"Homogeneous" second order differential equation: zeros separation principles

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Abstract. In this paper we study the following problems:

<u>Problem 1</u>. Let $I \subset \mathbb{R}$ be an open interval and $F : \mathbb{R}^3 \times I \to \mathbb{R}$ be a continuous function with, $F(0, 0, 0, x) = 0$, for all $x \in I$. We consider the following differential equations

$$F(y'', y', y, x) = 0. \tag{0.1}$$

Let $y \in C^2(I)$ be a nontrivial solution of (0.1). In which conditions we have that:
  (1) the zeros of y and y' separate each other?
  (2) the zeros of y and y'' separate each other?
  (3) the zeros of y' and y'' separate each other?

<u>Problem 2</u>. Let $y_1, y_2 \in C^2(I)$ be two linearly independent solutions of (0.1). In which conditions we have that:
  (1) the zeros of y_1 and y_2 separate each other?
  (2) the zeros of y_1' and y_2' separate each other?
  (3) the zeros of y_1'' and y_2'' separate each other?

<u>Problem 3</u>. Let $F, G : \mathbb{R}^3 \times I \to \mathbb{R}$ be two continuous functions with $F(0, 0, 0, x) = 0$, $G(0, 0, 0, x) = 0$, for all $x \in I$. We consider the following system of differential equations,

$$\begin{aligned} F(y', y, z, x) &= 0, \\ G(z', y, z, x) &= 0. \end{aligned} \tag{0.2}$$

Let $(y, z) \in C^1(I, \mathbb{R}^2)$ be a nontrivial solution of (0.2). In which conditions we have that:
  (1) the zeros of y and z separate each other?
  (2) the zeros of y' and z' separate each other?

<u>Problem 4</u>. Let (y_1, z_1) and (y_2, z_2) be two linearly independent solutions of (0.2). In which conditions we have that:
  (1) the zeros of y_1 and y_2 separate each other?
  (2) the zeros of z_1 and z_2 separate each other?
  (3) the zeros of y_1' and y_2' separate each other?
  (4) the zeros of z_1' and z_2' separate each other?

Some other problems are formulated.

1. Introduction and Preliminaries

Let $I \subset \mathbb{R}$ be an open interval (bounded or not!) of $\mathbb{R}$. For a function $y : I \to \mathbb{R}$ we denote by Z_y the zero set of y.

Definition 1.1. The zeros of two functions $y_1, y_2 \in C(I)$ separate each other iff:
(1) $Z_{y_1} \cap Z_{y_2} = \emptyset$;
(2) all zeros of y_1 and y_2 are simple;
(3) if x_1 and x_2 are two consecutive zeros of y_1, then y_2 has one zero in $]x_1, x_2[$, and if x_1, x_2 are two consecutive zeros of y_2, then y_1 has one zero in $]x_1, x_2[$.

The following results are well known:

<u>Wronskian Lemma</u> (1962; [49], [54], [8]). Let y_1, $y_2 \in C^1(I)$ be two functions. If

$$W(y_1, y_2)(x) := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} (x) \neq 0, \text{ for all } x \in I,$$

then the zeros of y_1 and y_2 separate each other.

<u>Tonelli Lemma</u> (1963; [50], [63], [39], [28]). Let y_1, $y_2 \in C^1[a, b]$ be such that
(1) $y_1(a) = y_1(b) = 0$ and $y_1(x) > 0$, for all $x \in]a, b[$;
(2) $y_2(x) > 0$, for all $x \in [a, b]$.
Then there exist $x_0 \in]a, b[$ and $\lambda > 0$ such that:
(i) $y_2(x_0) = \lambda y_1(x_0)$;
(ii) $y_2'(x_0) = \lambda y_1'(x_0)$.

Let $F, G : \mathbb{R}^3 \times I \to \mathbb{R}$ be two continuous functions with $F(0, 0, 0, x) = 0$, $G(0, 0, 0, x) = 0$, for all $x \in I$. We consider the following differential equation

$$F(y'', y', y, x) = 0 \tag{1.1}$$

and the following differential system

$$\begin{aligned} F(y', y, z, x) &= 0 \\ G(z', y, z, x) &= 0 \end{aligned} \tag{1.2}$$

There are many results for the zeros of the solutions $y \in C^2(I)$ of (1.1) and of the components y and z of the solutions $(y, z) \in C^1(I, \mathbb{R}^2)$ of (1.2) (see the books: [57], [58], [24], [25], [47], [61], [2], [4], [6], [10], [13], [15], [22], [23], [26], [27], [30], [31], [32],

[34], [35], [40], [41], [45], [55], [62], [64], [68], [17], ... and articles: [1], [3], [5], [7], [8], [14], [16]-[21], [28], [37], [39], [42], [43], [48]-[54], [63], ...).

In this paper we present new results, new proofs for some known results by techniques of real function theory: Bolzano Theorem, Rolle Theorem, Wronskian Lemma and Tonelli Lemma. Some research directions are also presented.

The structure of our paper is as follows:
2. Zeros separation of some real functions
3. Linear and homogeneous second order differential equations
4. Nonlinear equations in a normal form
5. Homogeneous nonlinear second order differential equations
6. Some research directions

2. Zeros Separation of Some Real Functions

Let $I \subset \mathbb{R}$ be an open interval and $y, z \in C(I)$ (or $y, z \in C(I, \mathbb{R}^2)$) be linearly independent functions. We denote by $H(y, z)$ the linear hull of $\{y, z\}$ over $\mathbb{R}$. We have:

Theorem 2.1 (Wronskian separation principle for scalar functions). *Let $y, z \in C^1(I)$ be two functions. We suppose that, $W(y, z)(x) \neq 0$, for all $x \in I$. Let $u, v \in H(y, z)$ be two linearly independent functions. Then the zeros of u, v separate each other.*

Proof. Since $W(y, z)(x) \neq 0$, for all $x \in I$, implies that $W(u, v)(x) \neq 0$, for all $x \in I$, the proof follows from the Wronskian Lemma. $\square$

Theorem 2.2 (Wronskian separation theorem for vectorial functions). *Let $y, z \in C(I, \mathbb{R}^2)$ be two vectorial functions. We suppose that:*

(1) $W(y, z)(x) := \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}(x) \neq 0$, *for all $x \in I$.*

(2) *The zeros of y_1 and y_2, and z_1 and z_2 separate each other.*

Then the zeros of y_1 and z_1, and y_2 and z_2 separate each other. Moreover, if the zero of each two linearly independent element of $H(y_1, y_2)$ and of $H(z_1, z_2)$ separate each other, then if $u \in H(y, z)$ is a nontrivial element, the zeros of u_1 and u_2 separate each other.

Proof. First of all we prove that the zeros of y_1 and y_2, and z_1 and z_2 separate each other. Let us prove this property for y_1 and z_1. Let, for example, x_1, $x_2 \in Z_{y_1}$ be two consecutive zeros. Then, from $W(y, z)(x) \neq 0$, for all $x \in I$, it follows that $sgn\ y_2(x_1)z_1(x_1) = sgn\ y_2(x_2)z_1(x_2)$. But $y_2(x_1)y_2(x_2) < 0$. So, from Bolzano Theorem, $Z_{z_1} \cap]x_1, x_2[\neq 0$. Now, let us prove the second part of the conclusions of the theorem. Let $v \in H(y, z)$ be such that the functions u, v are linearly independent. Then, $W(y, z)(x) \neq 0$, for all $x \in I$, implies that $W(u, v)(x) \neq 0$, for all $x \in I$. The proof follows from the first part of conclusions of the theorem. $\square$

Remark 2.3. From Theorem 2.1 and Theorem 2.2 it follows the following result:

Theorem 2.4. *Let $y, z \in C^1(I)$ be as in Theorem 2.1. If the zeros of y' and z' separate each other, then the zeros of y and y', and z and z' separate each other.*

3. Linear and Homogeneous Second Order Differential Equations

In this section we present some well known results in the zero separation theory for linear and homogeneous second order differential equations as consequence of Wronskian Lemma and Wronskian Theorems.

For p, q, p_i, $q_i \in C(I)$, $i = 1, 2$, we consider the differential equation

$$y'' + py' + qy = 0 \tag{3.1}$$

and the system of differential equations

$$\begin{aligned} y' + p_1 y + q_1 z &= 0 \\ z' + p_2 y + q_2 z &= 0 \end{aligned} \tag{3.2}$$

The basic results in zero separation theory for (3.1) and (3.2) are the following.

Theorem 3.1. *Let y, z be two linearly independent solutions of* (3.1). *Then we have:*

  (*i*) *(Sturm) The zeros of y and z separate each other.*
 (*ii*) *(Butlewski* [12]) *If $q(x) \neq 0$, for all $x \in I$, then the zeros of y' and z' separate each other.*
(*iii*) *(Butlewski* [12]) *If $p, q \in C^1(I)$ and $[q^2 + p'q - q'p](x) \neq 0$ for all $x \in I$, then the zeros of y'' and z'' separate each other.*

Proof. (*i*). Follows from Wronskian Lemma.

(*ii*). Since, $W(y', z') = qW(y, z)$, the proof follows from Wronskian Lemma.

(*iii*). Since, $W(y'', z'') = (q^2 + p'q - pq')W(y, z)$, the proof follows from Wronskian Lemma. □

Theorem 3.2 (Nicolescu [42] **and Butlewski** [12]). *Let y be a nontrivial solution of* (3.1). *We have:*

  (*i*) *If $q(x) \neq 0$, for all $x \in I$, then the zeros of y and y' separate each other.*
 (*ii*) *If $p, q \in C^1(I)$, $p(x) \neq 0$ and $[q^2 + p'q - pq'](x) \neq 0$, for all $x \in I$, then the zeros of y and y'' separate each other.*
(*iii*) *If $p, q \in C^1(I)$, $q(x) \neq 0$ and $[q^2 + p'q - q'p](x) \neq 0$, for all $x \in I$, then the zeros of y' and y'' separate each other.*

Proof. Let u, v be two linearly independent solutions of (3.1). Then $y \in H(u, v)$.

(*i*). Since $W(u, v)(x) \neq 0$ and $W(u', v')(x) \neq 0$, for all $x \in I$, by Theorem 3.1, the zeros of u, v, and u' and v' separate each other. Now the proof follows from Remark 3.5.

(*ii*)+(*iii*). The proof follows from Theorem 3.1 and Theorem 2.2. □

Theorem 3.3 (Nicolescu [42] **and Butlewski** [12]). *Let (y_1, z_1) and (y_2, z_2) be two linearly independent solutions of* (3.2). *Then we have:*

  (*i*) *If $q_1(x) \neq 0$, for all $x \in I$, then the zeros of y_1 and y_2 separate each other.*
 (*ii*) *If $p_2(x) \neq 0$, for all $x \in I$, then the zeros of z_1 and z_2 separate each other.*
(*iii*) *If $p_1, q_1 \in C^1(I)$ and $q_1(p_1 q_2 - p_2 q_1)(x) + (q_1 p_1' - p_1 q_1')(x) \neq 0$, for all $x \in I$, then the zeros of y_1' and y_2' separate each other.*
(*iv*) *If $p_2, q_2 \in C^1(I)$ and $p_2(p_1 q_2 - p_2 q_1)(x) + (p_2 q_2' - q_2 p_2')(x) \neq 0$, for all $x \in I$, then the zeros of z_1' and z_2' separate each other.*

Proof. (i). $W(y_1, y_2)(x) = -q_1 W((y_1, z_1), (y_2, z_2))(x) \neq 0$, for all $x \in I$. The proof follows from Wronskian Lemma.

(ii). $W(z_1, z_2)(x) = p_2 W((y_1, z_1), (y_2, z_2))(x) \neq 0$, for all $x \in I$. The proof follows from Wronskian Lemma.

(iii). $W(y_1', y_2') = [q_1(p_1 q_2 - p_2 q_1) + q_1 p_1' - p_1 q_1'](x) \, W((y_1, z_1), (y_2, z_2))(x) \neq 0$, for all $x \in I$. The proof follows from Wronskian Lemma.

(iv). $W(z_1', z_2') = [p_2(p_1 q_2 - p_2 q_1) + p_2 q_2' - q_2 p_2'](x) \, W((y_1, z_1), (y_2, z_2))(x) \neq 0$, for all $x \in I$. The proof follows from Wronskian Lemma. $\qquad\square$

Theorem 3.4. *Let (y, z) be a nontrivial solution of* (3.2). *Then we have:*

(i) *(Nicolescu) If $q_1(x) \neq 0$ and $p_2(x) \neq 0$, for all $x \in I$, then the zeros of y and z separate each other.*

(ii) *(Butlewski) If $p_i, q_i \in C^1(I)$, $i = 1, 2$, $(p_1 q_2 - p_2 q_1)(x) \neq 0$, $[q_1(p_1 q_2 - p_2 q_1) + q_1 p_1' - p_1 q_1'](x) \neq 0$ and $[p_2(p_1 q_2 - p_2 q_1) + p_2 q_2' - q_2 p_2'](x) \neq 0$, for all $x \in I$, then the zeros of y' and z' separate each other.*

Proof. Let (y_1, z_1) and (y_2, z_2) be two linearly independent solutions of (3.2). Then, $(y, z) \in H((y_1, z_1), (y_2, z_2))$.

(i). Since, $W((y_1, z_1), (y_2, z_2))(x) \neq 0$, for all $x \in I$, the proof follows from Theorem 3.3 and Theorem 2.2.

(ii). Since, $W((y_1', z_1'), (y_2', z_2'))(x) = (p_1 q_2 - p_2 q_1)(x) \, W((y_1, z_1), (y_2, z_2))(x) \neq 0$, for all $x \in I$, the proof follows from Theorem 3.3 and Theorem 2.1. $\qquad\square$

Remark 3.5. Let us consider the quasilinear equation

$$y'' + p(y', y, x)y' + q(y', y, x)y = 0 \tag{3.3}$$

and the quasilinear system

$$\begin{aligned} y' + p_1(y, z, x)y + q_1(y, z, x)z &= 0 \\ z' + p_2(y, z, x)y + q_2(y, z, x)z &= 0 \end{aligned} \tag{3.4}$$

From the Theorem 3.2 and Theorem 3.4 we have the following results:

Theorem 3.6. *We suppose that:*

(1) $p, q \in C(\mathbb{R}^2 \times I)$.

(2) $q(t) \neq 0$, *for all $t \in \mathbb{R}^2 \times I$.*

If $y \in C^2(I)$ is a nontrivial solution of (3.3), *then the zeros of y and y' separate each other.*

Proof. We remark that y is a solution of the following linear equation

$$u'' + p(y', y, x)u' + q(y', y, x)u = 0. \qquad\square$$

Theorem 3.7. *We suppose that:*

(1) $p_i, q_i \in C(\mathbb{R}^2 \times I)$, $i = 1, 2$.

(2) $q_1(t) \neq 0$, $p_2(t) \neq 0$, *for all $t \in \mathbb{R}^2 \times I$.*

If $(y, z) \in C^1(I, \mathbb{R}^2)$ is a nontrivial solution of (3.4), *then the zeros of y and z separate each other.*

Proof. We remark that (y, z) is a solution of the following linear system

$$u' + p_1(y, z, x)u + q_1(y, z, x)v = 0,$$
$$v' + p_2(y, z, x)u + q_2(y, z, x)v = 0.$$
(3.5)

□

Remark 3.8. For similar results, as above, using <u>linearization principle</u>, see [52].

4. Nonlinear Equations in a Normal Form

For $f, p, q \in C(\mathbb{R}^2 \times I)$ with, $f(0, 0, x) = 0$, $p(0, 0, x) = 0$ and $q(0, 0, x) = 0$, for all $x \in I$, we consider the following equation

$$y'' + f(y', y, x) = 0$$
(4.1)

and the following system,

$$y' + p(y, z, x) = 0,$$
$$z' + q(y, z, x) = 0.$$
(4.2)

In this case we have the following results.

Theorem 4.1. *We suppose that:*

(1) *For all $x \in I$, sgn $f(0, t, x) = $ sgn t, for all $t \in \mathbb{R}$ or sgn $f(0, t, x) = -$sgn t, for all $t \in \mathbb{R}$.*
(2) *All Cauchy problems for* (4.1) *have at most a solution.*

In this conditions if y is a nontrivial solution of (4.1), *then the zeros of y and y' separate each other.*

Proof. First of all we remark that, in the conditions of our theorem, the zeros of y and y' are all simple. Now, let $x_1, x_2 \in I$ be two consecutive zeros of y'. For example, for the first part of (1), we have that,

$$\text{sgn } y''(x_i) = -\text{sgn } y(x_i), \ i = 1, 2.$$

But $y''(x_1)y''(x_2) < 0$. from this we have that $y(x_1)y(x_2) < 0$. Now the proof follows from Bolzano Theorem.   □

Example 4.2 (see [1]). Let us consider the following nonlinear equation

$$y'' + y'^2 + e^t y^3 = 0, \ t \in \mathbb{R}_+.$$

This equation satisfies the condition in Theorem 4.1. So, if $y \in C^2(\mathbb{R}_+)$ is a solution of this equation, then the zeros of y and y' separate each other.

A similar result we have for the equation (see [57])

$$y'' + a|y'|y' + by = 0, \ a, b \in \mathbb{R}_+^*.$$

In a similar way we have,

Theorem 4.3. *We suppose that:*

(1) *For all $x \in I$, sgn $p(0, t, x) = $ sgn t, for all $t \in \mathbb{R}$ and sgn $q(t, 0, x) = $ sgn t, for all $t \in \mathbb{R}_+$.*
(2) *All Cauchy problems for (4.2) have at most a solution.*

Then, if $(y, z) \in C^1(I, \mathbb{R}^2)$ is a solution of (4.2), then the zeros of y and z separate each other.

5. Homogeneous Nonlinear Second Order Differential Equations

Let $F, G \in C(\mathbb{R}^3 \times I)$ be two functions with $F(0, 0, 0, x) = 0$ and $G(0, 0, 0, x) = 0$, for all $x \in I$. We consider the following differential equation

$$F(y'', y', y, x) = 0 \tag{5.1}$$

and the following system of differential equations

$$\begin{align} F(y', y, z, x) &= 0, \\ G(z', y, z, x) &= 0. \end{align} \tag{5.2}$$

We have the following results:

Theorem 5.1 (Tonelli [63]; see also [50] and [37]). *We suppose that:*
(1) *For each $x \in I$, the function, $F(\cdot, \cdot, \cdot, x) : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous function.*
(2) *All Cauchy problems for (5.1) have at most a solution.*

In this conditions, if y_1 and y_2 are two linearly independent solutions of (5.1), then the zeros of y_1 and y_2 separate each other.

Proof. First we remark that, $y = 0$ is a solution of (5.1). If y is a nontrivial solution of (5.1), then the zeros of y are simple and if y_1 and y_2 are two solutions such that $Z_{y_1} \cap Z_{y_2} \neq \emptyset$, then y_1 and y_2 are linearly dependent. Now let y_1 and y_2 be two linearly independent solutions. Let for example x_1, $x_2 \in Z_{y_1}$ be two consecutive zeros. If $Z_{y_2} \cap]x_1, x_2[= \emptyset$, then, $y_2(x) \neq 0$, for all $x \in [x_1, x_2]$. Since if y is a solution then, $-y$ is also a solution, we can suppose that: $y_1(x) > 0$, for all $x \in]x_1, x_2[$ and $y_2(x) > 0$, for all $x \in [x_1, x_2]$. Now the proof follows from Tonelli Lemma. $\square$

Theorem 5.2 (Tonelli [63]; see also [28]). *We suppose that:*
(1) *For each $x \in I$, $F(\cdot, \cdot, \cdot, x) : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous function.*
(2) *If y_1 and y_2 are solutions of (5.1) and for some $x_0 \in I$, $y_1(x_0) = y_2(x_0) > 0$ and $y_1'(x_0) = y_2'(x_0)$, then, $y_1 = y_2$.*

In these conditions, if y_1 and y_2 are two linearly independent solutions of (5.1) then:
(i) *if $x_1, x_2 \in I$ are two consecutive zeros of y_1, then $Z_{y_2} \cap [x_1, x_2] \neq \emptyset$;*
(ii) *if $x_1, x_2 \in I$ are two consecutive zeros of y_2, then $Z_{y_1} \cap [x_1, x_2] \neq \emptyset$.*

Proof. The proof follows from Tonelli Lemma. $\square$

Theorem 5.3 (see [28]). *We suppose that:*
(1) *For each $x \in I$, $F(\cdot, \cdot, \cdot, x) : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous function.*
(2) *If y_1 and y_2 are two solutions of (5.1) and for some $x_0 \in I$, $y_1'(x_0) = y_2'(x_0) > 0$ and $y_1''(x_0) = y_2''(x_0)$, then, $y_1 = y_2$.*

In these conditions, if y_1 and y_2 are two linearly independent solutions of (5.1) *then:*

(*i*) *if $x_1, x_2 \in I$ are two consecutive zeros of y_1', then $Z_{y_2'} \cap [x_1, x_2] \neq \emptyset$;*
(*ii*) *if $x_1, x_2 \in I$ are two consecutive zeros of y_2', then $Z_{y_1'} \cap [x_1, x_2] \neq \emptyset$.*

Proof. The proof follows from Tonelli Lemma (see [28]).   □

Theorem 5.4. *We suppose that:*

(1) *For each $x \in I$, the functions $F, G : \mathbb{R}^3 \to \mathbb{R}$ are homogeneous.*
(2) *If (y_1, z_1) and (y_2, z_2) are two solutions of* (5.2), *then:*
    (*a*) *if for some $x_0 \in I$, $y_1(x_0) = y_2(x_0)$ and $y_1'(x_0) = y_2'(x_0)$, then, $(y_1, z_1) = (y_2, z_2)$;*
    (*b*) *if for some $x_0 \in I$, $z_1(x_0) = z_2(x_0)$ and $z_1'(x_0) = z_2'(x_0)$, then, $(y_1, z_1) = (y_2, z_2)$.*

In the above conditions, if (y_1, z_1) and (y_2, z_2) are two linearly independent solutions of (5.2), *then the zeros of y_1 and y_2, and of z_1 and z_2 separate each other.*

Proof. The proof follows from Tonelli Lemma.   □

6. Some Research Directions

6.1. *Zero separation theory for nonlinear second order differential equations*
$\quad$ <u>Problem 6.1.</u> To complete the results of this paper with some results for (see Abstract):

(*i*) Problem 1: (2) and (3);
(*ii*) Problem 2: (3);
(*iii*) Problem 3: (2);
(*iv*) Problem 4: (3) and (4);

References: [1], [28], [31], [32], [34], [49], [50], [52], [67], ...

6.2. <u>Problem 6.2.</u> To use the implicit function theorem in [11] to study the uniqueness of solution for the following problem:

$$F(y'', y', y, x) = 0, \ y'(x_0) = y_0', \ y''(x_0) = y_0''.$$

6.3. *Zero separation principles for the solutions of two second order differential equations*
$\quad$ Let $p_i, q_i \in C(I)$, $i = 1, 2$. We consider the following differential equations:

$$y'' + p_1 y' + q_1 y = 0 \tag{6.1}$$

$$z'' + p_2 z' + q_2 z = 0 \tag{6.2}$$

Let y_1 and y_2 be two linearly independent solutions of (6.1), and z_1 and z_2 be two linearly independent solutions of (6.2). From the Theorem 2.2 we have the following result:

$\quad$ *If, $W((y_1, z_1), (y_2, z_2))(x) \neq 0$, for all $x \in I$, and y is a nontrivial solution of* (6.1) *and z is a nontrivial solution of* (6.2) *such that, $(y, z) \in H((y_1, z_1), (y_2, z_2))$, then the zeros of y and z separate each other.*

$\quad$ From this result the following problem appears.

Problem 6.3. Let y_1 and y_2 be two linearly independent solutions of (6.1), and z_1 and z_2 be two linearly independent solutions of (6.2). In which conditions we have that,

$$W((y_1, z_1), (y_2, z_2))(x) \neq 0, \text{ for all } x \in I?$$

Each solution for this problem has an impact in the zero separation theory for some special functions: Bessel functions, Legendre functions, Mathieu functions, ...

References: [59], [60], [38], [32], [57], [62], ...

6.4. *Wronskian Lemma and Wronskian Theorems techniques in zero separation of polynomial functions*

The following Markov Theorem is well known: *If the zeros of the polynomial functions P and Q are all real and separate each other, then the zeros of P' and Q' separate each other.*

In [16], F. Constantinescu proves this theorem using the Sturm separation theorem.

Problem 6.4. To use Wronskian Lemma and Wronskian Theorems in the zero separation theory of polynomial functions.

References: [16], [18], [34], [43], [51], [53], [65], [66], ...

6.5. Problem 6.5. To use the *angular function* of a planar vector field techniques in zero separation theory for nonlinear second order equations.

References: [34], [4], [56], [13], [41], ...

6.6. Problem 6.6. To use the Borůvka theory of functions defined in terms of zeros of solutions of linear second order equations to study the *zero distance functions*.

References: [9], [10], [5], [21], [41], [36], ...

6.7. *Zero separation theory in the case of linear and homogeneous second order differential equations in the complex domain*

Problem 6.7. To give a good notion of zero separation of two complex functions. In terms of this notion, to construct a Sturm zero separation theory in the complex domain.

References: [25], [26], [29], [33], [1], [6], [7], [14], [62], ...

6.8. Problem 6.8. The notion of the category of ordinary linear differential equations is given in [41]. An object of this category is a linear differential equation and a morphism for two objects is a global transformation. The problem is to consider some subcategories of this category which are relevant for zero point theory of solutions of differential equations.

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