Stud. Univ. Babeş-Bolyai Math. 64(2019), No. 1, 43–61
DOI: 10.24193/subbmath.2019.1.05

On some new integral inequalities concerning twice differentiable generalized relative semi-(m, h)-preinvex mappings

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Abstract. The authors first present some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-(m, h)-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set is derived. By using the notion of generalized relative semi-(m, h)-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via conformable fractional integrals are established. These new presented inequalities are also applied to construct inequalities for special means.

Mathematics Subject Classification (2010): 26A51, 26A33, 26D07, 26D10, 26D15.

Keywords: Hermite-Hadamard type inequality, fractional integrals, m-invex.

1. Introduction

The subsequent double inequality is known as Hermite-Hadamard inequality.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping on an interval I of real numbers and $a, b \in I$ with $a < b$. Then The subsequent double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, please see [3]-[14], [19], [20], [18], [24], [26], [29], [38], [43], [44] and the references mentioned in these papers.
Let us evoke some definitions as follows.

Definition 1.2. [42] A set $M_\varphi \subseteq \mathbb{R}^n$ is named as a relative convex (φ-convex) set, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi,\ \forall\ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1]. \tag{1.2}$$

Definition 1.3. [42] A function f is named as a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)), \tag{1.3}$$

$\forall\, x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]$.

Definition 1.4. [7] A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow [0,+\infty)$ is said to be P-function, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 1.5. [2] A set $K \subseteq \mathbb{R}^n$ is said to be invex respecting the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Definition 1.6. [25] Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.4}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 1.6, f becomes a preinvex function, see [31]. If the mapping $\eta(y,x) = y - x$ in Definition (1.6), then the non-negative function f reduces to h-convex mappings, see [41].

Definition 1.7. [40] Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, a function $f : K \longrightarrow \mathbb{R}$ is said to be a tgs-convex function on K if the inequality

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{1.5}$$

grips for all $x, y \in K$ and $t \in (0,1)$.

Definition 1.8. [5], [22] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if f it is non-negative and $\forall\, x, y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.6}$$

Definition 1.9. [27] A function: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.7}$$

Definition 1.10. [30] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y,x,m)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.8}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, for some fixed $m \in (0,1]$.

We need the subsequent Riemann-Liouville fractional calculus background.

Definition 1.11. [23] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide applications of Riemann-Liouville fractional integrals, many authors extended to research Riemann-Liouville fractional inequalities via different classes of convex mappings: for generalizations, variations and new inequalities for them, see for instance [23]-[32] and the references therein.
We also use here the subsequent conformable fractional integrals.

Definition 1.12. Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$, then the left conformable fractional integral starting at a is defined by

$$\left(I^a_{\alpha}f\right)(t) = \frac{1}{n!}\int_a^t (t-x)^n(x-a)^{\beta-1}f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$\left({}^bI_{\alpha}f\right)(t) = \frac{1}{n!}\int_t^b (x-t)^n(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$, where $n = 0, 1, 2, \ldots$, and hence $\left(I^a_{\alpha}f\right)(t) = \left(J^a_{n+1}f\right)(t)$.
In [33], Set et al. obtained a generalization of Hermite-Hadamard type inequality via conformable fractional integrals involving s-convex mappings.

Theorem 1.13. [33] *Enable $f : [a,b] \longrightarrow \mathbb{R}$ be a function with $0 \leq a < b$, $s \in (0,1]$, and $f \in L_1[a,b]$. If f is a convex mapping on $[a,b]$, then the coming inequalities for conformable fractional integrals clasp:*

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^s(b-a)^{\alpha}}\left[\left(I^a_{\alpha}f\right)(b) + \left({}^bI_{\alpha}f\right)(a)\right]$$

$$\leq \left[\frac{\beta(n+s+1, \alpha-n) + \beta(n+1, \alpha-n+s)}{n!}\right]\frac{f(a)+f(b)}{2^s},$$

together $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2, \ldots,$ where Γ is Euler gamma function.

In recent years, some researchers have studied bounds for Hermite-Hadamard inequality, Fejér type inequality and Ostrowski type inequality etc. via conformable fractional integrals. For more details about this topic, see [1], [15], [16], [34]-[37].

Let us recall the Gauss-Jacobi type quadrature formula as follows.

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star|f|, \tag{1.9}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star|f|$, see [39].
In [21], Liu obtained integral inequalities for P-function related to the left-hand side of (1.9), and in [28], Özdemir et al. also presented several integral inequalities concerning the left-hand side of (1.9) via some kinds of convexity.
Motivated by the above literatures, the main objective of this article is to establish integral inequalities for Gauss-Jacobi type quadrature formula and some new estimates on Hermite-Hadamard type inequalities via conformable fractional integrals associated with generalized relative semi-(m,h)-preinvex mappings. These new obtained inequalities are also applied to construct inequalities for special means.
To end this section, let us consider the following special functions:

(1) The Beta function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x,y>0,$$

(2) The incomplete Beta function:

$$\beta_a(x,y) = \int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, \;\; 0<a<1, \;\; x,y>0.$$

2. Main results involving Gauss-Jacobi type quadrature formula

The following definitions will be used in this section.

Definition 2.1. [8] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,mx) \in K$ grips for each $x,y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y,mx)$ could reduce to $\eta(y,x)$. For example when $m=1$, then the m-invex set degenerates an invex set on K.

We next introduce generalized relative semi-(m,h)-preinvex mappings.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, $h : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. A mapping $f : K \longrightarrow \mathbb{R}$ is said to be generalized relative semi-(m,h)-preinvex, if

$$f\big(m\varphi(x) + t\eta(\varphi(y),\varphi(x),m)\big) \le mh(1-t)f(x) + h(t)f(y) \tag{2.1}$$

holds for all $x,y \in I$ and $t \in [0,1]$ for some fixed $m \in (0,1]$.

Remark 2.4. Let us discuss some special cases in Definition 2.3 as follows.
(I) Taking $h(t) = t$, then we get generalized relative semi-m-preinvex mappings.
(II) Taking $h(t) = t^s$ for $s \in (0, 1]$, then we get generalized relative semi-(m, s)-Breckner-preinvex mappings.
(III) Taking $h(t) = t^{-s}$ for $s \in (0, 1]$, then we get generalized relative semi-(m, s)-Godunova-Levin-Dragomir-preinvex mappings.
(IV) Taking $h(t) = 1$, then we get generalized relative semi-(m, P)-preinvex mappings.
(V) Taking $h(t) = t(1 - t)$, then we get generalized relative semi-(m, tgs)-preinvex mappings.
(VI) Taking $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.
We claim the following integral identity.

Lemma 2.5. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and any fixed $p, q > 0$, we have*

$$\begin{aligned} &\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1 - t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt. \end{aligned} \tag{2.2}$$

Proof. It is easy to observe that

$$\begin{aligned} &\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\ &\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q \\ &\times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1 - t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt. \end{aligned}$$

This completes the proof of the lemma. □

With the help of Lemma 2.5, we have the following results.

Theorem 2.6. *Suppose $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous mapping on K° respecting $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. If $|f|^{\frac{k}{k-1}}$ for $k > 1$ is generalized relative semi-(m, h)-preinvex mapping on an open m-invex* *set K for some fixed $m \in (0,1]$, then for any fixed $p, q > 0$, we have*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\\&\times\left[m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}\left(\int_0^1 h(t)dt\right)^{\frac{k-1}{k}}.\end{aligned} \tag{2.3}$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is generalized relative semi-(m,h)-preinvex on K, combining with Lemma 2.5, Hölder inequality and properties of the modulus, we get

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\le|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}\\&\times\left[\int_0^1 |f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}\\&\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\\&\times\left[\int_0^1\left(mh(1-t)|f(a)|^{\frac{k}{k-1}}+h(t)|f(b)|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}}\\&=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\\&\times\left[m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}\left(\int_0^1 h(t)dt\right)^{\frac{k-1}{k}}.\end{aligned}$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 2.6.

Corollary 2.7. *In Theorem 2.6 for $h(t)=t^s$ where $s\in[0,1]$, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}}.\end{aligned}$$

Corollary 2.8. *In Theorem 2.6 for $h(t)=t^{-s}$ where $s\in[0,1)$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}}{1-s}\right]^{\frac{k-1}{k}}.$$

Corollary 2.9. *In Theorem 2.6 for $h(t)=t(1-t)$, we obtain the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}}{6}\right]^{\frac{k-1}{k}}.$$

Corollary 2.10. *In Theorem 2.6 for $h(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, we deduce the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\leq\left(\frac{\pi}{4}\right)^{\frac{k-1}{k}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\\&\times\left[m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}.\end{aligned}$$

Theorem 2.11. *Suppose $h:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ is a continuous mapping on K° respecting $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$, for $\eta(\varphi(b),\varphi(a),m)>0$. If $|f|^l$ for $l\geq 1$ is generalized relative semi-(m,h)-preinvex mapping on an open m-invex set K for some fixed $m\in(0,1]$, then for any fixed $p,q>0$, we have*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\leq\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\&\times\left[m|f(a)|^l I(h(t);p,q)+|f(b)|^l I(h(t);q,p)\right]^{\frac{1}{l}},\end{aligned}\tag{2.4}$$

where $I(h(t);p,q):=\displaystyle\int_0^1 t^p(1-t)^q h(1-t)dt.$

Proof. Since $|f|^l$ is generalized relative semi-(m,h)-preinvex on K, combining with Lemma 2.5, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\\
&\times\int_0^1\Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}}\Big[t^p(1-t)^q\Big]^{\frac{1}{l}}f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\\
&\le|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\\
&\times\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 t^p(1-t)^q|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))|^l dt\right]^{\frac{1}{l}}\\
&\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\left[\int_0^1 t^p(1-t)^q\left(mh(1-t)|f(a)|^l+h(t)|f(b)|^l\right)dt\right]^{\frac{1}{l}}\\
&=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\Big[m|f(a)|^l I(h(t);p,q)+|f(b)|^l I(h(t);q,p)\Big]^{\frac{1}{l}}.
\end{aligned}$$

So, the proof of this theorem is complete. □

Let us discuss some special cases of Theorem 2.11.

Corollary 2.12. *In Theorem* 2.11 *for* $h(t)=t^s$ *with* $s\in[0,1]$*, one can get the following inequality for generalized relative semi-*(m,s)*-Breckner-preinvex mappings:*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\Big[m|f(a)|^l\beta(p+1,q+s+1)+|f(b)|^l\beta(q+1,p+s+1)\Big]^{\frac{1}{l}}.
\end{aligned}$$

Corollary 2.13. *In Theorem* 2.11 *for* $h(t)=t^{-s}$ *with* $s\in(0,1]$*, we deduce the following inequality for generalized relative semi-*(m,s)*-Godunova-Levin-Dragomir preinvex mappings:*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\Big[m|f(a)|^l\beta(p+1,q-s+1)+|f(b)|^l\beta(q+1,p-s+1)\Big]^{\frac{1}{l}}.
\end{aligned}$$

Corollary 2.14. *In Theorem* 2.11 *for* $h(t) = t(1-t)$*, one can obtain the following inequality for generalized relative semi-*(m, tgs)*-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\beta^{\frac{1}{l}}(p+2,q+2)\Big[m|f(a)|^l+|f(b)|^l\Big]^{\frac{1}{l}}.$$

Corollary 2.15. *In Theorem* 2.11 *for* $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$*, we derive the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times \left[m|f(a)|^l\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(b)|^l\beta\left(q+\frac{1}{2},p+\frac{3}{2}\right)\right]^{\frac{1}{l}}.$$

3. Other results involving conformable fractional integrals

For establishing our main results regarding generalizations of Hermite-Hadamard type inequalities associated with generalized relative semi-(m, h)-preinvexity via conformable fractional integrals, we need the following lemma.

Lemma 3.1. *Let* $\varphi : I \longrightarrow K$ *be a continuous function. Suppose* $K \subseteq \mathbb{R}$ *be an open* m*-invex subset with respect to* $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0,1]$ *and let* $\eta(\varphi(b),\varphi(a),m) > 0$*. Assume that* $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ *be a twice differentiable function on* K° *and* $f'' \in L_1[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$*. Then for* $\alpha > 0$*, we have*

$$\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\Bigg\{\frac{\beta(n+2,\alpha-n)}{\eta(\varphi(x),\varphi(a),m)}\Big[f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-f'(m\varphi(a))\Big]$$

$$-\frac{\beta(n+2,\alpha-n)f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)}+\frac{(n+1)!}{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}$$

$$\times\Bigg[\left({}^{m\varphi(a)+\eta(\varphi(x),\varphi(a),m)}I_\alpha f\right)(m\varphi(a))$$

$$-(\alpha-n-1)\left({}^{m\varphi(a)+\eta(\varphi(x),\varphi(a),m)}I_{\alpha-1} f\right)(m\varphi(a))\Bigg]\Bigg\}+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{\frac{\beta(n+2,\alpha-n)}{\eta(\varphi(x),\varphi(b),m)}\Big[f'(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-f'(m\varphi(b))\Big]$$

$$-\frac{\beta(n+2,\alpha-n)f'(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(x),\varphi(b),m)}+\frac{(n+1)!}{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}$$

$$\times\Bigg[\left({}^{m\varphi(b)+\eta(\varphi(x),\varphi(b),m)}I_{\alpha}f\right)(m\varphi(b))$$

$$-(\alpha-n-1)\left({}^{m\varphi(b)+\eta(\varphi(x),\varphi(b),m)}I_{\alpha-1}f\right)(m\varphi(b))\Bigg]\Bigg\}$$

$$=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt. \quad (3.1)$$

We denote

$$\begin{aligned}&I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\\&:=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\\&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt\\&+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\\&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt.\end{aligned} \quad (3.2)$$

Proof. A simple proof of the equality can be done by performing two integration by parts in the integrals from the right side of (3.2) and changing the variables. □

Using relation (3.2) and Lemma (3.1), we now state the following theorem.

Theorem 3.2. *Suppose $h:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ be a twice differentiable mapping on K° (the interior of K). If $|f''|^q$ is generalized relative semi-(m,h)-preinvex mapping on K, $q>1$, $p^{-1}+q^{-1}=1$, then for $\alpha>0$, we have*

$$\begin{aligned}\left|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\right|&\le\frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1h(t)dt\right)^{\frac{1}{q}}\\&\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^q+|f''(x)|^q\Big]^{\frac{1}{q}}\\&+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^q+|f''(x)|^q\Big]^{\frac{1}{q}}\Bigg\},\end{aligned} \quad (3.3)$$

where $\delta(p,\alpha,n) := \int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]^p dt.$

Proof. Suppose that $q>1$. Using relation (3.2), Hölder inequality, generalized relative semi-(m,h)-preinvexity of $|f''|^q$ on K° and properties of the modulus, we have

$$\begin{aligned}
&\big|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\big| \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|} \\
&\times \int_0^1 |\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|} \\
&\times \int_0^1 |\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 |f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 |f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \Big[mh(1-t)|f''(a)|^q + h(t)|f''(x)|^q\Big] dt\right)^{\frac{1}{q}} \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \Big[mh(1-t)|f''(b)|^q + h(t)|f''(x)|^q\Big] dt\right)^{\frac{1}{q}} \\
&= \frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 h(t)dt\right)^{\frac{1}{q}} \\
&\times \Bigg\{ |\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^q + |f''(x)|^q\Big]^{\frac{1}{q}} \\
&+ |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^q + |f''(x)|^q\Big]^{\frac{1}{q}} \Bigg\}.
\end{aligned}$$

So, the proof of this theorem is complete. □

Corollary 3.3. *In Theorem 3.2, if choosing $\alpha \in (n, n+1]$ where $n = 0, 1, 2, \ldots,$ one can get the following inequality for conformable fractional integrals:*

$$\Bigg|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a)) - \eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_\alpha f\right)(m\varphi(a)) + \left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_\alpha f\right)(m\varphi(b))\right]\Bigg|$$
$$\leq \frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}$$
$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^q + |f''(x)|^q\Big]^{\frac{1}{q}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^q + |f''(x)|^q\Big]^{\frac{1}{q}}\Bigg\}.$$

Corollary 3.4. *In Corollary 3.3, if putting $\alpha = n+1$ where $n = 0, 1, 2, \ldots$ and $|f''| \leq K$, one can obtain the following inequality for fractional integrals:*

$$\Bigg|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a)) - \eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$
$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a)) + J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$
$$\leq K(m+1)^{\frac{1}{q}}\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}\right)^{\frac{1}{p}}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}$$
$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} + |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\right].$$

Theorem 3.5. *Suppose $h : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) > 0$. Assume that $f : K =$*

$[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ *be a twice differentiable mapping on* K°. *If* $|f''|^q$ *is generalized relative semi-*(m,h)*-preinvex mapping on* K, $q \geq 1$, *then for* $\alpha > 0$, *we have*

$$\left|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times \Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}\Bigg\}, \tag{3.4}$$

where $A(h(t);\alpha,n) := \int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]h(1-t)dt.$

Proof. Suppose that $q \geq 1$. Using relation (3.2), the well-known power mean inequality, the generalized relative semi-(m,h)-preinvexity of $|f''|^q$ and properties of the modulus, we have

$$\left|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\right| \leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$
$$\times \int_0^1 |\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$
$$\times \int_0^1 |\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]dt\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right]^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]dt\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right]^{\frac{1}{q}}$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]dt\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]\Big[mh(1-t)|f''(a)|^q+h(t)|f''(x)|^q\Big]dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]\left[mh(1-t)|f''(b)|^q+h(t)|f''(x)|^q\right]dt\right]^{\frac{1}{q}}$$

$$=\frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[m|f''(a)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\right]^{\frac{1}{q}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[m|f''(b)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\right]^{\frac{1}{q}}\right\}.$$

So, the proof of this theorem is complete. □

Corollary 3.6. *In Theorem* 3.5*, if the choice of* $\alpha\in(n,n+1]$ *where* $n=0,1,2,\dots$*, we get the following inequality for conformable fractional integrals:*

$$\left|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}\right.$$

$$-\frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b),\varphi(a),m)}$$

$$\left.\times\left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_\alpha f\right)(m\varphi(a))+\left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_\alpha f\right)(m\varphi(b))\right]\right|$$

$$\le\frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[m|f''(a)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\right]^{\frac{1}{q}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[m|f''(b)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\right]^{\frac{1}{q}}\right\}.$$

Corollary 3.7. *In Corollary* 3.6*, if the choice of* $\alpha=n+1$ *where* $n=0,1,2,\dots$ *and* $|f''|\le K$*, we obtain the following inequality for fractional integrals:*

$$\left|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\right.$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$

$$\leq\frac{K}{(\alpha+2)^{1-\frac{1}{q}}}\left[mA(h(t);\alpha,n)+A(h(1-t);\alpha,n)\right]^{\frac{1}{q}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

Remark 3.8. In Corollary 3.3 and Corollary 3.6, if taking $h(t)=t^{s}$, $h(t)=t^{-s}$, $h(t)=t(1-t)$ or $h(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, then one can get some special conformable fractional integral inequalities for generalized relative semi-(m,s)-Breckner-preinvex functions, generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex functions, generalized relative semi-(m,tgs)-preinvex functions, and generalized relative semi-m-MT-preinvex functions, respectively. For Corollary 3.4 and Corollary 3.7, we also derive some similar Riemann-Liouville fractional integral inequalities for these functions.

4. Applications to special means

Let us begin this section by considering some particular means for two positive real numbers α,β $(\alpha\neq\beta)$ and for this aim we recall the following means:

1. The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}.$$
2. The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}.$$
3. The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}.$$
4. The power mean:
$$P_r:=P_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\quad r\geq 1.$$
5. The identric mean:
$$I:=I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha\neq\beta;\\ \alpha, & \alpha=\beta.\end{cases}$$
6. The logarithmic mean:
$$L:=L(\alpha,\beta)=\frac{\beta-\alpha}{\ln(\beta)-\ln(\alpha)}.$$
7. The generalized log-mean:
$$L_p:=L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}};\ \ p\in\mathbb{R}\setminus\{-1,0\}.$$

8. The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \le \alpha_i \le 1$, $u_i > 0$ $(i = 1, 2, \dots, n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have $H \le G \le L \le I \le A$.
Now, let a and b be positive real numbers along with $a < b$. Consider the function $M := M(\varphi(x), \varphi(y)) : [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \times [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous mappings. Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(x), \varphi(y))$ and setting $\eta(\varphi(x), \varphi(y)) = M(\varphi(x), \varphi(y))$, $\forall x, y \in I$, together $m = 1$ in (3.3) and (3.4), one can obtain the subsequent interesting results involving means:

$$\left|I_{f,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)\right|$$

$$= \Bigg|\frac{M^{\alpha+2}(\varphi(a),\varphi(x))}{M(\varphi(a),\varphi(b))}\left\{\frac{\beta(n+2,\alpha-n)}{M(\varphi(a),\varphi(x))}\Big[f'(\varphi(a)+M(\varphi(a),\varphi(x)))-f'(\varphi(a))\Big]\right.$$

$$-\frac{\beta(n+2,\alpha-n)f'(\varphi(a)+M(\varphi(a),\varphi(x)))}{M(\varphi(a),\varphi(x))}+\frac{(n+1)!}{M^{\alpha+2}(\varphi(a),\varphi(x))}$$

$$\left.\times\left[\left({}^{\varphi(a)+M(\varphi(a),\varphi(x))}I_{\alpha}f\right)(\varphi(a))-(\alpha-n-1)\left({}^{\varphi(a)+M(\varphi(a),\varphi(x))}I_{\alpha-1}f\right)(\varphi(a))\right]\right\}$$

$$+\frac{M^{\alpha+2}(\varphi(b),\varphi(x))}{M(\varphi(a),\varphi(b))}\left\{\frac{\beta(n+2,\alpha-n)}{M(\varphi(b),\varphi(x))}\Big[f'(\varphi(b)+M(\varphi(b),\varphi(x)))-f'(\varphi(b))\Big]\right.$$

$$-\frac{\beta(n+2,\alpha-n)f'(\varphi(b)+M(\varphi(b),\varphi(x)))}{M(\varphi(b),\varphi(x))}+\frac{(n+1)!}{M^{\alpha+2}(\varphi(b),\varphi(x))}$$

$$\left.\times\left[\left({}^{\varphi(b)+M(\varphi(b),\varphi(x))}I_{\alpha}f\right)(\varphi(b))-(\alpha-n-1)\left({}^{\varphi(b)+M(\varphi(b),\varphi(x))}I_{\alpha-1}f\right)(\varphi(b))\right]\right\}\Bigg|$$

$$\le \frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{M(\varphi(a),\varphi(b))}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}\left\{M^{\alpha+2}(\varphi(a),\varphi(x))\Big[|f''(a)|^q+|f''(x)|^q\Big]^{\frac{1}{q}}\right.$$

$$\left.+M^{\alpha+2}(\varphi(b),\varphi(x))\Big[|f''(b)|^q+|f''(x)|^q\Big]^{\frac{1}{q}}\right\}, \tag{4.1}$$

$$\begin{aligned}
&\left|I_{f,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)\right| \\
&\leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{M(\varphi(a),\varphi(b))} \\
&\times\Bigg\{M^{\alpha+2}(\varphi(a),\varphi(x))\Big[|f''(a)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}} \qquad (4.2)\\
&+M^{\alpha+2}(\varphi(b),\varphi(x))\Big[|f''(b)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Letting $M(\varphi(x),\varphi(y)) := A, G, H, P_r, I, L, L_p, M_p, \forall\, x, y \in I$ in (4.1) and (4.2), we get the inequalities involving means for a particular choices of a twice differentiable generalized relative semi-$(1,h)$-preinvex mappings. The details are left to the interested reader.

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