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Approximations of bi-criteria optimization problem

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Abstract. In this article we study approximation methods for solving bi-criteria optimization problems. Initial problem is approximated by a new one consisting of the second order approximation of feasible set and components of objective function might be initial function, first or second approximation of it. Conditions such that efficient solution of the approximate problem will remain efficient for initial problem and reciprocally are studied. Numerical examples are developed to emphasize the importance of these conditions.

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1. Introduction

Bi-criteria optimization problems are quite often used to solve theoretical and practical problems from areas as portfolio theory [4], energy field [5], data analysis [3], logistics [6].
"Scalarization" methods [2] (weighting problem, k^{th} objective Lagrangian problem, k^{th} objective ε - constrained problem) are common methods for solving this type of problems. Highly complex mathematical models are reducing the efficiency of "scalarization" methods and approximation might represent a good alternative.
This article is analyzing conditions such that efficient solution of a certain approximate problem will remain efficient for the initial problem and reciprocally. Approximate problem consists of replacing components of objective function and also constraints with their approximate functions.

2. Basic concepts

Let X be a set in $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}$ functions. If f is differentiable at x_0 then we denote:

$$F^1(x) = f(x_0) + \nabla f(x_0)\,\eta(x, x_0)$$

and call it first $\eta-$approximation of f
and if f is twice differentiable at x_0 then we denote:

$$F^2(x) = f(x_0) + \nabla f(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\,\nabla^2 f(x_0)\,\eta(x, x_0).$$

and call it second $\eta-$approximation of f.

Definition 2.1. Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:
invex *at* x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \geq \nabla f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^1(x);$$

incave at x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \leq \nabla f(x_0)\,\eta(x, x_0)$$

or equivalently

$$f(x) \leq F^1(x);$$

avex at x_0 with respect to η if it is both invex and incave at x_0 w.r.t. η.

If function f is invex, respectively incave or avex we denote invex^1, respectively incave^1 or avex^1.

Definition 2.2. Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function twice differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:
second order invex *at* x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \geq \nabla f(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\,\nabla^2 f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^2(x);$$

second order incave *at* x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \leq \nabla f(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\,\nabla^2 f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \leq F^2(x);$$

second order avex at x_0 with respect to η if it is both second order invex and second order incave at x_0 w.r.t. η.

If function f is second order invex, respectively second order incave or second order avex we denote invex^2, respectively incave^2 or avex^2.

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

We consider the bi-criteria optimization problem $\left(P_0^{0,0}\right)$, defined as:

$$\begin{cases} \min\left(f_1, f_2\right)(x) \\ x = (x_1, x_2, ...x_n) \in X \\ g_t\left(x\right) \leq 0,\ t \in T \\ h_s\left(x\right) = 0,\ s \in S. \end{cases}$$

Assuming that functions f_1, f_2, are differentiable of order $i,\ j \in \{1,\ 2\}$ and functions $g_t,\ (t \in T)\,,\ h_s,\ (s \in S)$ are second order differentiable, we will approximate original problem $\left(P_0^{0,0}\right)$ by problems $\left(P_2^{i,j}\right)$:

$$\begin{cases} \min\left(F_1^i, F_2^j\right)(x) \\ x = (x_1, x_2, ...x_n) \in X \\ G_t^2\left(x\right) \leq 0,\ t \in T \\ H_s^2\left(x\right) = 0,\ s \in S \end{cases}$$

where $(i,j) \in \{(1,0)\,,\ (1,1)\,,\ (2,0)\,,\ (2,1)\,,\ (2,2)\}$ and $F_1^0 = f_1,\ F_2^0 = f_2$. We denote by

$$\mathcal{F}^k = \left\{x \in X :\ G_t^k\left(x\right) \leq 0,\ t \in T,\ H_s^k\left(x\right) = 0,\ s \in S\right\},\ k \in \{0,1,2\}$$

the set of feasible solutions for bi-criteria optimization problem $\left(P_k^{i,j}\right)$, where $(i,j) \in \{(1,0)\,,\ (1,1)\,,\ (2,0)\,,\ (2,1)\,,\ (2,2)\}$ and $k \in \{0,1,2\}$.

3. Approximate problems and relation to initial problem

In this section we will study the conditions such that efficient solution of approximated problems $\left(P_2^{1,0}\right)$, $\left(P_2^{2,0}\right)$, $\left(P_2^{2,1}\right)$ and $\left(P_2^{2,2}\right)$ will remain efficient also for original problem $\left(P_0^{0,0}\right)$ and reciprocally.

Case $\left(P_2^{1,1}\right)$ was studied in [1], where also conditions such that $\mathcal{F}^0 \subseteq \mathcal{F}^2$ and $\mathcal{F}^2 \subseteq \mathcal{F}^0$ were analyzed. We will use them in our work, so we will briefly present the Theorems stating these inclusions.

Theorem 3.1 (Boncea and Duca [1]**).** *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.*

Assume that:

- **a.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **b.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

then

$$\mathcal{F}^0 \subseteq \mathcal{F}^2.$$

Theorem 3.2 (Boncea and Duca [1]**).** *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.*

Assume that

a. *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
b. *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

then

$$\mathcal{F}^2 \subseteq \mathcal{F}^0.$$

Theorem 3.3. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

a. $x_0 \in \mathcal{F}^0$,
b. *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
c. *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
d. *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
e. *f_2 is differentiable at x_0 and invex^1 at x_0 with respect to η,*
f. $\eta(x_0,\ x_0) = 0.$

If x_0 is an efficient solution for $\left(P_2^{2,1}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. x_0 being an efficient solution for $\left(P_2^{2,1}\right)$, implies that

$$\nexists x \in \mathcal{F}^2 \ s.t.\ \left(F_1^2(x), F_2^1(x)\right) \leq \left(F_1^2(x_0), F_2^1(x_0)\right).$$

Conditions b) and c) imply that

$$\mathcal{F}^0 \subseteq \mathcal{F}^2$$

and thus

$$\nexists x \in \mathcal{F}^0 \ s.t.\ \left(F_1^2(x), F_2^1(x)\right) \leq \left(F_1^2(x_0), F_2^1(x_0)\right). \tag{3.1}$$

Let's assume that x_0 is not an efficient solution for $\left(P_0^{0,0}\right)$. Then

$$\exists y \in \mathcal{F}^0 \ s.t.\ (f_1(y), f_2(y)) \leq (f_1(x_0), f_2(x_0))$$

which implies that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} f_1(y) < f_1(x_0) \\ f_2(y) \leqq f_2(x_0) \end{cases} \tag{3.2}$$

or

$$\begin{cases} f_1(y) \leqq f_1(x_0) \\ f_2(y) < f_2(x_0). \end{cases} \tag{3.3}$$

Because f_1 is invex^2 at x_0 with respect to η we get $F_1^2(y) \leq f_1(y)$, $\forall y \in \mathcal{F}^0$.
Because f_2 is invex^1 at x_0 with respect to η we get $F_2^1(y) \leq f_2(y)$, $\forall y \in \mathcal{F}^0$.
Because $\eta(x_0\ x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$ and $f_2(x_0) = F_2^1(x_0)$.
Thus from (3.2) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} F_1^2(y) < F_1^2(x_0) \\ F_2^1(y) \leqq F_2^1(x_0) \end{cases}$$

which contradicts (3.1) and from (3.3) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} F_1^2(y) \leqq F_1^2(x_0) \\ F_2^1(y) < F_2^1(x_0) \end{cases}$$

which contradicts (3.1).
In conclusion x_0 is an efficient solution for $\left(P_0^{0,0}\right)$. □

Theorem 3.4. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^2$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **e.** *f_2 is differentiable at x_0 and incave^1 at x_0 with respect to η,*
- **f.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,1}\right)$.

Proof. x_0 being an efficient solution for $\left(P_0^{0,0}\right)$, implies that

$$\nexists x \in \mathcal{F}^0 \text{ s.t. } (f_1(x), f_2(x)) \leq (f_1(x_0), f_2(x_0)).$$

Conditions b) and c) imply that

$$\mathcal{F}^2 \subseteq \mathcal{F}^0$$

and thus

$$\nexists x \in \mathcal{F}^2 \text{ s.t. } (f_1(x), f_2(x)) \leq (f_1(x_0), f_2(x_0)). \tag{3.4}$$

Let's assume that x_0 is not an efficient solution for $\left(P_2^{2,1}\right)$. Then

$$\exists y \in \mathcal{F}^2 \text{ s.t. } \left(F_1^2(y), F_2^1(y)\right) \leq \left(F_1^2(x_0), F_2^1(x_0)\right)$$

which implies that $\exists y \in \mathcal{F}^2$ *s.t.*

$$\begin{cases} F_1^2(y) < F_1^2(x_0) \\ F_2^1(y) \leqq F_2^1(x_0) \end{cases} \tag{3.5}$$

or

$$\begin{cases} F_1^2(y) \leqq F_1^2(x_0) \\ F_2^1(y) < F_2^1(x_0). \end{cases} \tag{3.6}$$

Because f_1 is incave^2 at x_0 with respect to η we get $f_1(y) \leq F_1^2(y)$, $\forall y \in \mathcal{F}^2$.
Because f_2 is incave^1 at x_0 with respect to η we get $f_2(y) \leq F_2^1(y)$, $\forall y \in \mathcal{F}^2$.
Because $\eta(x_0\ x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$ and $f_2(x_0) = F_2^1(x_0)$.
Thus from (3.5) we get that $\exists y \in \mathcal{F}^2$ *s.t.*

$$\begin{cases} f_1(y) < f_1(x_0) \\ f_2(y) \leqq f_2(x_0) \end{cases}$$

which contradicts (3.4) and from (3.6) we get that $\exists y \in \mathcal{F}^2$ *s.t.*

$$\begin{cases} f_1(y) \leqq f_1(x_0) \\ f_2(y) < f_2(x_0) \end{cases}$$

which contradicts (3.4).
In conclusion x_0 is an efficient solution for $\left(P_2^{2,1}\right)$. □

Theorem 3.5. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^0$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is differentiable at x_0 and invex^1 at x_0 with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{1,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem* 3.3. □

Theorem 3.6. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^2$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is differentiable at x_0 and incave^1 at $x_0$with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{1,0}\right)$.

Proof. Proof is similar with *Theorem* 3.4. □

Theorem 3.7. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^0$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem* 3.3. □

Theorem 3.8. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^2$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,0}\right)$.

Proof. Proof is similar with *Theorem* 3.4. □

Theorem 3.9. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^0$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **e.** *f_2 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **f.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem* 3.3. □

Theorem 3.10. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

a. $x_0 \in \mathcal{F}^2$,
b. *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
c. *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
d. *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
e. *f_2 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
f. $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,2}\right)$.

Proof. Proof is similar with *Theorem* 3.4. □

4. Numerical examples

In the above theorems, conditions referring to invexity, incavity or avexity of functions are essential to ensure that efficient solution of the initial problem remains efficient for the approximate problem and reciprocally. If those conditions are not fulfill it is possible either that efficient solution of initial problem remains efficient for the approximate problem (and reciprocally) or it does not remain efficient.

Example 4.1. Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(-\left(x_1 - \frac{3\pi}{5}\right)^2 - \left(x_2 - \frac{2\pi}{5} - 1\right)^2; -x_1 + x_2\right) \\ -x_1 - \sin x_1 + x_2 \le 0 \\ x_1 - \frac{5\pi}{2} \le 0 \\ x_1; x_2 \ge 0 \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ is $x_0 = (\frac{\pi}{2},\ 1 + \frac{\pi}{2}) \in \mathcal{F}^0$.
Second order approximate functions for the constraints are:

$$G_t^2(x) = g_t(x_0) + \nabla g_t(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\,\nabla^2 g_t \eta(x, x_0), t \in \{1, 2, 3, 4\}$$

Considering $\eta(x, x_0) = x - x_0$ we get:

$$G_1^2(x) = -x_1 + x_2 + \frac{1}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - 1,$$

$$G_2^2(x) = x_1 - \frac{5\pi}{2},$$

$$G_3^2(x) = x_1,\ G_4^2(x) = x_2.$$

Consequently, the approximate problem $\left(P_2^{0,0}\right)$ is:

$$\begin{cases} \min\left(-\left(x_1-\frac{3\pi}{5}\right)^2-\left(x_2-\frac{2\pi}{5}-1\right)^2;-x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0 \end{cases}$$

Calculating the values of objective function for problem $\left(P_2^{0,0}\right)$ in

$$x_0=\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)\in\mathcal{F}^2 \text{ and } x=\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right)\in\mathcal{F}^2$$

we obtain:

$$f\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right)=\left(-\frac{58\pi^2}{400}+\frac{14\pi^3}{640}-\frac{\pi^4}{32};\ 1-\frac{\pi^2}{32}\right)$$

and

$$f\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)=\left(-\frac{\pi^2}{50},\ 1\right).$$

Because $\left(-\frac{58\pi^2}{400}+\frac{14\pi^3}{640}-\frac{\pi^4}{32};\ 1-\frac{\pi^2}{32}\right)<\left(-\frac{\pi^2}{50},\ 1\right)$ it follows that $x_0=\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)$ is not an efficient solution for approximate problem $\left(P_2^{0,0}\right)$.

Example 4.2. Let's consider the same initial problem as in *Example* 4.1. First order approximations for the components of the objective function are

$$F_p^1(x)=f_p(x_0)+\nabla f_p(x_0)\,\eta(x,x_0),\ p\in\{1,2\}.$$

Considering $\eta(x,x_0)=x-x_0$ we get:

$$F_1^1(x)=-\frac{\pi}{5}x_1-\frac{\pi}{5}x_2+\frac{9\pi^2}{50}+\frac{\pi}{5}$$

and

$$F_2^1(x)=-x_1+x_2.$$

Approximate functions for the constrains are the same computed at *Example* 4.1. Consequently the approximate problem $\left(P_2^{1,1}\right)$ is:

$$\begin{cases} \min\left(-\frac{\pi}{5}x_1-\frac{\pi}{5}x_2+\frac{9\pi^2}{50}+\frac{\pi}{5};\ -x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0 \end{cases}$$

Calculating the values for the objective function of problem $\left(P_2^{1,1}\right)$ in

$$x_0=\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)\in\mathcal{F}^2 \text{ and } x=\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right)\in\mathcal{F}^2$$

we get that

$$F^1\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right) < F^1\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)$$

which proves that $x_0 = \left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)$ is not an efficient solution for problem $\left(P_2^{1,1}\right)$.

Example 4.3. Let's consider the same initial problem as in *Example* 4.1. Second order approximations for the components of the objective function are

$$F_p^2(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x,x_0) + \frac{1}{2}\eta(x,x_0)^T\,\nabla^2 f_p(x_0)\,\eta(x,x_0),\ p \in \{1,2\}.$$

Considering $\eta(x,x_0) = x - x_0$ we get:

$$F_1^2(x) = -\frac{\pi}{2}\left(x_1-\frac{\pi}{2}\right)^2 - \frac{\pi+2}{2}\left(x_2-1-\frac{\pi}{2}\right)^2 - \frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5}$$

and

$$F_2^2(x) = -x_1 + x_2.$$

Approximate functions for the constrains are the same computed at *Example* 4.1. Consequently the approximate problem $\left(P_2^{2,2}\right)$ is:

$$\begin{cases} \min\left(-\frac{\pi}{2}\left(x_1-\frac{\pi}{2}\right)^2 - \frac{\pi+2}{2}\left(x_2-1-\frac{\pi}{2}\right)^2 - \frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5};\ -x_1+x_2\right) \\ -x_1 + x_2 + \frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2 - 1 \le 0 \\ x_1 - \frac{5\pi}{2} \le 0 \\ x_1; x_2 \ge 0 \end{cases}$$

Calculating the values for the objective function of problem $\left(P_2^{2,2}\right)$ in

$$x_0 = \left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right) \in \mathcal{F}^2 \text{ and } x = \left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right) \in \mathcal{F}^2$$

we get that

$$F^2\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right) < F^2\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)$$

which proves that $x_0 = \left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)$ is not an efficient solution for problem $\left(P_2^{2,2}\right)$.

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