag{3.1}$$

for all $u_1, u_2 \in X$, $p > -1$.

It is known that the space (X,d) is a generalized complete metric space. Here, we consider the stability of the equation (2.4) in the generalized Naguno-Perron-Van Kampen conditions.

Theorem 3.1. *If we have*

(i) $f : E \to \mathbb{R}$ is continuous and bounded on E;

(ii) $a < +\infty$, $b < +\infty$;

(iii) f verifies the generalized Naguno-Perron-Van Kampen conditions ([12])*:*

$$|f(x,y,u)| \leq \alpha(xy)^p, \ p > -1, \ \alpha > 0. \tag{3.2}$$

$$|f(x,y,u_1) - f(x,y,u_2)| \leq \frac{c}{(xy)^r}|u_1 - u_2|^q, \ q \geq 1, \ c > 0, \tag{3.3}$$

$$pq + q - r = p, \ xy \neq 0, \ \rho = \frac{c(2\alpha)^{q-1}}{(p+1)^{2q}} < 1, \ \text{for all } (x,y,u) \in E.$$

Then the equation (2.4) is Ulam-Hyers stable.

Proof. Evidently, in the conditions Naguno-Perron-Van Kampen, by Luxemburg-Jung theorem (Theorem 1.2), the equation (2.4) has a unique solution u in X_λ.

First we observe that

$$|v(x,y)-u(x,y)| \leq \frac{2\alpha}{(p+1)^2}(xy)^{p+1}. \tag{3.4}$$

From (2.1), (2.5), (3.2), (3.3) we have

$$|v(x,y)-u(x,y)| \leq \left|v(x,y)-h(x,y)-\int_0^x\int_0^y f(s,t,v(s,t))dsdt\right| + \int_0^x\int_0^y |f(s,t,v(s,t))-f(s,t,u(s,t))|dsdt.$$

From (3.3) we have

$$|v(x,y)-u(x,y)| \leq |v(x,y)-A(v)(x,y)| + \int_0^x\int_0^y \frac{c}{(st)^r}|v(s,t)-u(s,t)|^q dsdt$$

$$\leq |v(x,y)-A(v)(x,y)| + \int_0^x\int_0^y \frac{c}{(st)^r}\cdot\frac{|v(s,t)-u(s,t)|}{(st)^{p+1}}\cdot\frac{|v(s,t)-u(s,t)|^{q-1}}{(st)^{-p-1}}dsdt$$

$$\leq |v(x,y)-A(v)(x,y)| + cd(u,v)\int_0^x\int_0^y \frac{(2\alpha)^{q-1}}{(p+1)^{2(q-1)}}(st)^{pq+q-r}dsdt.$$

Then we have

$$d(u,v) \leq d(v,A(v)) + \rho d(u,v) \tag{3.5}$$

and

$$d(u,v) \leq \frac{\varepsilon}{1-\rho},$$

then

$$d(u,v) \leq C_f \cdot \varepsilon$$

where

$$C_f = \frac{1}{1-\rho}.$$

From Definition 2.1, the equation (2.4) is Ulam-Hyers stable.

Remark 3.2. For every $\lambda \in \Lambda$ there exists at least a solution v of (2.5) in X_λ and for each v exists a unique solution u of (2.4) which is Ulam-Hyers stable.

Remark 3.3. It is possible that the inequation (2.5) do not have a solution, but in this case the equation (2.4) is Ulam-Hyers stable.

Example 3.4. Let us consider the equation (2.1) in the Naguno-Perron-Van Kampen conditions (3.2)+(3.3), $p > -1$, $r = 1$, $q \geq 1$.

In this case $c_f = \dfrac{1}{1-\rho}$, where $\rho = \dfrac{c(2\alpha)^{q-1}}{(p+1)^{2q}}$ and the equation (2.1) is Ulam-Hyers stable. If $\rho = 1$ then the equation (2.1) is Ulam-Hyers instable.

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