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Existence and multiplicity of solutions to the Navier boundary value problem for a class of $(p(x), q(x))$-biharmonic systems

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Abstract. In this article, we study the following problem with Navier boundary conditions.

$$\begin{cases} \Delta(a(x,\Delta u)) = F_u(x,u,v), & \text{in } \Omega \\ \Delta(a(x,\Delta v)) = F_v(x,u,v), & \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

By using the Mountain Pass Theorem and the Fountain Theorem, we establish the existence of weak solutions of this problem.

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1. Introduction

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions was an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [15], Zhikov [20] and the reference therein; see also [4, 7, 8, 5].

Fourth-order equations appears in many context. Some of theses problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells (see [10]). In addition, this type of equations can describe the static from change of beam or the sport of rigid body.

In [1] the authors studied a class of $p(x)$-biharmonic of the form

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{q(x)-2}u \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \geq 0$.

In [3], A. El Amrouss and A. Ourraoui considered the below problem and using variational methods, by the assumptions on the Carathéodory function f, they establish the existence of Three solutions the problem of the form

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda g(x,u) \quad \text{in } \Omega,$$
$$Bu = Tu = 0 \quad \text{on } \partial\Omega.$$

Inspired by the above references, the work of L. Li [11]and [14], the aim of this article is to study the existence and multiplicity of weak solutions for $(p(x), q(x))-$biharmonic type system

$$\begin{cases} \Delta(a(x,\Delta u)) = F_u(x,u,v), & \text{in } \Omega \\ \Delta(a(x,\Delta v)) = F_v(x,u,v), & \text{in } \Omega, \\ u = \Delta u = 0, v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$,

$$\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u),$$

is the $p(x)$-biharmonic operator, p,q are continuous functions on $\overline{\Omega}$ with

$$\inf_{x\in\overline{\Omega}} p(x) > \max\left\{1, \frac{N}{2}\right\}, \inf_{x\in\overline{\Omega}} q(x) > \max\left\{1, \frac{N}{2}\right\}$$

and $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(.,s,t)$ is continuous in $\overline{\Omega}$, for all $(s,t) \in \mathbb{R}^2$, $F(x,.,.)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, and F_u, F_v denote the partial derivative of F, with respect to u, v respectively such that

(F_1) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$, we assume

$$\lim_{|s|\to 0} \frac{F_s(x,s,t)}{|s|^{p(x)-1}} = 0, \lim_{|t|\to 0} \frac{F_t(x,s,t)}{|s|^{q(x)-1}} = 0.$$

(F_2) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$, we assume

$$F(x,s,t) = o(|s|^{p(x)-1} + |t|^{q(x)-1}) \ as \ |(s,t)| \to \infty.$$

(F_3) There exists $\underline{u} > 0$, $\underline{v} > 0$ such that $F(x,\underline{u},\underline{v}) > 0$ for a.e $x \in \Omega$

(F_4) There exist $\lambda > 0$ such that $F(x,s,t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)})$ for all $(s,t) \in \mathbb{R}^2$, with

$$\alpha^- > r^+, \ 1 < \beta^- \leq \beta^+ < r^-.$$

(F_5) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$ $F(x,-s,-t) = -F(x,s,t)$.

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ to be a continuous potential derivative with respect to ξ of the mapping $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ where $a = DA = A^{'}$, with the assumption as below

(A_1) $A(x,0) = 0$, for all $x \in \Omega$.

(A_2) $a(x,\xi) \leq C_1(1 + |\xi|^{r(x)-1})$, $C_1 > 0$ and $r^- > p^+, r^- > q^+$.

(A_3) A is $r(x)$-uniformly convex: there exists a constant $k > 0$ such that

$$A\left(x, \frac{\xi+\eta}{2}\right) \leq \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\eta) - k|\xi-\eta|^{r(x)},$$

for all $x \in \Omega, \quad \xi, \eta \in \mathbb{R}^N$.

(A_4) A is $r(x)$-subhomogenuous, for all $(x,\xi) \in \Omega \times \mathbb{R}^N$,

$$|\xi|^{r(x)} \leq a(x,\xi) \leq r(x)A(x,\xi).$$

(A_5) For all $(x,s) \in \Omega \times \mathbb{R}^N$ $a(x,-s) = -a(x,s)$.

The main results of this paper are the following theorems.

Theorem 1.1. *Assume that* $(A_1) - (A_4)$ *and* $(F_1) - (F_3)$ *hold. Then the problem* (1.1) *has two weak solutions.*

Theorem 1.2. *Assume that* $(A_1) - (A_5)$ *and* $(F_1) - (F_5)$ *hold. Then the problem* (1.1) *has a sequence of weak solutions such that* $\phi(\pm(u_k, v_k)) \to +\infty$, *as* $k \to +\infty$ *with* ϕ *is a energy associated of the problem* (1.1) *defined in* (2.2).

This paper is organized as three sections. In Section 2, we recall some basic properties of the variable exponent Lebegue-Sobolev spaces. In Section3 we give the proof of main results.

2. Preliminaries

To study $p(x)$)-Laplacian problems, we need some results on the spaces $L^{p(x))}(\Omega)$ and $W^{k,p(x))}(\Omega)$, and properties of $p(x)$)-Laplacian, which we use later. Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{h(x); h(x) \in C(\overline{\Omega}), h(x) > 1, \forall x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\{h(x);\ x \in \overline{\Omega}\}, \quad h^- = \min\{h(x);\ x \in \overline{\Omega}\};$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x))}(\Omega) = \Big\{u; u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x))}dx < \infty\Big\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(x))} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x))} dx \leq 1\right\}.$$

Then $(L^{p(x))}(\Omega), |\cdot|_{p(x))})$ becomes a Banach space.

Proposition 2.1 **([9]).** *The space $(L^{p(x))}(\Omega), |\cdot|_{p(x))})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x))$, i.e.,*

$$\frac{1}{p(x))} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x))}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uvdx\right| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x))}|v|_{q(x)} \leq 2|u|_{p(x))}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x))}(\Omega)$ is defined as

$$W^{k,p(x))}(\Omega) = \{u \in L^{p(x))}(\Omega) : D^\alpha u \in L^{p(x))}(\Omega), |\alpha| \leq k\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_N^{\alpha_N}}u,$$

with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N}\alpha_i$. The space $W^{k,p(x))}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x))} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x))},$$

also becomes a separable and reflexive Banach space. For more details, we refer the reader to [6, 9, 13]. Denote

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \geq 1$.

Proposition 2.2 **([9]).** *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x))}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Then the function space $\left(\left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)\right), \|u\|_{p(x)}\right)$ is a separable and reflexive Banach space, where

$$\|u\|_{p(x)} = \inf\Big\{\mu > 0 : \int_\Omega \Big(\Big|\frac{\Delta u(x)}{\mu}\Big|^{p(x)} \leq 1\Big\}.$$

Remark 2.3. According to [[18] Theorem 4.4.], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $\|\cdot\|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $\|\cdot\|_{p(x)}$ are equivalent.

Proposition 2.4 **([2]).** *If we denote $\rho(u) = \int_\Omega |\Delta u|^{p(x)}dx$, then for $u, u_n \in X$, we have*

(1) *$\|u\|_p < 1$ (respectively=1; > 1) $\iff$ $\rho(u) < 1$ (respectively $= 1$; > 1);*

(2) $\|u\|_p \leq 1 \Rightarrow \|u\|_p^{p^+} \leq \rho(u) \leq \|u\|_p^{p^-}$;
(3) $\|u\|_p \geq 1 \Rightarrow \|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+}$;
(4) $\|u_n\|_p \to 0$ *(respectively* $\to \infty$*)* $\Longleftrightarrow \rho(u_n) \to 0$ *(respectively* $\to \infty$*).*

Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space

$$X = \left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)\right) \times \left(W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)\right)$$

equipped with the norm

$$\|(u,v)\| = \max\{\|u\|_{p(x)}, \|u\|_{q(x)}\}.$$

Remark 2.5 (see [19]**).** As the Sobolev space X is a reflexive and separable Banach space, there exist $(e_n)_{n\in\mathbb{N}^*} \subseteq X$ and $(f_n)_{n\in\mathbb{N}^*} \subseteq X^*$ such that $f_n(e_l) = \delta_{nl}$ for any $n, l \in \mathbb{N}^*$ and

$$X = \overline{\text{span}\{e_n : n \in \mathbb{N}^*\}}, \quad X^* = \overline{\text{span}\{f_n : n \in \mathbb{N}^*\}}^{w^*}.$$

For $k \in \mathbb{N}^*$, denote by

$$X_k = \text{span}\{e_k\},\ Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_k^\infty X_j}.$$

For every $m > 1$, $u, v \in L^m(\Omega)$, we define

$$|(u,v)|_m := \max\{|u|_m, |v|_m\}.$$

Lemma 2.6 (See [8]**).** *Define*

$$\beta_k := \sup\{|(u,v)|_m; \|(u,v)\| = 1, (u,v) \in Z_k\},$$

where $m := \max\limits_{x\in\overline{\Omega}}(p(x), q(x))$*. Then, we have*

$$\lim_{k\to\infty} \beta_k = 0.$$

2.1. Existence and multiplicity of weak solutions

Definition 2.7. We say that $(u,v) \in X$ is weak solution of (1.1) if

$$\int_\Omega a(x,\Delta u)\Delta\varphi dx + \int_\Omega a(x,\Delta v)\Delta\varphi dx = \int_\Omega F_u(x,u,v)\varphi dx + \int_\Omega F_v(x,u,v)\varphi dx, \quad (2.1)$$

for all $\varphi \in X$.

The functional associated to (1.1) is given by

$$\phi(u,v) = \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx - \int_\Omega F(x,u,v)dx. \quad (2.2)$$

It should be noticed that under the condition $(F_1) - (F_2)$ the functional ϕ is of class $C^1(X,\mathbb{R})$ and

$$\begin{aligned}\phi'(u,v)(\psi,\varphi) = &\int_\Omega a(x,\Delta u)\Delta\psi dx + \int_\Omega a(x,\Delta v)\Delta\varphi dx \quad (2.3)\\ &- \int_\Omega F_u(x,u,v)\psi dx - \int_\Omega F_v(x,u,v)\varphi dx,\ \forall(\psi,\varphi) \in X.\end{aligned}$$

Then, we know that the weak solution of (1.1) corresponds to critical point of the functional ϕ.

Definition 2.8. We say that

(1) The C^1-functional ϕ satisfies the Palais-Smale condition (in short (PS) condition) if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which, $(\phi(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $\phi^{'}(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.
(2) The C^1-functional ϕ satisfies the Palais-Smale condition at the level c (in short $(PS)_c$ condition) for $c \in \mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which, $\phi(u_n) \to c$ and $\phi^{'}(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.
(3) The C^1-functional ϕ satisfies the $(PS)_c^*$ condition for $c \in \mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which, $u_n \in Y_n$ for each $n \in \mathbb{N}$, $\phi(u_n) \to c$ and $\phi^{'}_{|Y_n})(u_n) \to 0$ as $n \to \infty$ with $Y_n,\ n \in \mathbb{N}$ as defined in Remark 2.5, has a subsequence convergent to a critical point of ϕ.

Remark 2.9. It is easy to see that if ϕ satisfies the (PS) condition, then ϕ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.

Proof of Theorem 1.1. To prove Theorem 1.1, we shall use the Mountain Pass theorem [16]. We first start with the following lemmas.

Lemma 2.10. *Under the assumptions* (F_1)-(F_3) *and* (A_1)-(A_3) ϕ *is sequentially weakly lower semi continuous and coercive .*

Proof. By (F_1)-(F_2), we see that

$$|F(x,s,t)| \leq C_3(1 + |s|^{p(x)} + |t|^{q(x)}),\ \forall (s,t) \in \mathbb{R}^2. \tag{2.4}$$

By the compact embeddings

$$X \hookrightarrow L^{p(x)}(\Omega),\ X \hookrightarrow L^{q(x)}(\Omega),$$

we deduce that $w \mapsto \int_\Omega F(x,w)dx$ is sequentially lower semi continuous $\forall w \in \mathbb{R}^2$. Since

$$w \mapsto \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx$$

is convex uniformly, so it is sequentially lower semi continuous.
Now we prove that ϕ is coercive. From (F_2) for ε small enough, there exist $\delta > 0$ such that

$$|\ F(x,s,t)\ | \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),\ \text{for}\ |(s,t)| > \delta,$$

and thus we have

$$|\ F(x,s,t)\ | \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}) + \max_{|(s,t)|\leq\delta} |\ F(x,s,t)\ |\,||(s,t)|, \forall (s,t) \in \mathbb{R}^2,$$

for a.e $x \in \Omega$. Consequently, for $\|(u,v)\| > 1$ we obtain

$$\begin{aligned}
\phi(u,v) &\geq \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx \\
&- \varepsilon \int_\Omega |u|^{p(x)}dx - \varepsilon \int_\Omega |v|^{q(x)}dx - \max_{|(u,v)|\leq\delta} \mid F(x,u,v) \mid \int_\Omega |(u,v)|dx \\
&\geq \int_\Omega \frac{1}{r(x)}|\Delta u|^{r(x)}dx + \int_\Omega \frac{1}{r(x)}|\Delta v|^{r(x)}dx \\
&- C\varepsilon \int_\Omega |u|^{p(x)}dx - C\varepsilon \int_\Omega |v|^{q(x)})dx - \max_{|(u,v)|\leq\delta} \mid F(x,u,v) \mid \int_\Omega |(u,v)|dx \\
&\geq \frac{1}{r^+} \max\left(\|u\|^{r^-}_{r(x)}, \|v\|^{r^-}_{r(x)}\right) - 2C\varepsilon \max\left(\|u\|^{p^+}_{p(x)}, \|v\|^{q^+}_{q(x)}\right) \\
&\quad -C\varepsilon|\Omega| \max_{|(u,v)|\leq\delta} \mid F(x,u,v) \mid \max\left(\|u\|^{p^+}_{p(x)}, \|v\|^{q^+}_{q(x)}\right).
\end{aligned}$$

Therefore, ϕ is coercive and has a global minimizer $(\overline{u_1}, \overline{v_1})$ which is a nontrivial because by (F_3)

$$\phi(\overline{u_1}, \overline{v_1}) \leq \phi(\underline{u}, \underline{v}) < 0.$$

Lemma 2.11. *Under the assumptions* (F_1)*-*(F_3) *and* (A_1)*-*(A_4)*. Then* ϕ *satisfies the Palais-smale condition.*

Proof. Let $w_n = (u_n, v_n) \subset X$ be a Palais-smale sequence, then

$$\phi^{'}(w_n) \to 0 \text{ in } X^*, \ \phi(w_n) \to l \in \mathbb{R}.$$

We show that (w_n) is bounded. By (A_5) we have

$$\begin{aligned}
\phi(w_n) &= \int_\Omega A(x,\Delta u_n)dx + \int_\Omega A(x,\Delta v_n)dx - \int_\Omega F(x,u_n,v_n)dx \\
&\geq \int_\Omega \frac{1}{r(x)}|\Delta u_n|^{r(x)}dx + \int_\Omega \frac{1}{r(x)}|\Delta v_n|^{r(x)}dx - \int_\Omega F(x,u_n,v_n)dx,
\end{aligned}$$

and we get

$$\begin{aligned}
&\phi^{'}(u_n,v_n)(u_n,v_n) \\
&= \int_\Omega a(x,\Delta u_n)\Delta u_n dx + \int_\Omega a(x,\Delta v_n)\Delta v_n dx \\
&- \int_\Omega F_{u_n}(x,u_n,v_n)u_n dx - \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx \\
&\leq \int_\Omega r(x)A(x,\Delta u_n)dx + \int_\Omega r(x)A(x,\Delta v_n)dx \\
&- \int_\Omega F_{u_n}(x,u_n,v_n)u_n dx - \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx.
\end{aligned}$$

Using the fact that $F_s, F_t \in C(\Omega \times \mathbb{R}^2, \mathbb{R})$ and with $(F_1) - (F_2)$, for $\varepsilon > 0$ there exists $\delta > 0$ and $\eta > 0$ such that

$$|F_s(x,s,t)| \leq \varepsilon |s|^{p(x)-1}, \ |F_t(x,s,t)| \leq \varepsilon |t|^{q(x)-1},$$

and

$$|F(x,s,t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),$$

for all $|s,t)| \leq \delta$, and for all $|s,t)| \geq \eta$.
Then we have

$$|F_s(x,s,t)s| \leq \varepsilon |s|^{p(x)}, \ |F_t(x,s,t)t| \leq \varepsilon |t|^{q(x)}, \tag{2.5}$$

and

$$|F(x,s,t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),$$

for all $|s,t)| \leq \delta$, and for all $|s,t)| \geq \eta$.
It yields,

$$\begin{aligned}
& -\frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n) \\
\geq\ & -\frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta u_n)dx - \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta v_n)dx \\
+\ & \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right] \\
\geq\ & -\frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta u_n)dx - \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta v_n)dx \\
+\ & \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
& \phi(u_n,v_n) - \frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n) \\
\geq\ & \int_\Omega A(x,\Delta u_n)dx + \int_\Omega A(x,\Delta v_n)dx - \int_\Omega F(x,u_n,v_n)dx \\
-\ & \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta u_n)dx - \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta v_n)dx \\
-\ & \int_\Omega F(x,u_n,v_n)dx + \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right] \\
\geq\ & \frac{1}{2}\left[\int_\Omega |\Delta u_n|^{r(x)}dx + \int_\Omega |\Delta v_n|^{r(x)}dx\right] - \int_\Omega F(x,u_n,v_n)dx \\
+\ & \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right] \\
\geq\ & \frac{1}{2}\max\left(\|u_n\|^{r^+}_{r(x)}, \|v_n\|^{r^+}_{r(x)}\right) - (C\varepsilon+\varepsilon)\int_\Omega |u_n|^{p(x)}dx - (C\varepsilon+\varepsilon)\int_\Omega |v_n|^{q(x)})dx.
\end{aligned}$$

Since $r^- > p^+ > 1,\ r^- > q^+ > 1$, by the compact embeddings

$$X \hookrightarrow L^{p(x)}(\Omega),\ X \hookrightarrow L^{q(x)}(\Omega),$$

we deduce

$$\begin{aligned} & \phi(u_n,v_n) - \frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n) \\ \geq \ & \frac{1}{2}\max\left(\|u_n\|_{r(x)}^{r^+}, \|v_n\|_{r(x)}^{r^+}\right) - 2(C^{'}\varepsilon+\varepsilon)\|(u_n,v_n)\| \\ \geq \ & \left[\frac{1}{2} - 2(C^{'}\varepsilon+\varepsilon)\right]\|(u_n,v_n)\|, \end{aligned}$$

where $C^{'}$ is positive constant.

For ε small enough with $R = \frac{1}{2} - 2(C^{'}\varepsilon+\varepsilon) > 0$, we get

$$\|(u_n,v_n)\| \leq \frac{1}{R}\left(\phi(u_n,v_n) - \frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n)\right).$$

Since $\phi(u_n,v_n)$ is bounded and $\phi^{'}(u_n,v_n)(u_n,v_n) \to 0$ as $n \to \infty$, then (u_n,v_n) is bounded in X, passing to a subsequence, so $(u_n,v_n) \rightharpoonup (u,v)$ in X and $(u_n,v_n) \to L^{p(x)}(\Omega) \times L^{q(x)}(\Omega)$. We show that $(u_n,v_n) \to (u,v)$ in X.

$$\begin{aligned} & \phi^{'}(u_n,v_n)\left((u_n,v_n)-(u,v)\right) \\ = \ & \int_\Omega a(x,\Delta u_n)\Delta(u_n-u)dx + \int_\Omega a(x,\Delta v_n)\Delta(v_n-v)dx \\ - \ & \int_\Omega F_{u_n}(x,u_n,v_n)(u_n-u)dx - \int_\Omega F_{v_n}(x,u_n,v_n)(v_n-v)dx. \end{aligned}$$

Since

$$\begin{aligned} & \left|\int_\Omega a(x,\Delta u_n)\Delta(u_n-u)dx + \int_\Omega a(x,\Delta v_n)\Delta(v_n-v)dx\right| \\ = \ & |\phi^{'}(u_n,v_n)\left((u_n,v_n)-(u,v)\right) + \int_\Omega F_{u_n}(x,u_n,v_n)(u_n-u)dx \\ + \ & \int_\Omega F_{v_n}(x,u_n,v_n)(v_n-v)dx| \\ \leq \ & \|\phi^{'}(u_n,v_n)\|_{X^\star}\|(u_n,v_n)-(u,v)\| \\ + \ & \int_\Omega |F_{u_n}(x,u_n,v_n)||(u_n-u)|dx + \int_\Omega |F_{v_n}(x,u_n,v_n)||(v_n-v)|dx. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} & \int_\Omega |F_{u_n}(x,u_n,v_n)||(u_n-u)|dx + \int_\Omega |F_{v_n}(x,u_n,v_n)||(v_n-v)|dx \\ \leq \ & \varepsilon\int_\Omega \left(|u_n-u|^{p(x)} + |v_n-v|^{q(x)}\right)dx, \end{aligned}$$

we get

$$\limsup_{n\to+\infty}\left(\int_\Omega a(x,\Delta u_n)\Delta(u_n-u)dx + \int_\Omega a(x,\Delta v_n)\Delta(v_n-v)dx\right) \leq 0.$$

Since $a(x,\xi)$ is of (S_+) type, we see that $(u_n,v_n) \to (u,v)$ in X.

Now, we verified the conditions of Mountain Pass Theorem. By Hölder's inequality, from (F_1) there exists $\delta > 0$ such that

$$\begin{aligned} |F(x,u,v)| &\leq \left| \int_0^u F_s(x,s,v)dx + \int_0^v F_t(x,0,t)dx + F(x,0,0) \right| \\ &\leq \varepsilon \left| \int_0^u |s|^{p(x)-1} dx + \int_0^v |t|^{q(x)-1} dx \right| + |F(x,0,0)| \\ &\leq \varepsilon(|u|^{p(x)} + |v|^{q(x)}) + M, \end{aligned}$$

for all $|u,v)| \leq \delta$, with $M := \max\limits_{x\in\overline{\Omega}} F(x,0,0)$ and by (F_2), there exists $M(\delta) > 0$ such that

$$| F(x,u,v) | \leq M(\delta)(|u|^{p(x)} + |v|^{q(x)}), \text{ for } |(u,v)| > \delta.$$

Therefore, for $\|(u,v)\| = \varrho$ small enough, we have

$$\begin{aligned} \phi(u,v) &\geq \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx - \varepsilon \int_{|(u,v)|<\delta} \left(|u|^{p(x)} + |v|^{q(x)} \right) dx \\ &- M(\delta) \int_{|(u,v)|>\delta} (|u|^{p(x)} + |v|^{q(x)}) - M meas\{|(u,v)| < \delta\} \\ &\geq \frac{1}{r^+} \max \left(\|u\|^{r^+}_{r(x)}, \|v\|^{r^+}_{r(x)} \right) \\ &- \min(\varepsilon C, M(\delta)C') \max \left(\|u\|^{p^-}_{p(x)}, \|v\|^{q^-}_{q(x)} \right) - M meas\{|(u,v)| < \delta\} \\ &= g(\varrho). \end{aligned}$$

There exists $\theta > 0$ such that $g(\varrho) > \theta > 0$. Since $\phi(0,0) = 0$, we conclude that ϕ satisfies the conditions of Mountain Pass Theorem. Then there exists $(\overline{u_2}, \overline{v_2})$ such that $\phi'(\overline{u_2}, \overline{v_2}) = 0$.

Proof of Theorem 1.2. To prove Theorem 1.2, above, will be based on a variational approach, using the critical points theory, we shall prove that the C^1-functional ϕ has a sequence of critical values. The main tools for this end are "Fountain theorem" (see Willem [16, Theorem 6.5]) which we give below.

Theorem 2.12 ("Fountain theorem", [16]**).** *Let X be a reflexive and separable Banach space, $\phi \in C^1(X,\mathbb{R})$ be an even functional and the subspaces X_k, Y_k, Z_k as defined in remark* 2.5*. If for each $k \in \mathbb{N}^*$ there exist $\rho_k > r_k > 0$ such that*

(1) $\inf_{x\in Z_k, \|x\|=r_k} \phi(x) \to \infty$ *as* $k \to \infty$,
(2) $\max_{x\in Y_k, \|x\|=\rho_k} \phi(x) \leq 0$,
(3) *I satisfies the $(PS)_c$ condition for every $c > 0$.*

Then I has a sequence of critical values tending to $+\infty$.

According to Lemma 2.6, (F_5) and (A_5), $\Phi \in \mathcal{C}^1(X,\mathbb{R})$ is an even functional. We will prove that if k is large enough, then there exist $\rho_k > \nu_k > 0$ such that

$$b_k := \inf\{\Phi(u)/u \in Z_k, \|u\| = \nu_k\} \to +\infty \quad \text{as } k \to +\infty; \tag{2.6}$$

$$a_k := \max\{\Phi(u)/u \in Y_k, \|u\| = \rho_k\} \to 0 \quad \text{as } k \to +\infty. \tag{2.7}$$

For any $(u,v) \in Z_k$, $\|v\|_{q(x)} > 1$, $\|u\|_{p(x)} > 1$ and $\|(u,v)\| = \eta_k$, (η_k will be specified later), by (2.4) we have

$$\begin{aligned}
\phi(u,v) &= \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx - \int_\Omega F(x,u,v)dx \\
&\geq \frac{1}{r^+}\max\left(\|u\|^{r^-}_{r(x)}, \|v\|^{r^-}_{r(x))}\right) - \int_\Omega C_3(1+|u|^{p(x)}+|v|^{q(x)})dx \\
&\geq \frac{1}{r^+}\max\left(\|u\|^{r^-}_{r(x)}, \|v\|^{r^-}_{r(x))}\right) - C_3\int_\Omega dx - C_3\int_\Omega |u|^{p(x)}dx - C_3\int_\Omega |v|^{q(x)}dx \\
&\geq \frac{1}{r^+}\|(u,v)\|^{r^-} - C_3(\beta_k\|(u,v)\|)^{p^+} - C_3(\beta_k\|(u,v)\|)^{q^+} - C_3|\Omega| \\
&\geq \frac{1}{r^+}\|(u,v)\|^{r^-} - C_4\beta_k\|(u,v)\|^m - C_3|\Omega|,
\end{aligned}$$

where m is defined in Lemma 2.6. We fix

$$\eta_k = \left(\frac{1}{r^+C_4\beta_k^b}\right)^{\frac{1}{m-r^-}} \to +\infty \text{ as } k \to +\infty.$$

Consequently

$$\phi(u,v) \geq \eta_k\left[\frac{1}{r^+}\eta_k^{r^- -1} - C_4\beta_k^b\eta_k^{m-1}\right] - C_3|\Omega|.$$

Then,

$$\phi(u,v) \to +\infty \text{ as } k \to +\infty.$$

Proof of (2.7). From (F_4), there exists $\lambda > 0$ such that

$$F(x,s,t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)}),$$

with $\alpha^- > r^+, \beta^+ < r^-$.

Therefore, by Lemma 2.1 [12] and Lemma 3.1 [17], for any $\omega := (u,v) \in Y_k$ with $\|\omega\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned}
\phi(t\omega) &= \int_\Omega A(x,t\Delta u)dx + \int_\Omega A(x,t\Delta v)dx - \int_\Omega F(x,t\omega)dx \\
&\leq \int_\Omega t^{r(x)}A(x,\Delta u)dx + \int_\Omega t^{r(x)}A(x,\Delta v)dx \\
&\quad - \lambda\int_\Omega |tu|^{\alpha(x)}dx + \lambda\int_\Omega |tv|^{\beta(x)}dx \\
&\leq t^{r^+}\left[\int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx\right] \\
&\quad - \lambda t^{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx + \lambda t^{\beta^-}\int_\Omega |v|^{\beta(x)}dx.
\end{aligned}$$

By $\alpha^- > r^+ > \beta^-$ and $dimY_k < \infty$, we conclude that $\phi(tu,tv) \to -\infty$ as $\|t\omega\| \to +\infty$ for $\omega \in Y_k$. By applying the fountain Theorem, we achieved the proof of Theorem 1.2.

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