Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 3, 413–416

Book reviews

Daniel Li and Hervé Queffélec, Introduction to Banach spaces: analysis and probability (2 volumes, translated from the French by Danièle Gibbons and Greg Gibbons), Cambridge Studies in Advanced Mathematics:
no. 166: vol. 1, xxx+431 p., ISBN: 978-1-107-16051-4/hbk ($ 94.99), 978-1-316-67576-2/ebook;
no. 167: vol. 2, xxx+374 p., ISBN: 978-1-107-16262-4/hbk ($ 99.99), 978-1-316-67739-1/ebook, ISBN 978-1-107-16263-1/set ($ 165.00),
Cambridge University Press, Cambridge, 2017.

The subject of the book is the interplay between the Banach space theory and probability theory, which has a long and fruitful history. Although the study of Banach space valued random variables started in the 1950s, their importance in the study of Banach spaces became clear only after the introduction, at the beginning of 1970s, of the notions of type and cotype (presented in volume I, Chapter 5), proving their intimately connections with Banach spaces. On the other hand, probability methods allow the proof of some deep results in Banach space theory as, for instance, Dvoretzky's theorem on the Euclidean sections of convex bodies in Banach spaces, the relevance of martingales in the study of Radon-Nikodým property, Davie's proof on the existence of Banach spaces without approximation property, Gowers' dichotomy theorem.

The characteristic of this textbook is that full proofs are given to all results, both from Banach spaces and from probability theory as well. As the authors mention in Preface – the proofs are given "from scratch", without referring in the proof to a "well-known result" or admitting an auxiliary difficult result. For this reason, proofs of some theorems in analysis or functional analysis as, for instance, Marcel Riesz' theorem on the continuity of Hilbert transform, Riesz-Thorin's interpolation theorem, Rademacher's theorem on the a.e. differentiability of Lipschitz functions, Eberlein-Shmulian and Krein-Milman theorems, etc, are included.

The book contains the necessary material from probability theory: Ch.1, *Fundamental notions of probability* (including Khinchin's inequality and martingales), Ch.4, *Banach space valued random variables* (Lévy's symmetry principle, Kahane's inequalities), in volume I, and Ch.3, *Gaussian processes* (including Brownian motion, Dudley's majoration and Fernique's minoration theorems), from volume II.

Concerning Banach spaces the following topics are treated in the first volume: Ch.2, *Bases in Banach space*, Ch.3, *Unconditional convergence* (Orlicz-Pettis' theorem, Gowers' dichotomy theorem), Ch.5, *Type and cotype of Banach spaces*, Ch.6, *p-Summing operators*, Ch.7, *Some properties of L^p-spaces*, Ch.8, *The space ℓ_1* (dedicated to Rosenthal's ℓ_1-theorem). In the second volume: Ch.1, *Euclidean sections* (Dvoretzky's theorem), Ch.2, *Separable Banach spaces without the approximation property* (the counterexamples of Enflo and Davie), Ch.4, *Reflexive subspaces of L^1* (the Kadec-Pełczyński theorem, Maurey's factorization theorem), Ch.5, *The method of selectors* (contains three results of Bourgain illustrating the method of selectors), Ch.6, *The Pisier space of almost surely continuous functions.*

In fact, the contents is explained in details in Preface (30 pages). The bibliography is divided in two sections - books and papers.

Each chapter ends with a section of Comments and one of Exercises. The comments refer to the origin of some results presented in the chapter or to complementary results. Many of exercises propose proofs of recent and important results. These proofs are decomposed in several steps, so that the reader can fill up in the details and, in most cases, the sources (an article or a book) are indicated.

The book was published as one "thick" volume (627 pages) in the collection *Cours Spécialisés de la Société Mathématique de France.* Danièle and Greg Gibbons provided an excellent translation of the French text, keeping the lively and pleasant style of the French original (e.g. quotations from George Brassens, or a reference to "ensemble flirtant" of Bourbaki). With respect to the French edition some mistakes were corrected and some missing arguments were added. At the end of the second volume four appendices were added, three surveys – A. *News in the theory of infinite-dimensional Banach spaces in the past 20 years*, by G. Godefroy (7 pages), B. *An update on some problems in high-dimensional convex geometry and related probabilistic results*, by O. Guédon (8 pages), C. *A few updates and pointers*, by G. Pisier (9 pages) – and a research paper, D. *On the mesh condition for Sidon sets*, by L. Rodríguez-Piazza (8 pages).

The book contains a lot of interesting and deep results on Banach spaces and harmonic analysis treated, with the methods of probability theory. It can be used for advanced courses in functional analysis, but also by professional mathematicians as a valuable source of information.

S. Cobzaş

A. R. Alimov and I. G. Tsar'kov; Geometric theory of approximation Geometricheskaya teoriya priblizhenii) (Russian).
Part I. **Classical notions and constructions in the approximation by sets** (Klassicheskie ponyatiya i kontruktsii priblizheniya mnozhestvami), 346 p, OntoPrint Moscow, 2017, ISBN 978-5-906886-91-0;
Part II. **Approximation by classes of sets, further developments of basic questions of the geometric approximation theory** (Priblizhenie klassami mnozhestv, dal'neishee razvitie osnovnykh voprosov geometricheskoi teorii priblizhenya), 350 p, OntoPrint Moscow, 2018, ISBN 978-5-00121-053-5.

The book is about the best approximation in normed linear spaces in connection with the geometric properties of the underlying space. For a long time the standard reference in this area was Ivan Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer 1970 (an updated translation of the Romanian version from 1967, see also, I. Singer, *The theory of best approximation and functional analysis*, SIAM, Philadelphia, PA, 1974). Singer's books stimulated the research in this area and, since ten, a lot of results were obtained, new notions emerged and many challenging problems were solved. But one, considered by some researchers the most important in best approximation theory, resisted to all attempts to solve it – the problem of the convexity of Chebyshev sets – is any Chebyshev subset of a Hilbert space convex? The authors of this book have important contributions to this problem, mainly concerning the class of the so-called solar sets (or suns), a recurrent theme of the book and an essential tool in the characterization of best approximation (e.g. Kolmogorov's criterium).

The authors treat best approximation problems both in abstract and in concrete normed spaces. For instance, the first volume contains some classical results on Chebyshev problem of best approximation – alternation results, Haar theorem, Chebyshev systems, rational approximation – in $C[a,b]$ and in L^p-spaces. Among the abstract problems studied in the first volume we mention: best approximation in Euclidean spaces (characterization, Phelps theorem on the nonexpansiveness of the metric projection), the role of approximative compactness and of Efimov-Stechkin spaces in the study of the existence of best approximation and continuity of the metric projection, solarity and characterization of best approximation. Five proofs (of Berdyshev-Klee-Vlasov, Asplund, Konyagin, Vlasov and Brosowski) are given for the convexity of Chebyshev sets in $\mathbb{R}^n$. Solarity and connectedness of Chebyshev sets in connection with continuity and selection properties of metric projection are also studied.

In the first chapter of the second volume, *Approximation of vector-valued functions*, one presents some results of Zukhovickii, Stechkin, Tsar'kov, Garkavi, Koshcheev, a.o., on the extension of the results from the first volume (characterization, Haar condition, Chebyshev systems, etc) to the case of the space $C(Q,X)$, where Q is a compact Hausdorff space and X a Banach space. The second chapter is devoted to a detailed study of Jung constant defined as the radius of the smallest set covering an arbitrary set of diameter 1. This is a very important tool in the geometry of Banach spaces with applications to fixed point theory for nonexpansive (the inverse of Jung constant is called the coefficient of normality of the corresponding Banach space) and condensing mappings. A consistent chapter, Chapter 3 (102 pages), contains a detailed study of Chebyshev centers, a notion related to best approximation (simultaneous approximation) and having important applications as, for instance, to optimal location problems. One studies the existence and uniqueness of Chebyshev centers, continuity, stability and selections for the Chebyshev center map, algorithms for finding Chebyshev centers and applications.

Chapter 6 is concerned with widths in the sense of Kolmogorov, a notion strongly related to approximation theory – one studies the approximation by classes of functions, comparing the efficiency of the approximation by various classes of approximating sets (e.g. algebraic or trigonometric polynomials, rational functions, etc). The last chapter of the second volume, Chapter 7, *Approximation properties of arbitrary sets in linear normed spaces. Almost Chebyshev sets and sets of almost uniqueness,* is concerned with genericity properties (in the sense of Baire category) and porosity results in best approximation problems and in the study of farthest points (existence and uniqueness), a direction of research initiated by S. B. Stechkin in 1963.

Written by two experts with substantial contributions to the domain, this two volume book incorporates a lot of results (including authors' results), both classical but also new ones situated in the focus of current research, the book is of interest to a large community of mathematicians interested in the applications of Banach space geometry and applications (and reading Russian). The book is clearly written, in a pleasant style, reflecting the erudition of the authors (not only in mathematics, but also in other areas of human knowledge, as illustrated by the mottos and the beginning of some chapters and sections).

S. Cobzaş