Stud. Univ. Babeş-Bolyai Math. 64(2019), No. 4, 477–486
DOI: 10.24193/subbmath.2019.4.03

Differential subordinations and superordinations for analytic functions defined by Sălăgean integro-differential operator

Ágnes Orsolya Páll-Szabó

Abstract. In this paper we consider the linear operator $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{L}^n f(z) = (1-\lambda)\, \mathscr{D}^n f(z) + \lambda I^n f(z),$$

where $\mathscr{D}^n$ is the Sălăgean differential operator and I^n is the Sălăgean integral operator. We give some results and applications for differential subordinations and superordinations for analytic functions and we will determine some properties on admissible functions defined with the new operator.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Sălăgean integro-differential operator, differential subordination, differential superordination, dominant, best dominant, "sandwich-type theorem".

1. Preliminaries

Let U be the unit disk in the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H} = \mathcal{H}(U)$ be the space of holomorphic functions in U and let

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U) :\ f(z) = z + a_{n+1} z^{n+1} + \cdots, z \in U\right\}$$

with $\mathcal{A}_1 = \mathcal{A}$. For $a \in \mathbb{C}$ and n a positive integer, let

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H}(U) :\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\right\}.$$

Denote by

$$K = \left\{f \in \mathcal{A} : \Re \frac{z f''(z)}{f'(z)} + 1 > 0, z \in U\right\}$$

the class of normalized convex functions in U.

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\}$$

and such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

Definition 1.1. ([9], Definition 3.5.1, [4]) Let $f,g\in\mathcal{H}$. We say that the function f is subordinate to the function g or g is superordinate to f, if there exists a function w, which is analytic in U and $w(0)=0; |w(z)|<1; z\in U$, such that $f(z)=g(w(z))$; $\forall z\in U$. We denote by $\prec$ the subordination relation. If g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f\left(U\right)\subseteq g\left(U\right)$.

We omit the requirement $''z\in U''$ because the definition and conditions of the functions, in the unit disk U.

Let $\psi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$ be a function and let h be univalent in U and $q\in\mathcal{Q}$. In article [6] it is studied the problem of determining conditions on admissible function ψ such that

$$\psi\left(p\left(z\right),zp'\left(z\right),z^2p''\left(z\right);z\right)\prec h\left(z\right),\ \left(z\in U\right)\tag{1.1}$$

(second-order) differential subordination, implies $p(z)\prec q(z)$, $\forall p\in\mathcal{H}\left[a,n\right]$. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p\prec q$ for all p satisfying (1.1).
A dominant $\widetilde{q}$, which is the "smallest" function with this property and satisfies $\widetilde{q}\prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\varphi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$ be a function and let $h\in\mathcal{H}$ and $q\in\mathcal{H}\left[a,n\right]$. If p and $\varphi\left(p\left(z\right),zp'\left(z\right),z^2p''\left(z\right);z\right)$ are univalent in U and satisfy the (second-order) differential superordination

$$h\left(z\right)\prec\varphi\left(p\left(z\right),zp'\left(z\right),z^2p''\left(z\right);z\right),\ \left(z\in U\right)\tag{1.2}$$

then p is called a solution of the differential superordination. In [7] the authors studied the dual problem of determining properties of functions p that satisfy the differential superordination (1.2). The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q\prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant of (1.2) and is the "largest" function with this property. The best subordinant is unique up to a rotation of U.

Definition 1.2. [11, 12] For $f\in\mathcal{A}, n\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,\ldots\}$, the Sălăgean differential operator $\mathscr{D}^n$ is defined by $\mathscr{D}^n:\mathcal{A}\to\mathcal{A}$,

$$\mathscr{D}^0f(z)=f(z),$$
$$\mathscr{D}^{n+1}f(z)=z\left(\mathscr{D}^nf\left(z\right)\right)',\ z\in U.$$

Remark 1.3. If $f\in\mathcal{A}$ and $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, then

$$\mathscr{D}^nf(z)=z+\sum_{k=2}^{\infty}k^na_kz^k, z\in U.$$

Definition 1.4. [11] For $f \in \mathcal{A}, n \in \mathbb{N}_0$, the operator I^n is defined by

$$I^0 f(z) = f(z),$$

$$I^n f(z) = I\left(I^{n-1} f(z)\right), \ z \in U, \ n \geq 1.$$

Remark 1.5. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \ z \in U, (n \in \mathbb{N}_0)$$

and $z\left(I^n f(z)\right)' = I^{n-1} f(z)$.

Definition 1.6. Let $\lambda \geq 0, n \in \mathbb{N}$. Denote by $\mathscr{L}^n$ the operator given by $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{L}^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z), z \in U.$$

Remark 1.7. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n}\right] a_k z^k, z \in U. \tag{1.3}$$

Lemma 1.8. [2] *Let q be an univalent function in U and $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If p is an analytic function in U, with $p(0) = q(0)$ and

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z), \tag{1.4}$$

then $p(z) \prec q(z)$ and q is the best dominant of (1.4).

Lemma 1.9. [2] *Let q be convex function in U, with $q(a) = 0$ and $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$ and $p(z) + \gamma z p'(z)$ is univalent in U, then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z) \Rightarrow q(z) \prec p(z)$$

and q is the best subordinant.

S. S. Miller and P. T. Mocanu obtained special results related to differential subordinations in [8] .

We follow Cotîrlă [3] and we generalise her results. Nechita obtained similar results in [10] for generalized Sălăgean differential operator (see also [1], [5]).

2. Main results

Theorem 2.1. *Let q be an univalent function in U with $q(0)=1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^{n} f(z)} + \gamma \left\{1 - \frac{\mathscr{L}^{n+1} f(z)\left[(1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)\right]}{\left[\mathscr{L}^{n} f(z)\right]^2} + \right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2} f(z) - \mathscr{D}^{n} f(z)\right]}{\mathscr{L}^{n} f(z)}\right\} \prec q(z) + \gamma z q'(z), \tag{2.1}$$

then

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^{n} f(z)} \prec q(z) \tag{2.2}$$

and q is the best dominant of (2.1).

Proof. We define the function

$$p(z) := \frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^{n} f(z)}.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)} = z\frac{\left[\mathscr{L}^{n+1} f(z)\right]'}{\mathscr{L}^{n+1} f(z)} - z\frac{\left[\mathscr{L}^{n} f(z)\right]'}{\mathscr{L}^{n} f(z)}. \tag{2.3}$$

By using the identity

$$z\left[\mathscr{L}^{n+1} f(z)\right]' = (1-\lambda)\mathscr{D}^{n+2} f(z) + \lambda I^{n} f(z) \tag{2.4}$$

we obtain from (2.3) that

$$\frac{zp'(z)}{p(z)} = \frac{1}{p(z)} - \frac{(1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\mathscr{L}^{n} f(z)}$$
$$+ \frac{(1-\lambda)\left(\mathscr{D}^{n+2} f(z) - \mathscr{D}^{n} f(z)\right)}{\mathscr{L}^{n+1} f(z)}$$

and

$$p(z) + \gamma z p'(z) = \frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^{n} f(z)} + \gamma \left\{1 - \frac{\mathscr{L}^{n+1} f(z)\left[(1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)\right]}{\left[\mathscr{L}^{n} f(z)\right]^2}\right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2} f(z) - \mathscr{D}^{n} f(z)\right]}{\mathscr{L}^{n} f(z)}\right\}.$$

The subordination (2.1) becomes

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z).$$

We obtain the conclusion of our theorem by applying Lemma 1.8. □

In the particular case $\lambda = 0$ and $n = 0$ we obtain:

Corollary 2.2. *Let q be an univalent function in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$(1+\gamma)\,\frac{zf'(z)}{f(z)} + \gamma\left[\frac{z^2f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2\right] \prec q(z) + \gamma zq'(z)$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

In the particular case $\lambda = 0$ and $n = 1$, we obtain:

Corollary 2.3. *Let q be an univalent function in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$1 + (1+3\gamma)\,\frac{zf''(z)}{f'(z)} + \gamma\left[1 - \left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + \frac{z^2f'''(z)}{f'(z)}\right] \prec q(z) + \gamma zq'(z)$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z)$$

and q is the best dominant.

When $\lambda = 1$ we get the Cotîrlă's result [3]:
We select in Theorem 2.1 a particular dominant q.

Corollary 2.4. *Let $A, B, \gamma \in \mathbb{C}, A \neq B$ such that $|B| \leq 1$ and $\Re\,\gamma > 0$. If for $f \in \mathcal{A}$*

$$\frac{\mathscr{L}^{n+1}f\,(z)}{\mathscr{L}^{n}f\,(z)} + \gamma\left\{1 - \frac{\mathscr{L}^{n+1}f\,(z)\left[(1-\lambda)\,\mathscr{D}^{n+1}f\,(z) + \lambda I^{n-1}f\,(z)\right]}{\left[\mathscr{L}^{n}f\,(z)\right]^2}\right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2}f\,(z) - \mathscr{D}^{n}f\,(z)\right]}{\mathscr{L}^{n}f\,(z)}\right\} \prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)\,z}{(1+Bz)^2},$$

then

$$\frac{\mathscr{L}^{n+1}f\,(z)}{\mathscr{L}^{n}f\,(z)} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.5. *Let q be a convex function in U with $q(0) = 1$ and $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $f \in \mathcal{A}$,*

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} + \gamma \left\{ 1 - \frac{\mathscr{L}^{n+1} f(z) \left[(1-\lambda) \mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z) \right]}{\left[\mathscr{L}^n f(z) \right]^2} + \frac{(1-\lambda) \left[\mathscr{D}^{n+2} f(z) - \mathscr{D}^n f(z) \right]}{\mathscr{L}^n f(z)} \right\}$$

is univalent in U and

$$q(z) + \gamma z q'(z) \prec \frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} + \gamma \left\{ 1 - \frac{\mathscr{L}^{n+1} f(z) \left[(1-\lambda) \mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z) \right]}{\left[\mathscr{L}^n f(z) \right]^2} + \frac{(1-\lambda) \left[\mathscr{D}^{n+2} f(z) - \mathscr{D}^n f(z) \right]}{\mathscr{L}^n f(z)} \right\}, \tag{2.5}$$

then $q(z) \prec \dfrac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)}$ and q is the best subordinant .

Proof. Let

$$p(z) := \frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)}.$$

If we proceed as in the proof of Theorem 2.1, the superordination (2.5) become

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

The conclusion of this theorem follows by applying the Lemma 1.9. □

From the combination of Theorem 2.1 and Theorem 2.5 we get the following "sandwich-type theorem".

Theorem 2.6. *Let q_1 and q_2 be convex functions in U with $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $f \in \mathcal{A}$,*

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} + \gamma \left\{ 1 - \frac{\mathscr{L}^{n+1} f(z) \left[(1-\lambda) \mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z) \right]}{\left[\mathscr{L}^n f(z) \right]^2} + \frac{(1-\lambda) \left[\mathscr{D}^{n+2} f(z) - \mathscr{D}^n f(z) \right]}{\mathscr{L}^n f(z)} \right\}$$

is univalent in U and

$$q_1(z)+\gamma z q_1'(z) \prec \frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)}+\gamma\left\{1-\frac{\mathscr{L}^{n+1} f(z)\left[(1-\lambda)\mathscr{D}^{n+1} f(z)+\lambda I^{n-1} f(z)\right]}{\left[\mathscr{L}^n f(z)\right]^2}\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2} f(z)-\mathscr{D}^n f(z)\right]}{\mathscr{L}^n f(z)}\right\} \prec q_2(z)+\gamma z q_2'(z), \tag{2.6}$$

then

$$q_1(z) \prec \frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} \prec q_2(z),$$

q_1 is the best subordinant and $q_2(z)$ is the best dominant.

Theorem 2.7. *Let q be a convex function in U with $q(0)=1$, $\gamma\in\mathbb{C}^*$ such that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0,-\Re\frac{1}{\gamma}\right\}.$$

If $f\in\mathcal{A}$ and

$$(1+\gamma)\, z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1} f(z)+\lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2} f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3} \prec q(z)+\gamma z q'(z), \tag{2.7}$$

then

$$z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \prec q(z),$$

q is the best dominant.

Proof. Let

$$p(z):=z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)}=1+\frac{(1-\lambda)\mathscr{D}^{n+1} f(z)+\lambda I^{n-1} f(z)}{\mathscr{L}^n f(z)}-2\frac{(1-\lambda)\mathscr{D}^{n+2} f(z)+\lambda I^n f(z)}{\mathscr{L}^{n+1} f(z)}. \tag{2.8}$$

It follows that

$$p(z)+\gamma z p'(z)=(1+\gamma)\, z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1} f(z)+\lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2} f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3}.$$

The subordination (2.7) becomes

$$p(z)+\gamma z p'(z) \prec q(z)+\gamma z q'(z).$$ □

We consider $n=0$ and $\lambda=0$.

Corollary 2.8. *Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$(1-\gamma)\frac{f(z)}{z\left[f'(z)\right]^2} + \gamma\left[\frac{1}{f'(z)} - \left(\frac{2f(z)\cdot f''(z)}{\left[f'(z)\right]^3}\right)^2\right] \prec q(z) + \gamma z q'(z)$$

then

$$\frac{f(z)}{z\left[f'(z)\right]^2} \prec q(z)$$

and q is the best dominant.

Corollary 2.9. *Let $A, B, \gamma \in \mathbb{C}, A \neq B$ such that $|B| \leq 1$ and $\Re\, \gamma > 0$. If for $f \in \mathcal{A}$*

$$(1+\gamma)\, z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} + \gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z) + \lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3} \prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)\,z}{(1+Bz)^2}, \tag{2.9}$$

then

$$z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.10. *Let q be a convex function in U with $q(0) = 1$, $\gamma \in \mathbb{C}$ such that $\Re\, \gamma > 0$. If $f \in \mathcal{A}$*

$$z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

$$(1+\gamma)\, z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} + \gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z) + \lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3}$$

is univalent in U and

$$q(z) + \gamma z q'(z) \prec (1+\gamma)\, z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} + \gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z) + \lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3}, \tag{2.10}$$

then

$$q(z) \prec z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2},$$

q is the best subordinant.

From Theorem 2.7 and Theorem 2.10 we get the following "sandwich-type theorem".

Theorem 2.11. *Let q_1 and q_2 be convex functions in U with $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$ such that $\Re\, \gamma > 0$. If $f \in \mathcal{A}$*

$$z \frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

$$(1+\gamma) z \frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} + \gamma z \frac{(1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z \frac{\mathscr{L}^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2} f(z) + \lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3}$$

is univalent in U and

$$q_1(z) + \gamma z q_1'(z) \prec (1+\gamma) z \frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} + \gamma z \frac{(1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z \frac{\mathscr{L}^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2} f(z) + \lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3} \prec q_2(z) + \gamma z q_2'(z), \quad (2.11)$$

then

$$q_1(z) \prec z \frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \prec q_2(z),$$

and q_1 is the best subordinant and $q_2(z)$ is the best dominant.

Acknowledgement. The present work has received financial support through the project: Entrepreneurship for innovation through doctoral and postdoctoral research, POCU/360/6/13/123886 co-financed by the European Social Fund, through the Operational Program for Human Capital 2014-2020.

References

[1] Aouf, M.K., Seoudy, T.M., *On differential Sandwich theorems of analytic functions defined by generalized Sălăgean integral operator,* Appl. Math. Lett., **24**(2011), 1364-1368.

[2] Bulboacă, T., *Differential Subordinations and Superordinations,* Casa Cărţii de Ştiinţă, Cluj-Napoca, 2005.

[3] Cotîrlă, L.I., *Differential subordination and superordination for analytic functions defined by integral operator,* Carpathian J. Math., **25**(2009), no. 1, 49-54.

[4] Crişan, O., *Differential subordinations obtained by using Al-Oboudi and Ruscheweyh operators*, Stud. Univ. Babeş-Bolyai Math., **56**(2011), no. 3, 45-51.

[5] El-Ashwah, R.M., Aouf, M.K., *Differential subordination and superordination for certain subclasses of analytic functions involving an extended integral operator*, Acta Univ. Apulensis Math. Inform., **28**(2011), 341-350.

[6] Miller, S.S., Mocanu, P.T., *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.

[7] Miller, S.S., Mocanu, P.T., *Subordinants of differential superordinations*, Complex Var. Elliptic Equ., **48**(2003), 815-826.

[8] Miller, S.S., Mocanu, P.T., *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[9] Mocanu, P.T., Bulboacă, T., Sălăgean, G.S., *The Geometric Theory of Univalent Functions*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006.

[10] Nechita, V.O., *Differential subordinations and superordinations for analytic functions defined by the generalized Sălăgean derivative*, Acta Univ. Apulensis Math. Inform., **16**(2008), 143-156.

[11] Sălăgean, G.S., *Subclasses of Univalent Functions*, Lecture Notes in Math., Springer Verlag, **1013**(1983), 362-372.

[12] Sălăgean, G.S., Yaguchi, T., *Relations between two kinds of derivatives on analytic functions II*, Stud. Univ. Babeş-Bolyai Math., **57**(2012), no. 2, 269-276.

Ágnes Orsolya Páll-Szabó
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca, Romania
e-mail: pallszaboagnes@math.ubbcluj.ro