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Quasilinear parabolic equations with $p(x)$-Laplacian diffusion terms and nonlocal boundary conditions

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Abstract. In this study, we prove the existence of local solution for a quasi linear generalized parabolic equation with nonlocal boundary conditions for an elliptic operator involving the variable-exponent nonlinearities, using Faedo-Galerkin arguments and compactness method.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary $\Gamma = \partial\Omega$. We consider the following quasi linear parabolic equations with nonlocal boundary conditions:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right) + |u|^{p(x)-2} u = f(x,t) \text{ in } Q_T = \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \;\; x \in \Omega, \tag{1.2}$$

$$u(x,t) = \int_\Omega K(x,y)\, u(y,t)\, dy, \;\; x \in \Gamma, \; t \in (0,T), \tag{1.3}$$

where the exponent $p(\cdot)$ is a given measurable function on $\overline{\Omega}$ such that:

$$2 \leq n < p_1 \leq p(x) \leq p_2 \leq \infty, \tag{1.4}$$

where

$$p_2 = ess \sup_{x\in\Omega} p(x), \quad p_1 = ess \inf_{x\in\Omega} p(x).$$

We also assume that $p(\cdot)$ satisfies the following Zhikov-Fan uniform local continuity condition :

$$|p(x)-p(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y|<\frac{1}{2},\ M>0. \tag{1.5}$$

In recent years, many authors have paid attention to the study of nonlinear hyperbolic, parabolic and elliptic equations with nonstandard growth condition. For instance, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, thermoelasticity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. More details on these problems can be found in [5, 8, 1, 3, 4, 15, 17, 18] and references therein.

Constant exponent. In (1.1), when $p(\cdot)=p$ is constant, local, global existence and long-time behavior have been considered by many authors.

For instance, in the absence of the term $|u|^{p-2}u$ and when the kernel datum function $K(x,y)=0$, using the compactness method and Faedo-Galerkin techniques, the existence and uniqueness of a weak solution has been proved see [16].

Baili Chen in [7] generalized the result of Lions to the situation when the presence of $|u|^{p-2}u$ and when $K(x,y)\neq 0$ in problem (1.1), applying exactly the same technique introduced in [16, Problème 12, page 140.], the author by constructing the approximate Galerkin solution, he proved the existence of generalized solution, the uniqueness questions are still open.

Problem (1.1)-(1.3) is the extension of the problems in Lion's book [16, p.140] in which the boundary conditions are homogeneous and in [7] in which the variable-exponent is constant. The uniqueness questions in problem (1.1)-(1.3) are more complicated than in [7] and are still open.

The main difficulty of this problem, concerns the weak converging approximate solution, is related to the presence of the quasilinear terms in (1.1) in the variable-exponent.

In this paper a class of quasi linear generalized parabolic equation with nonlocal boundary conditions for an elliptic operator involving the variable-exponent nonlinearities was considered. Hence by using Faedo-Galerkin arguments and compactness method as in [16], we will show the local existence of problem (1.1)-(1.3).

2. Preliminaries

In this section we list and recall some well-known results and facts from the theory of the Sobolev spaces with variable exponent. (For the details see [9, 11, 10, 12, 13, 14]).

Throughout the rest of the paper we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n\geq 2$ with smooth boundary Γ, Let $p:\overline{\Omega}\to[1,\infty]$ be a measurable function. We denote by $L^{p(\cdot)}(\Omega)$ the set of measurable functions u on Ω such that

$$A_{p(\cdot)}(u)=\int_{\Omega}|u(x)|^{p(x)}\,dx<\infty.$$

The variable-exponent space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{p(\cdot),\Omega} = \|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0,\ A_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

is a Banach space.

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects, see the first discussed the $L^{p(x)}(\Omega)$ spaces and $W^{k,p(x)}(\Omega)$ spaces by Kovàcik and Rákosnik in [14].

Let us list some properties of the spaces $L^{p(\cdot)}(\Omega)$ which will be used in the study of the problem (1.1)-(1.3).

- It follows directly from the definition of the norm that (see [9]),

$$\min\left(\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right) \leq A_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right).$$

- Let $p,\ q,\ s \geq 1$ be measurable functions defined on $\overline{\Omega}$ such that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \text{ for a.e. } x \in \Omega.$$

  if $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{q(\cdot)}(\Omega)$ then $u.v \in L^{s(\cdot)}(\Omega)$ and the following generalized Hölder inequality

$$\|uv\|_{s(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}.$$

  holds.

  Let us consider the following variable-exponent Lebesgue Sobolev space (see [9]),

$$W^{1,p(\cdot)}(\Omega) = \left\{v \in L^{p(\cdot)}(\Omega) : \text{such that } |\nabla v| \text{ exists and } |\nabla v| \in L^{p(\cdot)}(\Omega)\right\}.$$

This space is a Banach space with respect to the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \sum_i \|\nabla u_i\|_{p(\cdot)}.$$

Furthermore, we set $W_0^{1,p(\cdot)}(\Omega)$, to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Here we note that the space $W_0^{1,p(\cdot)}(\Omega)$ is usually defined in a different way for the variable exponent case. However (see Diening et al [9]), both definitions are equivalent under (1.5). The $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$ is the dual space of $W_0^{1,p(\cdot)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$ and is defined as $W^{-1,p'(\cdot)}(\Omega)$, in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(\cdot)} = 1$, the function $p'(\cdot)$ is called the dual variable exponent of $p(\cdot)$.

- Let $p,\ q : \Omega \to [1, +\infty)$ be measurable functions satisfying condition (1.5). If $p(x) \leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 2.1. ([9]) *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\Gamma = \partial\Omega$, $p(\cdot)$ is a given measurable function on $\overline{\Omega}$ satisfy conditions* (1.5) *and $q = const \geq 1$. If $q \leq p(x)$ a.e. in Ω, then*

$$\|v\|_q \leq C_{q,\Omega}\|v\|_{p(\cdot)} \quad \textit{with the constant } C_{q,\Omega} = (1 + |\Omega|)^{\frac{1}{q}}. \tag{2.1}$$

3. Notations and preliminaries

In this article, on f, u_0 and $K(x,y)$ we make the following assumptions

$$f \in L^{p_2'}(0,T;L^{p_2'}(\Omega)),\ \frac{1}{p_2}+\frac{1}{p_2'}=1, \tag{3.1}$$

$$u_0 \in L^{\infty}(\Omega), \tag{3.2}$$

$$\text{for any } x\in\Gamma,\quad K(x)<\infty,\ \ K_i(x)<\infty, \tag{3.3}$$

$$\sum_{i=1}^{n}\int_{\Gamma} K(x)^{p_2-1}K_i(x)\,d\Gamma<\infty,\ \ \sum_{i=1}^{n}\int_{\Gamma} K(x)^{p_1-1}K_i(x)\,d\Gamma<\infty, \tag{3.4}$$

$$\gamma=\max\begin{pmatrix} C_{p_1,\Omega}^{p_2}\left(\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_2-1}K_i(x)\,d\Gamma\right), \\ \left(C_{p_1,\Omega}^{p_1}\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_1-1}K_i(x)\,d\Gamma\right)\end{pmatrix}\le\frac{p_1-1}{p_2}, \tag{3.5}$$

for any $x\in\Gamma$, where

$$K(x)=\left(\int_{\Omega}|K(x,y)|^{p_1'}\,dy\right)^{\frac{1}{p_1'}}:$$

$$\text{norm of } k(x,y) \text{ in } L^{p_1'}(\Omega) \text{ with respect to } y,\ \frac{1}{p_1}+\frac{1}{p_1'}=1$$

$$K_i(x)=\left(\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial}{\partial x_i}K(x,y)\right|^{p_1'}dy\right)^{\frac{1}{p_1'}}:$$

$$\text{norm of } \frac{\partial}{\partial x_i}k(x,y) \text{ in } L^{p_1'}(\Omega) \text{ with respect to } y,\ \frac{1}{p_1}+\frac{1}{p_1'}=1$$

and $C_{p_1,\Omega}$ defined in (2.1).
Moreover, we assume that

$$r>\frac{n}{2}+2. \tag{3.6}$$

Let

$$\alpha_1=\left(\frac{p_2}{p_1}\left(\gamma+\frac{1}{p_2}\right)\right)^{\frac{p_2}{p_1-p_2}}. \tag{3.7}$$

We define the polynomial Q by

$$Q(\alpha)=\min\left(\alpha,\alpha^{\frac{p_1}{p_2}}\right)-\left(\gamma+\frac{1}{p_2}\right)\max\left(\alpha,\alpha^{\frac{p_1}{p_2}}\right)\quad\forall\alpha\in[0,+\infty].$$

Let

$$h(\alpha)=\alpha^{\frac{p_1}{p_2}}-\left(\gamma+\frac{1}{p_2}\right)\alpha.$$

Notice that $h(\alpha)=Q(\alpha)$, for $1\le\alpha\le\infty$. It is easy to check that the function $h(\alpha)$ is increasing for $1\le\alpha<\alpha_1$ and decreasing for $\alpha_1<\alpha\le+\infty$, where α_1 is its unique local maximum defined by (3.7). We will assume that:

$$1\le\|u_0\|_{p(\cdot)}^{p_2}=\alpha_0<\alpha_1 \tag{3.8}$$

and

$$\frac{1}{2}|u_0|^2 + C_{2,\Omega}^{\frac{p_2}{p_2-1}} \frac{p_2-1}{p_2} |\Omega|^{\frac{1}{2}} \int_0^T |f|_{p_2}^{\frac{p_2}{p_2-1}} dt < \int_0^T Q(\alpha_1)\, dt. \tag{3.9}$$

The classical formulation of the problem is as follows. Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}$, such that:

$$(u', v) - \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right), v \right) + \left(|u|^{p(x)-2} u, v \right) = (f, v), \ \forall v \in V \tag{3.10}$$

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

Where

$$V = \left\{ v \in H^r(\Omega) : v(x) = \int_\Omega K(x,y)\, v(y)\, dy \ \text{ for } x \in \Gamma \right\},$$

With assumption (1.4)-(3.6), using Sobelev embedding theorems, see [2], we have

$$H^r(\Omega) \hookrightarrow W^{2,p_2}(\Omega) \hookrightarrow W^{1,p_2}(\Omega) \hookrightarrow L^{p_2}(\Omega) \hookrightarrow L^2(\Omega)$$

It is easy to see that V is a subspace of $H^r(\Omega)$.

Whenever it doesn't cause a confusion, we use the following shorthand notations:

$L^q(\Omega)$: L^q space defined on Ω; $|.|_q = |.|_{q,\Omega}$: norm in $L^q(\Omega)$; $|.|_{q,\Gamma}$: norm in $L^q(\Gamma)$; $H^{-r}(\Omega)$: dual space of $H^r(\Omega)$; $|.|_{H^{-r}(\Omega)}$ norm in $H^{-r}(\Omega)$; C: nonnegative constant which may take different values on each occurrence.

4. Local existence

Theorem 4.1. *Under hypothesis* (1.4)-(3.9), *for any finite* $T > 0$, *the problem* (1.1)-(1.3) *admits a weak solution* u *such that*

$$u \in L^\infty\left(0,T; L^2(\Omega)\right) \cap C\left([0,T]; H^{-r}(\Omega)\right) \cap L^{p(\cdot)}\left(\Omega \times (0,T)\right), \tag{4.1}$$

$$\frac{\partial u}{\partial t} \in L^{p_2'}\left(0,T; H^{-r}(\Omega)\right), \tag{4.2}$$

$$|u|^{\frac{p(\cdot)-2}{2}} u \in \ L^2\left(0,T; H^1(\Omega)\right), \tag{4.3}$$

for all $v \in V$ *and a.e.* $t \in [0,T]$,

$$(u', v) - \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right), v \right) + \left(|u|^{p(x)-2} u, v \right) = (f, v), \tag{4.4}$$

$$u(x,0) = u_0(x), \ x \in \Omega.$$

Proof. Since V is a subspace of $H^r(\Omega)$ which is separable. We can choose a countable set of distinct basis elements w_j $(j = 1, 2, ...)$ which generate V and are orthonormal in $L^2(\Omega)$. Let V_m be the subspace of V generated by the first m elements: $w_1, w_2, ..., w_m$. We search u of the form:

$$u_m(x,t) = \sum_{i=1}^{m} K_{im}(t) w_i(x), \tag{4.5}$$

satisfying:

$$\begin{cases} (u_m', w_j) - \left(\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}\right), w_j\right) \\ + \left(|u_m|^{p(x)-2} u_m, w_j\right) = (f(t), w_j), \quad 1 \le j \le m, \\ u_m(0) = u_{0m}. \end{cases} \tag{4.6}$$

with

$$u_{0m} = \sum_{i=1}^m \alpha_{im} w_i \longrightarrow u_0 \quad \text{when } m \longrightarrow \infty \text{ in } L^{p(\cdot)}(\Omega). \tag{4.7}$$

Integrating by parts on the second term of left-hand side of (4.6), we have

$$\begin{cases} (u_m', w_j) + \left(\sum_{i=1}^n \left(|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}\right), \frac{\partial}{\partial x_i} w_j\right) + \left(|u_m|^{p(x)-2} u_m, w_j\right) \\ = \int_\Gamma |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i} w_j d\Gamma + (f(t), w_j), \quad 1 \le j \le m, \\ u_m(0) = u_{0m}. \end{cases} \tag{4.8}$$

By Peano's Theorem, for every finite m the problem (4.6), (4.8) has a solution on $(0, T_m)$ for each m. The following estimates permit us to confirm that T_m is independent of m.

a) A priori estimates

Multiplying the equation (4.8) by $K_{jm}(t)$, summing over $j = 1, ..., m$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \sum_{i=1}^n \int_\Omega \frac{4}{p^2(x)}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}} u_m\right)\right)^2 dx + \int_\Omega |u_m|^{p(x)} dx \tag{4.9}$$
$$= \int_\Gamma |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i} u_m(t) d\Gamma + (f(t), u_m)$$

Integrating on $(0, T)$ on both sides of (4.9), we get

$$\frac{1}{2}|u_m(T)|^2 + \int_0^T \sum_{i=1}^n \int_\Omega \frac{4}{p^2(x)}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}} u_m\right)\right)^2 dxdt$$
$$+ \int_0^T \min\left(||u_m||_{p(\cdot)}^{p_2}, ||u_m||_{p(\cdot)}^{p_1}\right) dt$$
$$\le \int_0^T \int_\Gamma \left| |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i} u_m(t)\right| d\Gamma dt + \int_0^T |(f(t), u_m)| dt + \frac{1}{2}|u_{0m}|^2. \tag{4.10}$$

The second term in the right-hand side of (4.10) can be estimated as follows

$$|(f(t), u_m)| \le |f|_2 |u_m|_2 \le C_{2,\Omega} |f|_2 ||u_m||_{p(\cdot)} \text{ (holder's inequality) and (2.1)}$$
$$\le C_{2,\Omega}^{\frac{p_2}{p_2-1}} \frac{p_2-1}{p_2} |f|_2^{\frac{p_2}{p_2-1}} + \frac{1}{p_2} ||u_m||_{p(\cdot)}^{p_2} \quad \text{(Young's inequality)}$$
$$\le C_{2,\Omega}^{\frac{p_2}{p_2-1}} \frac{p_2-1}{p_2} |\Omega|^{\frac{1}{2}} |f|_{p_2}^{\frac{p_2}{p_2-1}} + \frac{1}{p_2}\max\left(\|u_m\|_{p(\cdot)}^{p_2}, \|u_m\|_{p(\cdot)}^{p_1}\right).$$

Next, we estimate first term in the right-hand side of (4.10) using (2.1): For $x \in \Gamma$, we have

$$|u_m(x,t)| \le K(x) |u_m|_{p_1} \le C_{p_1,\Omega} K(x) \|u_m\|_{p(\cdot)}. \tag{4.11}$$

Similarly, for $x \in \Gamma$ we have

$$\left| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} u_m(x,t) \right| \leq K_i(x) \left| u_m \right|_{p_1} \leq C_{p_1,\Omega} K_i(x) \left\| u_m \right\|_{p(\cdot)} \tag{4.12}$$

Then using holder's inequality and assumptions (3.3) and (3.5), we have:

$$\begin{aligned}
\sum_{i=1}^{n} \int_{\Gamma} \left| |u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} u_m(t) \right| d\Gamma &\leq \sum_{i=1}^{n} \int_{\Gamma} |u_m|^{p(x)-1} \left| \frac{\partial u_m}{\partial x_i} \right| d\Gamma \\
&\leq \max\left(\sum_{i=1}^{n} \int_{\Gamma} |u_m|^{p_2-1} \left| \frac{\partial u_m}{\partial x_i} \right| d\Gamma, \sum_{i=1}^{n} \int_{\Gamma} |u_m|^{p_1-1} \left| \frac{\partial u_m}{\partial x_i} \right| d\Gamma \right) \\
&\leq \max \begin{pmatrix} C_{p_1,\Omega}^{p_2} \sum_{i=1}^{n} \int_{\Gamma} K(x)^{p_2-1} \left\| u_m \right\|_{p(\cdot)}^{p_2-1} K_i(x) \left\| u_m \right\|_{p(\cdot)} d\Gamma, \\ C_{p_1,\Omega}^{p_1} \sum_{i=1}^{n} \int_{\Gamma} K(x)^{p_1-1} \left\| u_m \right\|_{p(\cdot)}^{p_1-1} K_i(x) \left\| u_m \right\|_{p(\cdot)} d\Gamma \end{pmatrix} \\
&= \max \begin{pmatrix} C_{p_1,\Omega}^{p_2} \left(\sum_{i=1}^{n} \int_{\Gamma} K(x)^{p_2-1} K_i(x)\, d\Gamma \right) \left\| u_m \right\|_{p(\cdot)}^{p_2}, \\ C_{p_1,\Omega}^{p_1} \left(\sum_{i=1}^{n} \int_{\Gamma} K(x)^{p_1-1} K_i(x)\, d\Gamma \right) \left\| u_m \right\|_{p(\cdot)}^{p_1} \end{pmatrix} \\
&\leq \max \begin{pmatrix} C_{p_1,\Omega}^{p_2} \left(\sum_{i=1}^{n} \int_{\Gamma} K(x)^{p_2-1} K_i(x)\, d\Gamma \right), \\ \left(C_{p_1,\Omega}^{p_1} \sum_{i=1}^{n} \int_{\Gamma} K(x)^{p_1-1} K_i(x)\, d\Gamma \right) \end{pmatrix} \\
&\qquad \times \max\left(\left\| u_m \right\|_{p(\cdot)}^{p_2}, \left\| u_m \right\|_{p(\cdot)}^{p_1} \right)
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{1}{2} \left| u_m(t) \right|^2 + \int_0^T \sum_{i=1}^{n} \int_{\Omega} \frac{4}{p^2(x)} \left(\frac{\partial}{\partial x_i} \left(|u_m|^{\frac{p(x)-2}{2}} u_m \right) \right)^2 dx dt + \int_0^T Q\left(||u_m||_{p(\cdot)}^{p_2} \right) dt \\
\leq \frac{1}{2} \left| u_{0m} \right|^2 + C_{2,\Omega}^{\frac{p_2}{p_2-1}} \frac{p_2-1}{p_2} |\Omega|^{\frac{1}{2}} \int_0^T |f|_{p_2}^{\frac{p_2}{p_2-1}} dt,
\end{aligned} \tag{4.13}$$

at this step we will assume that $Q\left(||u_m||_{p(\cdot)}^{p_2} \right) \geq 0$, so from (3.9) and (4.13), we have the following a priori estimates:

$$|u_m| \leq C \ (C \text{ is independent of } m); \tag{4.14}$$

$$\int_0^T \sum_{i=1}^{n} \int_{\Omega} \frac{4}{p^2(x)} \left(\frac{\partial}{\partial x_i} \left(|u_m|^{\frac{p(x)-2}{2}} u_m \right) \right)^2 dx dt \leq C \ (C \text{ independent of } m). \tag{4.15}$$

So the solution $u_m(t)$ of problem (1.1)-(1.3) exists on $[0,T]$ for each m, and

$$u_m \quad \text{is bounded in } L^{\infty}\left(0,T;L^2(\Omega)\right); \tag{4.16}$$

$$|u_m|^{\frac{p(\cdot)-2}{2}} u_m \ \text{ is bounded in } L^2\left(0,T;H^1(\Omega)\right)$$

Claim 4.2. There exists an integer N such that

$$||u_m||_{p(\cdot)}^{p_2} < \alpha_1 \quad \forall t \in [0,T_m) \qquad m > N. \tag{4.17}$$

Proof of the Claim. Suppose (4.17) false. Then for each $m > N$, there exists $t \in [0, T_m)$ such that $||u_m(t)||_{p(\cdot)}^{p_2} \geq \alpha_1$. We note that from (3.8) and (4.7) there exists N_0 such that

$$1 \leq ||u_m(0)||_{p(\cdot)}^{p_2} < \alpha_1 \quad \forall m > N_0$$

Then by continuity there exists a first $T_m^* \in (0, T_m)$ such that

$$||u_m(T_m^*)||_{p(\cdot)}^{p_2} = \alpha_1, \tag{4.18}$$

from where

$$Q\left(||u_m||_{p(\cdot)}^{p_2}\right) = h\left(||u_m(t)||_{p(\cdot)}^{p_2}\right) \geq 0 \quad \forall t \in [0, T_m^*].$$

Now from (3.9) and (4.13), there exist $N > N_0$ and $\beta \in (1; \alpha_1)$ such that

$$0 \leq \frac{1}{2}|u_m(t)|^2 + \int_0^t \sum_{i=1}^n \int_\Omega \frac{4}{p^2(x)} \left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}} u_m\right)\right)^2 dxds$$

$$+ \int_0^t Q\left(||u_m||_{p(\cdot)}^{p_2}\right) ds \leq \int_0^t Q(\beta)\, ds \quad \forall t \in [0, T_m^*], \quad \forall m > N$$

Then the monotonicity of Q implies that

$$||u_m(t)||_{p(\cdot)}^{p_2} \leq \beta < \alpha_1 \quad \forall t \in [0, T_m^*]$$

and in particular, $||u_m(T_m^*)||_{p(\cdot)}^{p_2} < \alpha_1$, which is a contradiction to (4.18). And then the supposition $Q\left(||u_m||_{p(\cdot)}^{p_2}\right) \geq 0$ is true. □

From (4.17) the solution $u_m(t)$ of problem (1.1)-(1.3) satisfies other of (4.16),

$$u_m \text{ is bounded in } L^{p(\cdot)}(\Omega \times (0, T)). \tag{4.19}$$

Lemma 4.3. *Let u_m, constructed in* (4.5), *be the approximate solution of* (1.1)-(1.3). *Then*

$$\frac{\partial}{\partial t} u_m(t) \text{ is bounded in } L^{p_2'}(0, T; H^{-r}(\Omega)). \tag{4.20}$$

Proof. Let $v \in H^r(\Omega)$, from (4.6) we have

$$\left(\frac{\partial u_m(t)}{\partial t}, v\right) + \left(\sum_{i=1}^n \left(|u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i}\right), \frac{\partial}{\partial x_i} v\right) + \left(|u_m|^{p(x)-2} u_m, v\right) \tag{4.21}$$

$$= \sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} v d\Gamma + (f(t), v),$$

The last term in the left-hand side can be estimated as follows:

$$\left|\left(|u_m|^{p(x)-2} u_m, v\right)\right| \leq \left||u_m|^{p(x)-1}\right|_{p_2'} |v|_{p_2} \leq C\left||u_m|^{p(x)-1}\right|_{p'(\cdot)} |v|_{p_2} \quad (p_2' \leq p'(\cdot) \leq p_1')$$

$$\leq C \max\left(\left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_1'}}, \left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_2'}}\right) |v|_{p_2}$$

$$\leq C \max\left(\left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_1'}}, \left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_2'}}\right) |v|_{H^r}$$

Hence,

$$\left||u_m|^{p(\cdot)-2}u_m\right|_{H^{-r}(\Omega)} \leq C\max\left(\left(\int_\Omega |u_m|^{p(x)}dx\right)^{\frac{1}{p_1'}},\left(\int_\Omega |u_m|^{p(x)}dx\right)^{\frac{1}{p_2'}}\right) < \infty.$$

The norm of $|u_m|^{p(\cdot)-2}u_m$ in $L^{p_2'}(0,T;H^{-r}(\Omega))$ is bounded by

$$C\left(\int_0^T \max\left(\left(\int_\Omega |u_m|^{p(x)}dx\right)^{\frac{p_2'}{p_1'}},\int_\Omega |u_m|^{p(x)}dx\right)\right)^{\frac{1}{p_2'}} < \infty$$

Therefore, $|u_m|^{p(\cdot)-2}u_m$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$.

Next, we consider the term $\sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}vd\Gamma$ in the left-hand side of (4.21):

$$\begin{aligned}
&\left|\sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}vd\Gamma\right| \leq \left(\sum_{i=1}^n \left||u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}\right|_{p_1',\Gamma}\right)|v(t)|_{p_1,\Gamma}\\
&= \sum_{i=1}^n \left|\left|\int_\Omega K(x,y)\,u_m(y)\,dy\right|^{p(x)-2}\int_\Omega \frac{\partial}{\partial x_i}K_i(x,y)\,u_m(y)\,dy\right|_{p_1',\Gamma}\\
&\qquad\times\left|\int_\Omega K(x,y)\,v(y)dy\right|_{p_1,\Gamma}\\
&\leq C\sum_{i=1}^n \left|K(x)^{p(x)-2}K_i(x)\,|u_m(y)|_{p_1}^{p(x)-1}\right|_{p_1',\Gamma}\left|K(x)\,|v(y)|_{p_1}\right|_{p_1,\Gamma}\\
&\leq C\sum_{i=1}^n \left|K(x)^{p(x)-2}K_i(x)\right|_{p_1',\Gamma}|K(x)|_{p_1,\Gamma}\,|u_m(y)|_{p_1}^{p(x)-1}\,|v(y)|_{p_1}\\
&\leq C\max\left(\sum_{i=1}^n \left|K(x)^{p_1-2}K_i(x)\right|_{p_1',\Gamma},\sum_{i=1}^n \left|K(x)^{p_2-2}K_i(x)\right|_{p_1',\Gamma}\right)|K(x)|_{p_1,\Gamma}\\
&\qquad\times\max\left(|u_m(y)|_{p_1}^{p_1-1},|u_m(y)|_{p_1}^{p_2-1}\right)|v(y)|_{H^r(\Omega)}\\
&\leq C\max\left(\sum_{i=1}^n \left|K(x)^{p_1-2}K_i(x)\right|_{p_1',\Gamma},\sum_{i=1}^n \left|K(x)^{p_2-2}K_i(x)\right|_{p_1',\Gamma}\right)|K(x)|_{p_1,\Gamma}\\
&\qquad\times\max\left(|u_m(y)|_{p(\cdot)}^{p_1-1},|u_m(y)|_{p(\cdot)}^{p_2-1}\right)|v(y)|_{H^r}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}d\Gamma\right|_{H^{-r}(\Omega)}\\
&\leq C\max\left(\sum_{i=1}^n \left|K(x)^{p_1-2}K_i(x)\right|_{p_1',\Gamma},\sum_{i=1}^n \left|K(x)^{p_2-2}K_i(x)\right|_{p_1',\Gamma}\right)\\
&\qquad\times\max\left(|u_m(y)|_{p(\cdot)}^{p_1-1},|u_m(y)|_{p(\cdot)}^{p_2-1}\right)|K(x)|_{p_1,\Gamma} < \infty.
\end{aligned}$$

Then the norm of $\sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}d\Gamma$ in $L^{p_2'}(0,T;H^{-r}(\Omega))$ is bounded by

$$C\left(\begin{array}{c}\int_0^T\max\left(\sum_{i=1}^{n}\left|K(x)^{p_1-2}K_i(x)\right|_{p_1',\Gamma}^{p_2'},\sum_{i=1}^{n}\left|K(x)^{p_2-2}K_i(x)\right|_{p_1',\Gamma}^{p_2'}\right)\\ \times\max\left(|u_m(y)|_{p(\cdot)}^{(p_1-1)p_2'},|u_m(y)|_{p(\cdot)}^{(p_2-1)p_2'}\right)|K(x)|_{p_1,\Gamma}^{p_2'}dt\end{array}\right)^{\frac{1}{p_2'}}<\infty$$

Hence $\displaystyle\sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}d\Gamma$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$.

Next, we consider the second term in the left-hand side of (4.21). Integrating by parts gives

$$\left(\sum_{i=1}^{n}\left(|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}\right),\frac{\partial v}{\partial x_i}\right)=\int_{\Omega}\sum_{i=1}^{n}\frac{1}{p(x)-1}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{p(x)-2}u_m\right)\right)\frac{\partial v}{\partial x_i}dx \tag{4.22}$$

$$=\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\frac{\partial v}{\partial x_i}d\Gamma-\int_{\Omega}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\Delta v dx.$$

First, we have

$$\begin{aligned}&\left|\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\frac{\partial v}{\partial x_i}d\Gamma\right|\le\frac{1}{p_2-1}\sum_{i=1}^{n}\left||u_m|^{p(x)-2}u_m\right|_{p_1',\Gamma}\left|\frac{\partial v}{\partial x_i}\right|_{p_1,\Gamma}\\
&=\frac{1}{p_2-1}\sum_{i=1}^{n}\left|\left(\int_{\Omega}K(x,y)u_m(y)dy\right)^{p(x)-1}\right|_{p_1',\Gamma}\left|\int_{\Omega}\frac{\partial}{\partial x_i}K(x,y)v(y)dy\right|_{p_1,\Gamma}\\
&\le C\sum_{i=1}^{n}\left|K(x)^{p(x)-1}|u_m|_{p_1}^{p(x)-1}\right|_{p_1',\Gamma}\left|K_i(x)|v|_{p_1}\right|_{p_1,\Gamma}\\
&\le C\sum_{i=1}^{n}\left|K(x)^{p(x)-1}\right|_{p_1',\Gamma}|K_i(x)|_{p_1,\Gamma}|u_m|_{p_1}^{p(x)-1}|v|_{p_1}\\
&\le C\max\left(\sum_{i=1}^{n}\left|K(x)^{p_1-1}\right|_{p_1',\Gamma},\sum_{i=1}^{n}\left|K(x)^{p_2-1}\right|_{p_1',\Gamma}\right)\\
&\quad\times\max\left(|u_m|_{p_1}^{p_1-1},|u_m|_{p_1}^{p_2-1}\right)|K_i(x)|_{p_1,\Gamma}|v(y)|_{H^r}\end{aligned}$$

So we have

$$\begin{aligned}&\left|\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m d\Gamma\right|_{H^{-r}(\Omega)}\\
&\le C\max\left(\sum_{i=1}^{n}\left|K(x)^{p_1-1}\right|_{p_1',\Gamma},\sum_{i=1}^{n}\left|K(x)^{p_2-1}\right|_{p_1',\Gamma}\right)\\
&\quad\times\max\left(|u_m|_{p_1}^{p_1-1},|u_m|_{p_1}^{p_2-1}\right)|K_i(x)|_{p_1,\Gamma}<\infty\end{aligned}$$

consequently, $\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m d\Gamma$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$.

Next, consider $\int_\Omega \frac{1}{p(x)-1} |u_m|^{p(x)-2} u_m \Delta v dx$, by the same manner, we have

$$\begin{aligned}\left|\int_\Omega \frac{1}{p(x)-1} |u_m|^{p(x)-2} u_m \Delta v dx\right| &\le \frac{1}{p_1-1} \left||u_m|^{p(x)-1}\right|_{p_2'} |\Delta v|_{p_2} \\ &\le C \left||u_m|^{p(x)-1}\right|_{p'(\cdot)} |\Delta v|_{p_2} \\ &\le C \max\left(\left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_1'}}, \left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_2'}}\right) |v|_{H^r}\end{aligned}$$

therefore,

$$\int_\Omega \frac{1}{p(x)-1} |u_m|^{p(x)-2} u_m \Delta v dx$$

is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$. Since $f \in L^{p_2'}(0,T;L^{p_2'}(\Omega)) \subset L^{p_2'}(0,T;H^{-r}(\Omega))$, from this discussion and (4.21) it yields that $\frac{\partial}{\partial t}u_m$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$. □

Theorem 4.4. *Let B, B_1 be Banach spaces, and S be a set. Define*

$$M(v) = \max\left(\left(\sum_{i=1}^n \int_\Omega |v|^{p_1-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_1}}, \left(\sum_{i=1}^n \int_\Omega |v|^{p_2-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_2}}\right)$$

on S with:

a) $S \subset B \subset B_1$, and $M(v) \ge 0$ on S,

$$\begin{aligned}M(\lambda v) &= \max\left(\left(\sum_{i=1}^n \int_\Omega |v|^{p_1-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_1}}, \left(\sum_{i=1}^n \int_\Omega |v|^{p_2-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_2}}\right) \\ &= |\lambda| M(v)\end{aligned}$$

b) the set $\{v \mid v \in S,\ M(v) \le 1\}$ is relatively compact in B.

Define the set

$$F = \left\{ \begin{array}{l} v : v \text{ is locally summable on } [0,T] \text{ with value in } B_1; \\ \int_0^T (M(v(t)))^{q_0} dt \le C,\ v' \text{ bounded in } L^{q_1}(0,T;B_1), \end{array} \right\}$$

where $1 < q_i < \infty$, $i = 0,1$. Then $F \subset L^{q_0}(0,T;B)$, and F is relatively compact in $L^{q_0}(0,T;B)$.

We need Theorem (4.4) to prove the following lemma (4.5).

Lemma 4.5. *Let u_m, constructed as in (4.5), be the approximate solution of (1.1)-(1.3), then $u_m \to u$ in $L^{p_2}(0,T;L^{p_2}(\Omega))$ strongly and almost everywhere.*

Proof. Let

$$S = \left\{v : \max\left(|v|^{\frac{p_1-2}{2}} v, |v|^{\frac{p_2-2}{2}} v\right) \in H^1(\Omega)\right\}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the proof of [16, Proposition 12.1,p. 143] also works for both $|v|^{\frac{p_1-2}{2}} v$ and $|v|^{\frac{p_2-2}{2}} v$, then (b) holds.

Let $B = L^{p_2}(\Omega)$, $B_1 = H^{-r}(\Omega)$, $q_0 = p_2$, $q_1 = p_2'$, we have

$$\int_0^T (M(v(t)))^{q_0}\,dt \le C\int_0^T \max\begin{pmatrix} \left(\sum_{i=1}^n \int_\Omega |v|^{p_1-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{p_2}{p_1}}, \\ \left(\sum_{i=1}^n \int_\Omega |v|^{p_2-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)\end{pmatrix} dt$$

$$\le C\int_0^T \max\begin{pmatrix} \left(\sum_{i=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i}\left(|v|^{\frac{p_1-2}{2}}v\right)\right)^2 dx\right)^{\frac{p_2}{p_1}}, \\ \left(\sum_{i=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i}\left(|v|^{\frac{p_2-2}{2}}v\right)\right)^2 dx\right)\end{pmatrix} dt < \infty$$

□

Now with Lemma (4.3) and a priori estimates, conclusion follows easily from application of Theorem (4.4).

Next, we prove that we can pass the limit in (4.21). Lemmas (4.6)-(4.10), below, show that we can pass the limit in each term in the left-hand side of (4.21)

Lemma 4.6. *Let u_m, constructed as in* (4.5), *be the approximate solution of* (1.1)-(1.3), *then* $\left(|u_m|^{p(x)-2}u_m, v\right) \to \left(|u|^{p(x)-2}u, v\right)$ *as* $m \to \infty$.

Proof. Since u_m is bounded in $L^{p(\cdot)}(\Omega \times (0,T))$ then $|u_m|^{p(\cdot)-2}u_m$ is bounded in $L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega \times (0,T))$; hence, using same arguments as in [16, Lemma 1.3], we have

$$|u_m|^{p(\cdot)-2}u_m \to |u|^{p(\cdot)-2}u \text{ weakly in } L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega \times (0,T)).$$

□

Lemma 4.7. *Let u_m, constructed as in* (4.5), *be the approximate solution of* (1.1)-(1.3), *then*

$$\int_\Gamma \sum_{i=1}^n \frac{1}{p(x)-1}\left(|u_m|^{p(x)-2}\frac{\partial}{\partial x_i}u_m\right) v\,d\Gamma \to \int_\Gamma \sum_{i=1}^n \frac{1}{p(x)-1}\left(|u|^{p(x)-2}\frac{\partial}{\partial x_i}u\right) v\,d\Gamma$$

as $m \to \infty$.

Proof. By a priori estimates, u_m is bounded in $L^{p(\cdot)}(\Omega)$ for almost every t, then there exists subsequence of u_m, still denoted as u_m, converges to u_m weak star in $L^{p(\cdot)}(\Omega)$ (Alaoglu's Theorem) for almost every $t \in [0,T]$. Under the assumption that for fixed $x \in \Gamma$, we have

$$\int_\Omega K(x,y)\,u_m(y)\,dy \to \int_\Omega K(x,y)\,u(y)\,dy \quad \text{as } m \to \infty$$

Similarly

$$\int_\Omega \frac{\partial}{\partial x_i}K(x,y)\,u_m(y)\,dy \to \int_\Omega \frac{\partial}{\partial x_i}K(x,y)\,u(y)\,dy \quad \text{as } m \to \infty$$

Therefore, for $x \in \Gamma$, we have

$$|u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m \to |u|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u \quad \text{a.e.}$$

Since

$$\max\left(\int_{\Gamma} K^{p_1}(x)\,d\Gamma, \int_{\Gamma} K^{p_2}(x)\,d\Gamma\right) < \infty,$$
$$\text{and } \max\left(\int_{\Gamma} K_i^{p_1}(x)\,d\Gamma, \int_{\Gamma} K_i^{p_2}(x)\,d\Gamma\right) < \infty,$$

we have

$$|u_m|_{p(\cdot),\Gamma} \le C\max\left(\int_{\Gamma} K^{p_1}(x)\,d\Gamma, \int_{\Gamma} K^{p_2}(x)\,d\Gamma\right)\max\left(\|u_m\|_{p(\cdot)}^{p_1}, \|u_m\|_{p(\cdot)}^{p_2}\right) < \infty$$

and

$$\left|\frac{\partial}{\partial x_i}u_m\right|_{p(\cdot),\Gamma} \le C\max\left(\int_{\Gamma} K_i^{p_1}(x)\,d\Gamma, \int_{\Gamma} K_i^{p_2}(x)\,d\Gamma\right)\max\left(\|u_m\|_{p(\cdot)}^{p_1}, \|u_m\|_{p(\cdot)}^{p_2}\right) < \infty.$$

Then

$$\begin{aligned}
\left||u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m\right|_{p_2',\Gamma} &\le C\left||u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m\right|_{p'(\cdot),\Gamma} \quad \text{since } (p_2' \le p'(\cdot) \le p_1') \\
&\le \left||u_m|^{p(\cdot)-2}\right|_{\frac{p(\cdot)}{p(\cdot)-2},\Gamma}\left|\frac{\partial}{\partial x_i}u_m\right|_{p(\cdot),\Gamma} \quad \text{since } \left(\frac{1}{p'(\cdot)} = \frac{p(\cdot)-2}{p(\cdot)} + \frac{1}{p(\cdot)}\right) \\
&\le \max\left(\left(\int_{\Gamma}|u_m|^{p(x)}\,d\Gamma\right)^{\frac{1}{p_1}}, \left(\int_{\Omega}|u_m|^{p(x)}\,d\Gamma\right)^{\frac{1}{p_2}}\right) \\
&\times \max\left(\left(\int_{\Omega}\left|\frac{\partial}{\partial x_i}u_m\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_1}}, \left(\int_{\Gamma}\left|\frac{\partial}{\partial x_i}u_m\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_2}}\right) < \infty.
\end{aligned}$$

So, applying the same arguments as in [16, Lemma 1.3] to conclude that

$$|u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m \to |u|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u \ \text{ weakly in } L^{p_2'}(\Gamma).$$

for a.e. $t \in [0,T]$. Since,

$$\max\left(\left(\int_{\Omega}\left|\frac{\partial}{\partial x_i}v\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_1}}, \left(\int_{\Omega}\left|\frac{\partial}{\partial x_i}v\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_2}}\right) < \infty,$$

the proof is complete. □

Lemma 4.8. *Let u_m, constructed as in* (4.5), *be the approximate solution of* (1.1)-(1.3), *then*

$$\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\frac{\partial v}{\partial x_i}d\Gamma \to \int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u|^{p(x)-2}u\frac{\partial v}{\partial x_i}d\Gamma$$

as $m \to \infty$.

Proof. From the proof of Lemma (4.7), we have, for $x \in \Gamma$, $|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u$ almost everywhere, and

$$\left| |u_m|^{p(\cdot)-2} u_m \right|_{p_2',\Gamma} \leq C \left| |u_m|^{p(\cdot)-2} u_m \right|_{p'(\cdot),\Gamma}$$
$$\leq \max\left(\left(\int_\Gamma |u_m|^{p(x)} d\Gamma \right)^{\frac{1}{p_1}}, \left(\int_\Gamma |u_m|^{p(x)} d\Gamma \right)^{\frac{1}{p_2}} \right) < \infty.$$

Therefore, by applying [16, Lemma 1.3] we conclude that

$$|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u \text{ weakly in } L^{p_2'}(\Gamma).$$

Since $\frac{\partial v}{\partial x_i} \in L^{p_2'}(\Gamma)$, the proof is complete.

Lemma 4.9. *Let u_m, constructed as in* (4.5)*, be the approximate solution of* (1.1)-(1.3)*, then*

$$\int_\Omega \frac{1}{p(x)-1} \left(|u_m|^{p(x)-2} u_m \right) \Delta v dx \to \int_\Omega \frac{1}{p(x)-1} \left(|u|^{p(x)-2} u \right) \Delta v dx$$

as $m \to \infty$.

Proof. From lemma ((4.5)), we have $|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u$ almost everywhere, for $x \in \Omega$, since

$$\left| |u_m|^{p(\cdot)-2} u_m \right|_{p_2',\Omega} \leq C \left| |u_m|^{p(\cdot)-2} u_m \right|_{p'(\cdot),\Omega}$$
$$\leq \max\left(\left(\int_\Omega |u_m|^{p(x)} dx \right)^{\frac{1}{p_1}}, \left(\int_\Omega |u_m|^{p(x)} dx \right)^{\frac{1}{p_2}} \right) < \infty.$$

by [16, Lemma 1.3], we have $|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u$ weakly in $L^{p_2'}(\Omega)$. Since $\Delta v \in L^{p_2}(\Omega)$, the proof is complete. □

Lemma 4.10. *Let u_m, constructed as in* (4.5)*, be the approximate solution of* (1.1)-(1.3)*, then*

$$\left(\sum_{i=1}^n \left(|u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} \right), \frac{\partial}{\partial x_i} v \right) \to \left(\sum_{i=1}^n \left(|u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right), \frac{\partial}{\partial x_i} v \right)$$

as $m \to \infty$.

Proof. Replacing the results of (4.8) and (4.9) in (4.22), the proof is complete. □

Lemma 4.11. *Let u_m, constructed as in* (4.5)*. be the approximate solution of* (1.1)-(1.3)*, then $\left(\frac{\partial}{\partial t} u_m, v\right) \to \left(\frac{\partial}{\partial t} u, v\right)$ and $u(t)$ is continuous on $[0, T]$.*

Proof. Since $\frac{\partial}{\partial t} u_m(t)$ is bounded in $L^{p_2'}(0, T; H^{-r}(\Omega))$, by Alaoglu's theorem, there exists a subsequence, still denoted by $\frac{\partial}{\partial t} u_m(t)$, converging to χ weak star in $L^{p_2'}(0, T; H^{-r}(\Omega))$. By slightly modifying the proof of [6, Theorem 1] (with the space $L^{p_2'}(0, T; H^{-r}(\Omega))$ instead of $L^2(0, T; B_2^1(0, 1))$.), we have $\chi = u'$ and u is continuous on $[0, T]$. This ends the proof of Lemma (4.11). □

Combining all above results, the existence theorem (4.1) follows. □

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