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Λ^2-statistical convergence and its application to Korovkin second theorem

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Abstract. In this paper, we use the notion of strong $(N, \lambda^2)-$summability to generalize the concept of statistical convergence. We call this new method a λ^2-statistical convergence and denote by S_{λ^2} the set of sequences which are λ^2-statistically convergent. We find its relation to statistical convergence and strong $(N, \lambda^2)-$summability. We will define a new sequence space and will show that it is Banach space. Also we will prove the second Korovkin type approximation theorem for λ^2-statistically summability and the rate of λ^2-statistically summability of a sequence of positive linear operators defined from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

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1. Introduction

By w, we denote the space of all real or complex valued sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=1}^{\infty}$. Let $\Lambda = \{\lambda_k : k = 0, 1, \ldots\}$ be a nondecreasing sequence of positive numbers tending to ∞, as $k \to \infty$ and $\Delta^2\lambda_n \geq 0$, for each $n \in \mathbb{N}$. The first difference is defined as follows: $\Delta\lambda_k = \lambda_k - \lambda_{k-1}$, where $\lambda_{-1} = \lambda_{-2} = 0$, and the second difference is defined as

$$\Delta^2(\lambda_k) = \Delta(\Delta(\lambda_k)) = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2}.$$

Let $x = (x_k)$ be a sequence of complex numbers, such that $x_{-1} = x_{-2} = 0$. We will denote by

$$\Lambda^2(x) = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}). \tag{1.1}$$

A sequence $x=(x_k)$, is said to be strongly $(N,\lambda^2)-$ summable to a number L (see [8]) if

$$\lim_n \frac{1}{\lambda_n-\lambda_{n-1}}\sum_{k=0}^{n}|(\lambda_k x_k-2\lambda_{k-1}x_{k-1}+\lambda_{k-2}x_{k-2})-L|=0.$$

Let us denote by

$$[N,\lambda^2]=\Big\{x=(x_n):\exists L\in\mathbb{C},$$

$$\lim_{n\to\infty}\frac{1}{\lambda_n-\lambda_{n-1}}\sum_{k=1}^{n}|(\lambda_k x_k-2\lambda_{k-1}x_{k-1}+\lambda_{k-2}x_{k-2})-L|=0\Big\}$$

for the sets of sequences $x=(x_n)$ which are strongly (N,λ^2) summable to L, i.e., $x_k\to L[N,\lambda^2]$. The idea of statistical convergence was introduced by Fast [12] and studied by various authors (see [10], [13], [20], [5], [6]). A sequence $x=(x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon>0$,

$$\lim_n \frac{1}{n}|\{k\le n:|x_k-L|\ge\varepsilon\}|=0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S-\lim_n x=L$ or $x_k\to L(S)$ and S denotes the set of all statistically convergent sequences. In this paper, we introduce and study the concept of λ^2-statistical convergence and determine how it is related to $[N,\lambda^2]$ and S.

Definition 1.1. A sequence $x=(x_n)$ is said to be λ^2-statistically convergent or $S_{\lambda^2}-$convergent to L if for every $\varepsilon>0$

$$\lim_n \frac{1}{\lambda_n-\lambda_{n-1}}|\{k\le n:|(\lambda_k x_k-2\lambda_{k-1}x_{k-1}+\lambda_{k-2}x_{k-2})-L|\ge\varepsilon\}|=0.$$

In this case we write $S_{\lambda^2}-\lim_n x_n=L$ or $x_n\to L(S_{\lambda^2})$, and

$$S_{\lambda^2}=\{x=(x_n):\exists L\in\mathbb{C},S_{\lambda^2}-\lim_n x_n=L\}.$$

Definition 1.2. A sequence $x=(x_n)$ is said to be λ^2-statistically Cauchy if for every $\varepsilon>0$ exists a number $N=N(\varepsilon)$, such that

$$\lim_n \frac{1}{\lambda_n-\lambda_{n-1}}\left|\left\{k\le n:\left|\Delta^2\lambda_k(x_k)-\Delta^2\lambda_N(x_N)\right|\ge\varepsilon\right\}\right|=0.$$

A sequence of positive integers $\theta=(k_r)$ is called lacunary if $k_0=0, 0<k_r<k_{r+1}$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. And with I_r we will denote the following interval:$I_r=(k_{r-1},k_r]$, respectively q_r the ration: $\frac{k_r}{k_{r-1}}$.

Definition 1.3. A sequence $x=(x_n)$ is said to be lacunary λ^2-statistically convergent or $S^{\theta}_{\lambda^2}-$convergent to L if for every $\varepsilon>0$

$$\lim_r \frac{1}{h_r}|\{k\in I_r:|(\lambda_k x_k-2\lambda_{k-1}x_{k-1}+\lambda_{k-2}x_{k-2})-L|\ge\varepsilon\}|=0.$$

In this case we write $S^{\theta}_{\lambda^2}-\lim_n x_n=L$ or $x_n\to L(S^{\theta}_{\lambda^2})$, and

$$S^{\theta}_{\lambda^2}=\{x=(x_n):\exists L\in\mathbb{C},S^{\theta}_{\lambda^2}-\lim_n x_n=L\}.$$

Definition 1.4. A sequence $x = (x_n)$ is said to be lacunary λ^2-statistically Cauchy if for every $\varepsilon > 0$ exists a number $N = N(\varepsilon)$, such that

$$\lim_r \frac{1}{h_r} \left|\left\{k \in I_r : \left|\Delta^2\lambda_k(x) - \Delta^2\lambda_N(x)\right| \geq \varepsilon\right\}\right| = 0.$$

2. Some properties of $[N, \lambda^2]$ and S_{λ^2}

In this section we will show relation between $[N, \lambda^2]$ and S_{λ^2}.

Theorem 2.1. *Let (λ_n) be a sequence from Λ, then:*

1. $x_n \to L[N, \lambda^2]$, *then* $x_n \to L(S_{\lambda^2})$ *and the inclusion is proper.*
2. *If* $\Delta^2\lambda(x) \in l_\infty$ *and* $x_n \to L(S_{\lambda^2})$, *then* $x_n \to L[N, \lambda^2]$.
3. $S_{\lambda^2} \cap l_\infty = [N, \lambda^2] \cap l_\infty$.

Proof. (1) Let us suppose that $x_n \to L[N, \lambda^2]$. Then for every $\varepsilon > 0$ we have:

$$\sum_{k=1}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$\geq \sum_{\substack{k=1 \\ |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon}}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$\geq \varepsilon \left|\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon\}\right|.$$

Therefore $x_n \to L[N, \lambda^2] \Rightarrow x_n \to L(S_{\lambda^2})$. To prove the second part of the (1), we will show this.

Example 2.2. Let $x = x_n$ defined as follows:

$$x_n = \begin{cases} [\sqrt{\lambda_n - \lambda_{n-1}}], & 0 \leq k \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Then $x = (x_n) \notin l_\infty$ and for every $\varepsilon > 0$, we get that

$$\lim_n \frac{1}{\lambda_n - \lambda_{n-1}} \left|\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - 0| \geq \varepsilon\}\right|$$

$$\leq \lim_n \frac{[\sqrt{\lambda_n - \lambda_{n-1}}]}{\lambda_n - \lambda_{n-1}} = 0.$$

On the other hand

$$\lim_{n\to\infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=1}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - 0|$$

$$= \lim_n \frac{\lambda_n[\sqrt{\lambda_n - \lambda_{n-1}}] - 2\lambda_{n-1}[\sqrt{\lambda_{n-1} - \lambda_{n-2}}] + \lambda_{n-2}[\sqrt{\lambda_{n-2} - \lambda_{n-3}}]}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} = \infty.$$

(2) Let us suppose that $x_n \to L(S_{\lambda^2})$ and $\Delta^2\lambda(x) \in l_\infty$, then we can consider that

$$|\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2} - L| \leq M.$$

For any given $\varepsilon > 0$ we get the following estimation:

$$\frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=1}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{\substack{k=1 \\ |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon}}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$+ \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{\substack{k=1 \\ |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \leq \varepsilon}}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$\leq \frac{M}{\lambda_n - \lambda_{n-1}} |\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon\}| + \varepsilon,$$

which implies that $x_k \to L[N, \lambda^2]$.
(3) Follows immediately from (1) and (2). □

Proposition 2.3. *If $x = (x_n)$ is λ^2-statistically convergent to L, then it follows that x is λ^2-statistically Cauchy sequence.*

Proof. Let us suppose that x converges Λ^2-statistically to L and $\varepsilon > 0$. Then

$$\frac{1}{\lambda_n - \lambda_{n-1}} |\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon\}| \leq \frac{\varepsilon}{2}$$

satisfies for almost all k, and if N is chosen such that

$$\frac{1}{\lambda_N - \lambda_{N-1}} |\{k \leq N : |(\lambda_N x_N - 2\lambda_{N-1}x_{N-1} + \lambda_{N-2}x_{N-2}) - L| \geq \varepsilon\}| \leq \frac{\varepsilon}{2},$$

then we have:

$$\frac{1}{\lambda_n - \lambda_{n-1}} \left|\{k \leq n : \left|\Delta^2\lambda_k(x) - \Delta^2\lambda_N(x)\right| \geq \varepsilon\}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for almost k. Hence x is λ^2-statistically Cauchy sequence. □

Proposition 2.4. *If $x = (x_n)$ is lacunary λ^2-statistically convergent to L, then it follows that x is λ^2-statistically lacunary Cauchy sequence.*

Proposition 2.5. *If $x = (x_n)$ is a sequence for which there is a λ^2-statistically convergent sequence $y = (y_n)$ such that $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k, then it follows that x is λ^2-statistically convergent sequence.*

Proof. Let us consider that $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k. And $y_k \to L(S_{\lambda^2})$. Then for each $\varepsilon > 0$ and for every n we have:

$$\left\{k \leq n : |\Delta^2\lambda(x_k) - L| \geq \varepsilon\right\}$$
$$\subset \left\{k \leq n : \Delta^2\lambda(x_k) \neq \Delta^2\lambda(y_k)\right\} \cup \left\{k \leq n : |\Delta^2\lambda(y_k) - L| \geq \varepsilon\right\}.$$

From fact that $y_k \to L(S_{\lambda^2})$, it follows that set $\{k \le n : |\Delta^2\lambda(y_k) - L| \ge \varepsilon\}$ has finite numbers which are not depended from n, hence

$$\frac{\left|\{k \le n : |\Delta^2\lambda(y_k) - L| \ge \varepsilon\}\right|}{\lambda_n - \lambda_{n-1}} \to 0, n \to \infty.$$

On the other hand, from $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k, we get:

$$\frac{\left|\{k \le n : \Delta^2\lambda(x_k) \ne \Delta^2\lambda(y_k) \ge \varepsilon\}\right|}{\lambda_n - \lambda_{n-1}} \to 0, n \to \infty.$$

From last two relations follows that:

$$\frac{\left|\{k \le n : |\Delta^2\lambda(x_k) - L| \ge \varepsilon\}\right|}{\lambda_n - \lambda_{n-1}} \to 0, n \to \infty. \qquad \square$$

Proposition 2.6. *If $x = (x_n)$ is a sequence for which there is a lacunary λ^2-statistically convergent sequence $y = (y_n)$ such that $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k, then it follows that x is lacunary λ^2-statistically convergent sequence.*

Theorem 2.7. *Let θ be a lacunary sequence, then*

1. $L(S_{\lambda^2}) \subset L(S^{\theta}_{\lambda^2})$ *if and only if* $\lim_r \inf q_r > 1$.
2. $L(S^{\theta}_{\lambda^2}) \subset L(S_{\lambda^2})$ *if and only if* $\lim_r \sup q_r < \infty$.
3. $L(S_{\lambda^2}) = L(S^{\theta}_{\lambda^2})$ *if and only if* $1 < \lim_r \inf q_r \le \lim_r \sup q_r < \infty$.

Proof. Proof of the Proposition is omitted, because it is similar to Lemmas 2,3 in [14]. $\square$

We will denote by $\Lambda^2(X) = \{x = (x_n) \in w : \Lambda^2(x) \in X\}$. It is know that $(\Lambda^2(X), ||\cdot||_{\Lambda^2(X)})$ is a normed space where norm is given by (see [8]):

$$||x||_{\Lambda^2(X)} := \sup_{n\ge 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} |\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}|,$$

where $x = (x_k)$.

Theorem 2.8. $\Lambda^2(X)$ *is Banach space.*

Proof. Let (x_n) be any Cauchy sequence in $\Lambda^2(X)$, where $x^s = (x_1^s, x_2^s, \cdots, x_n^s, \cdots)$. Then there it follows that:

$$||x^s - x^t||_{\Lambda^2(X)} \to 0, s, t \to \infty.$$

From last relation we get:

$$\sup_{n\ge 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k^s - x_k^t) - 2\lambda_{k-1}(x_{k-1}^s - x_{k-1}^t) + \lambda_{k-2}(x_{k-2}^s - x_{k-2}^t)\right| \to 0,$$
$$t, s \to \infty.$$

Hence we obtain,

$$|x_k^t - x_k^s| \to 0, t, s \to \infty,$$

for every $k \in \mathbb{N}$. Therefore $(x_k^1, x_k^2, \cdots)$ is a Cauchy sequences in $\mathbb{C}$, the set of complex numbers. Since $\mathbb{C}$ is complete, it is convergent. Let us say

$$\lim_s x_k^s = x_k,$$

for every $k \in \mathbb{N}$. Since (x^s) is a Cauchy sequence, for each $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that

$$||x^s - x^t||_{\Lambda^2(X)} < \varepsilon$$

for all $s, t \geq N$ and for all $k \in \mathbb{N}$. Hence

$$\sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k^s - x_k^t) - 2\lambda_{k-1}(x_{k-1}^s - x_{k-1}^t) + \lambda_{k-2}(x_{k-2}^s - x_{k-2}^t)\right| < \varepsilon,$$

for all $s, t \geq N$ and for all $k \in \mathbb{N}$. If we pass with limit, in the last relation, when $t \to \infty$, we get:

$$\lim_t \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k^s - x_k^t) - 2\lambda_{k-1}(x_{k-1}^s - x_{k-1}^t) + \lambda_{k-2}(x_{k-2}^s - x_{k-2}^t)\right|$$

$$= \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k^s - x_k) - 2\lambda_{k-1}(x_{k-1}^s - x_{k-1}) + \lambda_{k-2}(x_{k-2}^s - x_{k-2})\right| < \varepsilon,$$

for all $s \geq N$ and for all $k \in \mathbb{N}$. This implies that

$$||x^s - x||_{\Lambda^2(X)} < \varepsilon,$$

for all $s \geq N$, that is $x^s \to x$, as $s \to \infty$ where $x = (x_k)$.
Since

$$||x||_{\Lambda^2(x)} = \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} |\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}|$$

$$= \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} |\lambda_k(x_k - x_k^N + x_k^N) - 2\lambda_{k-1}(x_{k-1} - x_{k-1}^N + x_{k-1}^N)$$

$$+\lambda_{k-2}(x_{k-2} - x_{k-2}^N + x_{k-2}^N)|$$

$$\leq \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k - x_k^N) - 2\lambda_{k-1}(x_{k-1} - x_{k-1}^N) + \lambda_{k-2}(x_{k-2} - x_{k-2}^N)\right|$$

$$+ \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k x_k^N - 2\lambda_{k-1}x_{k-1}^N + \lambda_{k-2}x_{k-2}^N\right|$$

$$\leq ||x^N - x||_{\Lambda^2(X)} + ||x^N||_{\Lambda^2(X)} = O(1),$$

we obtain $x \in \Lambda^2(X)$. $\square$

3. A Korovkin second type theorem

We say that the sequence (x_n) is Λ^2- summable to L if $\lim_n \Lambda^2 = L$.

Definition 3.1. We say that the sequence (x_n) is statistically summable to L by the weighted method determined by the sequence Λ^2 if $st - \lim_n \Lambda^2 = L$.

And we denote by $\Lambda^2(st)$ the set of all sequences which are statistically summable Λ^2. In the sequel we will use some notation related to the function spaces. With $F(\mathbb{R})$ we will denote the linear space of all real-valued functions defined in $\mathbb{R}$. And with $C(\mathbb{R})$ we will denote the space of all bounded and continuous functions defined in $\mathbb{R}$. It is know fact that $C(\mathbb{R})$ is a Banach space equipped with norm

$$||f||_\infty = \sup_{x\in\mathbb{R}} |f(x)|, f \in C(\mathbb{R}).$$

The space of all continuous and periodic functions with period 2π we will denote by $C_{2\pi}(\mathbb{R})$, which is a Banach space under norm given by

$$||f||_{2\pi} = \sup_{x\in\mathbb{R}} |f(x)|.$$

The classical Korovkin first and second theorems are given as follows (see [16, 17, 3]):

Theorem 3.2. *Let (T_n) be a sequence of positive linear operators from $C[0,1]$ into $F[0,1]$. Then*

$$\lim_{n\to\infty} ||T_n(f,x) - f(x)||_\infty = 0,$$

for all $f \in C[0,1]$ if and only if

$$\lim_{n\to\infty} ||T_n(f_i,x) - f_i(x)||_\infty = 0,$$

for $i \in \{0,1,2\}$ where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Theorem 3.3. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $F(\mathbb{R})$. Then*

$$\lim_{n\to\infty} ||T_n(f,x) - f(x)||_{2\pi} = 0,$$

for all $f \in C_{2\pi}(\mathbb{R})$ if and only if

$$\lim_{n\to\infty} ||T_n(f_i,x) - f_i(x)||_{2\pi} = 0,$$

for $i \in \{0,1,2\}$ where $f_0(x) = 1$, $f_1(x) = \cos x$ and $f_2(x) = \sin x$.

The Korovkin type theorems are investigated by several mathematicians in generalization of them in many ways and several settings such as function spaces, abstract Banach latices, Banach algebras, and so on. This theory is useful in real analysis, functional analysis, harmonic analysis, and so on. For more results related to the Korovkin type theorems see ([4, 11, 19, 21, 22, 24, 9, 7, 18, 2, 1, 23, 15]). In this paper we will prove the second Korovkin-type theorem with the help of Λ^2-statistically summability method which is a generalization of that given in [19] and [16, 17].

For given sequence of linear operators L_n we say that they are positive if $L_n(f(x)) \geq 0$ for all $f(x) \geq 0$, for given x.

Theorem 3.4. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then*

$$\Lambda^2(st)\text{-}\lim_{n\to\infty} ||T_n(f,x) - f(x)||_{2\pi} = 0, \ forall \ f \in C_{2\pi}(\mathbb{R}) \tag{3.1}$$

if and only if

$$\Lambda^2(st)\text{-}\lim_{n\to\infty} ||T_n(f_i,x) - f_i(x)||_{2\pi} = 0, \ i = 0,1,2, \tag{3.2}$$

where $f_0(x) = 1$, $f_1(x) = \cos x$ and $f_2(x) = \sin x$.

Proof. Let us consider that relation (3.1) is valid for all $f \in C_{2\pi}(\mathbb{R})$. Then it is valid especially for the $f(x) = 1, f(x) = \cos x$ and $f(x) = \sin x$, and condition (3.2) is valid. Now we will prove the contrary. Let us suppose that relations (3.2) is valid and we will prove that (3.1) is valid, too. Let $I = (a, a + 2\pi]$ any subinterval of length 2π from $\mathbb{R}$. Let us fix $x \in I$. By the conditions given for $f(x)$ it follows that:

$$(\forall\varepsilon > 0)(\exists\delta(\varepsilon) > 0) \to |f(t) - f(x)| < \varepsilon, \tag{3.3}$$

for all t, whenever $|t-x| < \delta$. If $|t-x| \geq \delta$. Let us consider that $t \in (x+\delta, 2\pi+x+\delta]$, then we get:

$$|f(t) - f(x)| \leq 2||f||_{2\pi} \leq \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\psi(t) \tag{3.4}$$

where $\psi(t) = \sin^2\left(\frac{t-x}{2}\right)$. From relations (3.3) and (3.4) for any fixed $x \in I$ and for any t we obtain:

$$|f(t) - f(x)| \leq \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\psi(t) + \varepsilon. \tag{3.5}$$

Respectively,

$$-\varepsilon - \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\psi(t) < f(t) - f(x) < \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\psi(t) + \varepsilon.$$

Applying the operator $T_n(1,x)$ in this inequality we have:

$$T_k(1,x)\left(-\varepsilon - \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\psi(t)\right) < T_k(1,x)\left(f(t) - f(x)\right) < T_k(1,x)\left(\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\psi(t) + \varepsilon\right).$$

Value of x is fixed, which means that $f(x)$ is a constant and above relation takes this form:

$$\begin{aligned} -\varepsilon T_k(1,x) - \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}} T_k(\psi(t),x) &< T_k(f,x) - f(x)T_k(1,x) \\ &< \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}} T_k(\psi(t),x) + \varepsilon T_k(1,x). \end{aligned} \tag{3.6}$$

On the other hand

$$T_k(f,x) - f(x) = T_k(f,x) - f(x)T_k(1,x) + f(x)[T_k(1,x) - 1]. \tag{3.7}$$

From relations (3.6) and (3.7) we have:

$$T_k(f,x) - f(x) < \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}} T_k(\psi(t),x) + \varepsilon T_k(1,x) + f(x)[T_k(1,x) - 1]. \tag{3.8}$$

Let us now estimate the following expression:

$$\begin{aligned}T_k(\psi(t),x) &= T_k\left(\sin^2\left(\frac{t-x}{2}\right),x\right) = T_k\left(\frac{1}{2}(1-\cos t\cos x-\sin t\sin x),x\right)\\ &= \frac{1}{2}\left\{T_k(1,x)-\cos xT_k(\cos t,x)-\sin xT_k(\sin t,x)\right\}\\ &= \frac{1}{2}\left\{[T_k(1,x)-1]-\cos x[T_k(\cos t,x)-\cos x]-\sin x[T_k(\sin t,x)-\sin x]\right\}.\end{aligned}$$

Now, from the last relation and (3.8), we obtain that

$$\begin{aligned}T_k(f,x)-f(x) &< \frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\frac{1}{2}\Big\{[T_k(1,x)-1]-\cos x[T_k(\cos t,x)-\cos x]\\ &\quad -\sin x[T_k(\sin t,x)-\sin x]\Big\}+\varepsilon T_k(1,x)+f(x)[T_k(1,x)-1]\\ &= \varepsilon+\varepsilon[T_k(1,x)-1]+f(x)[T_k(1,x)-1]+\frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\frac{1}{2}\Big\{[T_k(1,x)-1]\\ &\quad -\cos x[T_k(\cos t,x)-\cos x]-\sin x[T_k(\sin t,x)-\sin x]\Big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|T_k(f,x)-f(x)| &\le \varepsilon+\left(\varepsilon+|f(x)|+\frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\right)|T_k(1,x)-1|\\ &\quad +\frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\Big\{|\cos x|\cdot|T_k(\cos t,x)-\cos x|\\ &\quad +|\sin x|\cdot|T_k(\sin t,x)-\sin x|\Big\}\\ &\le \varepsilon+\left(\varepsilon+|f(x)|+\frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\right)|T_k(1,x)-1|\\ &\quad +\frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\Big\{|T_k(\cos t,x)-\cos x|+|T_k(\sin t,x)-\sin x|\Big\}.\end{aligned}$$

Now taking the $\sup_{x\in I}$ in the above relation, we get:

$$\begin{aligned}\|T_k(f,x)-f(x)\|_{2\pi} &\le \varepsilon+K\Big(\|T_k(1,x)-1\|_{2\pi}+\|T_k(\cos t,x)-\cos x\|_{2\pi}\\ &\quad +\|T_k(\sin t,x)-\sin x\|_{2\pi}\Big),\end{aligned}$$

where

$$K=\max\left\{\varepsilon+\|f\|_{2\pi}+\frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}},\frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\right\}.$$

Now replacing $T_k(.,x)$ by

$$\Lambda^2(.,x)=\frac{1}{\lambda_n-\lambda_{n-1}}\sum_{k=0}^{n}(\lambda_kT_k(.,x)-2\lambda_{k-1}T_{k-1}(.,x)+\lambda_{k-2}T_{k-2}(.,x))$$

in the above inequality on both sides. For a given $r > 0$, we can choose ε_1 such that $\varepsilon_1 < r$. Now we will define the following sets:

$$D = \Big\{k \leq \mathbb{N} : ||\Lambda^2(f,x) - f(x)||_{2\pi} \geq r\Big\},$$

$$D_i = \Big\{k \leq \mathbb{N} : ||\Lambda^2(f_i,x) - f_i(x)||_{2\pi} \geq \frac{r-\varepsilon_1}{3K}\Big\},\ i = 0,1,2.$$

Then $D \subset \cup_{i=0}^{2} D_i$ and for their densities is satisfied relation:

$$\delta(D) \leq \delta(D_0) + \delta(D_1) + \delta(D_2).$$

Finally, from relations (3.2) and the above estimation we get:

$$\Lambda^2(st)\text{-}\lim_n ||\Lambda^2(f,x) - f(x)||_{2\pi} = 0,$$

which completes the proof. □

Remark 3.5. If we take $\lambda_n = n^2$, then our Theorem 3.4 reduce to Theorem 2.1 of [19].

4. Rate of Λ^2- statistically convergence

In this section we will show the rate of the Λ^2- statistical convergence of positive linear operators in $C_{2\pi}(\mathbb{R})$ spaces.

Definition 4.1. Let (a_n) be any positive, nondecreasing sequence of positive numbers. We say that sequence $x = (x_n)$ is Λ^2- statistical convergent to number L with rate of convergence $o(a_n)$, if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{a_n} |\{m \leq n : |T_m - L| \geq \varepsilon\}| = 0.$$

In this case, we write $x_n - L = \Lambda^2(st) - o(a_n)$.

Lemma 4.2. *Let (a_n) and (b_n) be two positive nondecreasing positive numeric sequences. Let $x = (x_n)$ and $y = (y_n)$ be two sequences such that $x_n - L_1 = \Lambda^2(st) - o(a_n)$ and $y_n - L_2 = \Lambda^2(st) - o(b_n)$. Then*

1. $\alpha(x_n - L) = \Lambda^2(st) - o(a_n)$, *for any scalar* α.
2. $(x_n - L_1) \pm (y_n - L_2) = \Lambda^2(st) - o(c_n)$.
3. $(x_n - L_1)(y_n - L_2) = \Lambda^2(st) - o(a_n b_n)$,

where $c_n = \max\{a_n, b_n\}$.

Now let us recall the notion of the modules of continuity. The modulus of continuity for function $f(x) \in C_{2\pi}(\mathbb{R})$, is defined as follows:

$$\omega(f,\delta) = \sup_{|h|<\delta} |f(x+h) - f(x)|.$$

It is known that, for any value of the $|x-y|$, we get:

$$|f(x) - f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{4.1}$$

We have the following result:

Theorem 4.3. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Suppose that*

1. $||T_n(1,x) - 1||_{2\pi} = \Lambda^2(st) - o(a_n)$.
2. $\omega(f, \lambda_k) = \Lambda^2(st) - o(b_n)$, *where* $\lambda_n = \sqrt{T_n(\phi_x, x)}$ *and* $\phi_x(y) = (y-x)^2$.

Then for all $f \in C_{2\pi}(\mathbb{R})$, we have:

$$||T_n(f,x) - f(x)||_{2\pi} = \Lambda^2(st) - o(c_n),$$

where $c_n = \max\{a_n, b_n\}$.

Proof. Let $f \in C_{2\pi}(\mathbb{R})$ and $x \in [-\pi, \pi]$. From relations (3.7) and (4.1) we get this estimation:

$$\begin{aligned}
|T_n(f,x) - f(x)| &\le |T_n(|f(y) - f(x)|, x)| + |f(x)| \cdot |T_n(1,x) - 1| \\
&\le T_n\left(\frac{|x-y|}{\delta} + 1, x\right)\omega(f,\delta) + |f(x)| \cdot |T_n(1,x) - 1| \\
&\quad \text{(by Cauchy-Schwartz inequality)} \\
&\le \frac{1}{\delta}(T_n((x-y)^2, x))^{\frac{1}{2}}(T_n(1,x))^{\frac{1}{2}}\omega(f,\delta) + |f(x)| \cdot |T_n(1,x) - 1|.
\end{aligned}$$

If we are putting $\delta = \lambda_n = \sqrt{T_n(\phi_x, x)}$ in the last relation we obtain:

$$\begin{aligned}
||T_n(f,x) - f(x)||_{2\pi} &\le ||f||_{2\pi}||T_n(1,x) - 1||_{2\pi} + 2\omega(f,\lambda_n) \\
&\quad + \omega(f,\lambda_n)||T_n(1,x) - 1||_{2\pi} + \omega(f,\lambda_n)\sqrt{||T_n(1,x) - 1||_{2\pi}} \\
&\le C\Big\{||T_n(1,x) - 1||_{2\pi} + \omega(f,\lambda_n) + \omega(f,\lambda_n)||T_n(1,x) - 1||_{2\pi} \\
&\quad + \omega(f,\lambda_n)\sqrt{||T_n(1,x) - 1||_{2\pi}}\Big\},
\end{aligned}$$

where $C = \max\big\{||f||_{2\pi}, 2\big\}$. Now replacing $T_k(.,x)$ by

$$\Lambda^2(.,x) = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k T_k(.,x) - 2\lambda_{k-1}T_{k-1}(.,x) + \lambda_{k-2}T_{k-2}(.,x)),$$

we get

$$\begin{aligned}
||\Lambda^2(f,x) - f(x)||_{2\pi} &\le C\Big\{||\Lambda^2(1,x) - 1||_{2\pi} + \omega(f,\lambda_n) + \omega(f,\lambda_n)||\Lambda^2(1,x) - 1||_{2\pi} \\
&\quad + \sqrt{\omega(f,\lambda_n)||\Lambda^2(1,x) - 1||_{2\pi}}\Big\}.
\end{aligned}$$

The proof follows from the conditions (1) and (2). □

In the following example we show that Theorem 3.4 is stronger than Theorem 3.3.

Example 4.4. For any $n \in \mathbb{N}$ we will denote by $S_n(f)$ the $n-$th partial sum of the Fourier series of f, i.e.,

$$S_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

Let us consider the following expression:

$$\Lambda^2(f,x) = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) S_k(f).$$

We know that $\lim_{n\to\infty} \Lambda^2(f,x) = f$ (see [8]). Let us denote by $L_n : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ defined by:

$$L_n(f,x) = (1 + x_n)\Lambda^2(f,x)$$

where (x_n) is defined as follow:

$$x_n := \begin{cases} 1 & (n \quad \text{odd}) \\ -1 & (n \quad \text{even}). \end{cases} \tag{4.2}$$

After some calculations we have:

$$\Lambda^2(1,x) = 1,$$
$$\Lambda^2(\cos t, x) = \cos x,$$
$$\Lambda^2(\sin t, x) = \sin x.$$

We see that conditions (3.2) are satisfied, and by Theorem 3.4, it follows that

$$\Lambda^2(st)\text{-}\lim_n ||L_n(f,x) - f||_{2\pi} = 0,$$

but Theorem 3.3 does't hold.

Remark 4.5. Based in the previous example and Remark 3.5, we show that our Theorem 3.4 is also stronger than Theorem 2.1 due to Mohiuddine and Alotaibi [19].

References

[1] Acar, T., Mohiuddine, S.A., *Statistical* $(C,1)(E,1)$ *summability and Korovkin's theorem*, Filomat, **30**(2016), no. 2, 387-393.

[2] Acar, T., Dirik, F., *Korovkin type theorems in weighted* L_p *spaces via summation process*, Sci. World Jour., 2013, Article ID 53454.

[3] Altomare, F., *Korovkin-type theorems and approximation by positive linear operators*, Survey in Approximation Theory, **5**(2010), 92-164.

[4] Bhardwaj, V.K., Dhawan, S., *Korovkin type approximation theorems via* f*-statistical convergence*, J. Math. Anal., **9**(2018), no. 2, 99-117.

[5] Braha, N.L., *A new class of sequences related to the* l_p *spaces defined by sequences of Orlicz functions*, J. Inequal. Appl. 2011, Art. ID 539745, 10 pp.

[6] Braha, N.L., Et, M., *The sequence space* $E_n^q(M,p,s)$ *and* N_k-*lacunary statistical convergence*, Banach J. Math. Anal., **7**(2013), no. 1, 88-96.

[7] Braha, N.L., Loku, V., Srivastava, H.M., Λ^2-*weighted statistical convergence and Korovkin and Voronovskaya type theorems*, Appl. Math. Comput., **266**(2015), 675-686.

[8] Braha, N.L., Mansour, T., *On* Λ^2*-strong convergence of numerical sequences and Fourier series*, Acta Math. Hungar., **141**(2013), no. 1-2, 113-126.

[9] Braha, N.L., Srivastava, H.M., Mohiuddine, S.A., *A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée-Poussin mean*, Appl. Math. Comput., **228**(2014), 162-169.

[10] Connor, J.S., *The statistical and strong p-Cesaro convergence of sequences*, Analysis, **8**(1988), 47-63.

[11] Edely, O.H.H., Mohiuddine, S.A., Noman, A.K., *Korovkin type approximation theorems obtained through generalized statistical convergence*, Applied Math. Letters, **23**(2010), 1382-1387.

[12] Fast, H., *Sur la convergence statistique*, Colloq. Math., **2**(1951), 241-244.

[13] Fridy, J.A., *On statistical convergence*, Analysis, **5**(1985), 301-313.

[14] Fridy, J.A., Orhan, C., *Lacunary statistical convergences*, Pacific J. Math., **160**(1993), no. 1, 43-51.

[15] Kadak, U., Braha, N.L., Srivastava, H.M., *Statistical weighted $\mathcal{B}$-summability and its applications to approximation theorems*, Appl. Math. Comput., **302**(2017), 80-96.

[16] Korovkin, P.P., *Convergence of linear positive operators in the spaces of continuous functions*, (Russian), Doklady Akad. Nauk. SSSR (N.S.), **90**(1953), 961-964.

[17] Korovkin, P.P., *Linear Operators and Approximation Theory*, Hindustan Publ. Co., Delhi, 1960.

[18] Loku, V., Braha, N.L., *Some weighted statistical convergence and Korovkin type-theorem*, J. Inequal. Spec. Funct., **8**(2017), no. 3, 139-150.

[19] Mohiuddine, S.A., Alotaibi, A., Mursaleen, M., *Statistical summability $(C, 1)$ and a Korovkin type approximation theorem*, J. Inequa. Appl., 2012, 2012:172.

[20] Mursaleen, M., *$\lambda-$statistical convergences*, Math. Slovaca, **50**(2000), no. 1, 111-115.

[21] Mursaleen, M., Alotaibi, A., *Statistical lacunary summability and a Korovkin type approximation theorem*, Ann. Univ. Ferrara, **57**(2011), no. 2, 373-381.

[22] Mursaleen, M., Karakaya, V., Erturk, M., Gursoy, F., *Weighted statistical convergence and its application to Korovkin type approximation theorem*, Appl. Math. Comput., **218**(2012), 9132-9137.

[23] Sevda, O., Tuncer, A., Fadime, D., *Korovkin type theorems in weighted L_p-spaces via statistical A-summability*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), **62**(2016), no. 2, 2, 537-546.

[24] Srivastava, H.M., Mursaleen, M., Khan, A., *Generalized equi-statistical convergence of positive linear operators and associated approximation theorems*, Math. Comput. Modelling, **55**(2012), 2040-2051.

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