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Properties of absolute-$*$-k-paranormal operators and contractions for $*$-$\mathcal{A}(k)$ operators

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Abstract. First, we see if T is absolute-$*$-k-paranormal for $k \geq 1$, then T is a normaloid operator. We also see some properties of absolute-$*$-k-paranormal operator and $*$-$\mathcal{A}(k)$ operator. Then, we will prove the spectrum continuity of the class $*$-$\mathcal{A}(k)$ operator for $k > 0$. Moreover, it is proved that if T is a contraction of the class $*$-$\mathcal{A}(k)$ for $k > 0$, then either T has a nontrivial invariant subspace or T is a proper contraction, and the nonnegative operator

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

is a strongly stable contraction. Finally if $T \in *$-$\mathcal{A}(k)$ is a contraction for $k > 0$, then T is the direct sum of a unitary and $C_{\cdot 0}$ (c.n.u) contraction.

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1. Introduction

Throughout this paper, let H and K be infinite dimensional separable complex Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. We denote by $L(H, K)$ the set of all bounded operators from H into K. To simplify, we put $L(H) := L(H, H)$. For $T \in L(H)$, we denote by $\ker(T)$ the null space and by $T(H)$ the range of T. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* denotes its adjoint. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. An operator $T \in L(H)$, is a positive operator, $T \geq O$, if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. We write by $\sigma(T)$, $\sigma_p(T)$, and $\sigma_a(T)$ spectrum, point spectrum and approximate point spectrum respectively. Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by $\mathrm{iso}\sigma(T)$ and $\mathrm{acc}\sigma(T)$, respectively. We write $r(T)$ for the spectral radius. It is well known that $r(T) \leq \|T\|$. The operator T is called normaloid if $r(T) = \|T\|$.

A contraction is an operator T such that $\|Tx\| \leq \|x\|$ for all $x \in H$. A proper contraction is an operator T such that $\|Tx\| < \|x\|$ for every nonzero $x \in H$. A strict contraction is an operator such that $\|T\| < 1$ (*i.e.*, $\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} < 1$). Obviously, every strict contraction is a proper contraction and every proper contraction is a contraction. An operator T is said to be completely non-unitary (c.n.u) if T restricted to every reducing subspace of H is non-unitary.

An operator T on H is uniformly stable, if the power sequence $\{T^m\}_{m=1}^{\infty}$ converges uniformly to the null operator (*i.e.*, $\|T^m\| \to O$). An operator T on H is strongly stable, if the power sequence $\{T^m\}_{m=1}^{\infty}$ converges strongly to the null operator (*i.e.*, $\|T^m x\| \to 0$, for every $x \in H$).

A contraction T is of class $C_{0\cdot}$ if T is strongly stable (*i.e.*, $\|T^m x\| \to 0$ and $\|Tx\| \leq \|x\|$ for every $x \in H$). If T^* is a strongly stable contraction, then T is of class $C_{\cdot 0}$. T is said to be of class $C_{1\cdot}$ if $\lim_{m\to\infty} \|T^m x\| > 0$ (equivalently, if $T^m x \not\to 0$ for every nonzero x in H). T is said to be of class $C_{\cdot 1}$ if $\lim_{m\to\infty} \|T^{*m} x\| > 0$ (equivalently, if $T^{*m} x \not\to 0$ for every nonzero x in H). We define the class $C_{\alpha\beta}$ for $\alpha, \beta = 0, 1$ by $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$. These are the Nagy-Foiaş classes of contractions [21, p.72]. All combinations are possible leading to classes C_{00}, C_{01}, C_{10} and C_{11}. In particular, T and T^* are both strongly stable contractions if and only if T is a C_{00} contraction. Uniformly stable contractions are of class C_{00}.

For an operator $T \in L(H)$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$. An operator T is said to be a normal operator if $T^*T = TT^*$ and T is said to be hyponormal, if $|T|^2 \geq |T^*|^2$. An operator $T \in L(H)$, is said to be paranormal [11], if $\|T^2 x\| \geq \|Tx\|^2$ for every unit vector x in H. Further, T is said to be $*$-paranormal [1], if $\|T^2 x\| \geq \|T^* x\|^2$ for every unit vector x in H.

In [13] authors Furuta, Ito and Yamazaki introduced the class $\mathcal{A}$ operator, respectively the class $\mathcal{A}(k)$ operator defined as follows: For each $k > 0$, an operator T is from class $\mathcal{A}(k)$ operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2,$$

(for $k = 1$ it defines class $\mathcal{A}$ operator), and they showed that the class $\mathcal{A}$ is a subclass of paranormal operators.

In the same paper, authors introduced the absolute-k-paranormal operators as follows: For each $k > 0$, an operator T is absolute-k-paranormal if

$$\left\||T|^k Tx\right\| \geq ||Tx||^{k+1},$$

for every unit vector $x \in H$. In case where $k = 1$ it defines the paranormal operator. The class $\mathcal{A}(k)$ operator is included in the absolute-k-paranormal operator for any $k > 0$, [13, Theorem 2]).

B. P. Duggal, I. H. Jeon, and I. H. Kim [5], introduced $*$-class $\mathcal{A}$ operator. An operator $T \in L(H)$ is said to be a $*$-class $\mathcal{A}$ operator, if $|T^2| \geq |T^*|^2$. A $*$-class $\mathcal{A}$ is a generalization of a hyponormal operator, [5, Theorem 1.2], and $*$-class $\mathcal{A}$ is a subclass of the class of $*$-paranormal operators, [5, Theorem 1.3]. We denote the set of $*$-class

$\mathcal{A}$ by $\mathcal{A}^*$. An operator $T \in L(H)$ is said to be a k-quasi-$*$-class $\mathcal{A}$ operator [20], if

$$T^{*k}\left(|T^2| - |T^*|^2\right)T^k \geq O,$$

for a nonnegative integer k.

In [24] authors, S. Panayappan and A. Radharamani introduced the class $*$-$\mathcal{A}(k)$ operator and absolute-$*$-k-paranormal operator.

Definition 1.1. For each $k > 0$, an operator T is absolute-$*$-k-paranormal if

$$|||T|^kTx|| \geq ||T^*x||^{k+1}$$

for every unit vector $x \in H$.

In case where $k = 1$ it defines the $*$-paranormal operator.

Definition 1.2. For each $k > 0$, an operator T is class $*$-$A(k)$, if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T^*|^2.$$

In case where $k = 1$ it defines the $\mathcal{A}^*$ class operators.

In this paper, we shall show behavior of the class $*$-$\mathcal{A}(k)$ operator and absolute-$*$-k-paranormal operator.

2. Properties of absolute-$*$-k-paranormal operator and $*$-$\mathcal{A}(k)$ operator

Theorem 2.1. *If T is an absolute-$*$-k-paranormal operator for $k > 0$, then T is a normaloid operator.*

Proof. Let T be an absolute-$*$-k-paranormal operator. In case where $k = 1$, T is a $*$-paranormal operator, then by [1, Theorem 1.1] it follows that T is a normaloid operator. Following, it will be proved that for $k > 1$ the operator T is a normaloid operator, because for $0 < k < 1$, it was proved in [3](Theorem 2.9). Without losing the generality, assume $\|T\| = 1$. Since T is an absolute-$*$-k-paranormal, then

$$\|T^*x\|^{k+1} \leq \||T|^kTx\|\|x\|^k \leq \||T|^{k-1}\|\||T|Tx\|\|x\|^k \leq \|T^2x\|\|x\|^k$$

for all $x \in H$. Therefore,

$$\frac{\|T^*x\|^{k+1}}{\|x\|^k} \leq \|T^2x\| \leq \|x\| \tag{2.1}$$

for all $x \in H$.

By definition of $\|T^*\|$, there exists a sequence $\{x_i\}$ of unit vectors such that

$$\|T^*x_i\| \to \|T^*\| = \|T\| = 1. \tag{2.2}$$

Put $x = x_i$ in (2.1), then we have.

$$\frac{\|T^*x_i\|^{k+1}}{\|x_i\|^k} \leq \|T^2x_i\| \leq \|x_i\| = 1 \tag{2.3}$$

so, $\|T^2x_i\| \to 1$, by (2.2) and (2.3), that is

$$\|T^2\| = 1 = \|T\|^2.$$

Let us now suppose that

$$\|T^{n-1}x_i\| \to 1\ ,\ \|T^{n-2}x_i\| \to 1 \text{ and } \|T^{n-3}x_i\| \to 1 \text{ for } n \geq 3. \tag{2.4}$$

Put $x = T^{n-2}x_i$ in (2.1), then we have

$$\frac{\|T^*T^{n-2}x_i\|^{k+1}}{\|T^{n-2}x_i\|^k} \leq \|T^n x_i\| \leq \|T^{n-2}x_i\|. \tag{2.5}$$

From Cauchy-Schwarz inequality we have

$$\frac{\|T^{n-2}x\|^2}{\|T^{n-3}x\|} \leq \|T^*T^{n-2}x\|. \tag{2.6}$$

From relations (2.5) and (2.6) we have

$$\frac{\|T^{n-2}x_i\|^{k+2}}{\|T^{n-3}x_i\|^{k+1}} \leq \frac{\|T^*T^{n-2}x_i\|^{k+1}}{\|T^{n-2}x_i\|^k} \leq \|T^n x_i\| \leq \|T^{n-2}x_i\|.$$

respectively:

$$\frac{\|T^{n-1}x_i\|^{k+2}}{\|T^{n-3}x_i\|^{k+1}} \leq \frac{\|T^{n-2}x_i\|^{k+2}}{\|T^{n-3}x_i\|^{k+1}} \leq \frac{\|T^*T^{n-2}x_i\|^{k+1}}{\|T^{n-2}x_i\|^k} \leq \|T^n x_i\| \leq \|T^{n-2}x_i\|. \tag{2.7}$$

Hence, $\|T^n x_i\| \to 1$, by (2.4) and (2.7) that is $\|T^n\| = 1 = \|T\|^n$. Consequently

$$\|T^n\| = 1 = \|T\|^n,$$

for all positive integers n by induction. □

Example 2.2. An example of non-absolute-$*$-k-paranormal operator which is a normaloid operator. Let us denote by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then $||T^n|| = ||T||^n$ for all positive integers n. However, the relation

$$\left\||T|^k Tx\right\| \geq ||T^*x||^{k+1}$$

does not hold for the unit vector $e_3 = (0, 0, 1)$. With which was proved that T is a non-absolute-$*$-k-paranormal operator, but it is a normaloid operator.

It is known that there exists a linear operator T, so that T^n is compact operator for some $n \in \mathbb{N}$, but T itself is not compact. For instance, take any nilpotent noncompact operator (If $(e_n)_n$ is an orthonormal basis of H then the shift defined by $T(e_{2n}) = e_{2n+1}$ and $T(e_{2n+1}) = 0$ is not a compact operator for which $T^2 = O$).

In this context, we will show that in cases where an operator T is an absolute-$*$-k-paranormal operator and if its exponent T^n is compact, for some $n \in \mathbb{N}$, then T is compact too.

Theorem 2.3. *If T is an absolute-$*$-k-paranormal operator for $k > 0$ and if T^n is compact for some $n \in \mathbb{N}$, then it follows that T is compact too.*

Proof. Compactness of T^n implies countable spectrum (consisting of mutually orthogonal eigenvalues ([26], Theorem 6)), this then implies T^n normal compact, hence T is normal compact. □

Corollary 2.4. *If T, R are absolute-$*$-k-paranormal operators for $k > 0$ and if T^n and R^m are compact for some $n, m \in \mathbb{N}$, then it follows that $T \oplus R$ is compact too.*

Corollary 2.5. *If T, R are absolute-$*$-k-paranormal operators for $k > 0$ and if T^n is a compact operator for some $n \in \mathbb{N}$ or R^m is a compact operator for some $m \in \mathbb{N}$, then it follows that $T \otimes R$ is compact too.*

Lemma 2.6. [16, Hansen Inequality] *If $A, B \in L(H)$, satisfying $A \geq O$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\delta \geq B^*A^\delta B \text{ for all } \delta \in (0,1].$$

Lemma 2.7. [12, Löwner-Heinz Inequality] *If $A, B \in L(H)$, satisfying $A \geq B \geq O$, then $A^\delta \geq B^\delta$ for all $\delta \in [0,1]$.*

A subspace M of space H is said to be nontrivial invariant(alternatively, T-invariant) under T if $\{0\} \neq M \neq H$ and $T(M) \subseteq M$.

Theorem 2.8. *If T is a class $*$-$\mathcal{A}(k)$ operator for $0 < k \leq 1$ and M is its invariant subspace, then the restriction $T\mid_M$ of T to M is also a class $*$-$\mathcal{A}(k)$ operator.*

Proof. Since M is an invariant subspace of T, T has the matrix representation

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on} \quad H = M \oplus M^\perp.$$

Let P be the projection of H onto M, where $A = T\mid_M$ and $\begin{pmatrix} A & O \\ O & O \end{pmatrix} = TP = PTP$. Since T is a class $*$-$\mathcal{A}(k)$ operator, we have

$$P\left(\left(T^*|T|^2T\right)^{\frac{1}{k+1}} - |T^*|^2\right)P \geq O.$$

By Hansen inequality, we have

$$\begin{aligned} \begin{pmatrix} |A^*|^2 & O \\ O & O \end{pmatrix} &\leq \begin{pmatrix} |A^*|^2 + |B^*|^2 & O \\ O & O \end{pmatrix} \\ &\leq \left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}}. \end{aligned}$$

Since

$$P|T|^{2k}P \leq (P|T|^2P)^k,$$

then

$$PT^*P|T|^{2k}PTP \leq PT^*(P|T|^2P)^kTP.$$

By Löwner-Heinz inequality we have

$$\left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}} \leq \left(PT^*(P|T|^2P)^kTP\right)^{\frac{1}{k+1}}.$$

So, we have

$$\begin{pmatrix} |A^*|^2 & O \\ O & O \end{pmatrix} \leq \begin{pmatrix} \left(A^*|A^*|^{2k}A\right)^{\frac{1}{k+1}} & O \\ O & O \end{pmatrix}.$$

Hence, A is a class $*$-$\mathcal{A}(k)$ operator on M. □

Theorem 2.9. *If T is a class $*$-$\mathcal{A}(k)$ operator, has the representation $T=\lambda\oplus A$ on $\ker(T-\lambda)\oplus(\ker(T-\lambda))^{\perp}$, where $\lambda\neq 0$ is an eigenvalue of T, then A is a class $*$-$\mathcal{A}(k)$ operator with $\ker(A-\lambda)=\{0\}$.*

Proof. Since $T=\lambda\oplus A$, then $T=\begin{pmatrix}\lambda & O\\ O & A\end{pmatrix}$ and we have:

$$\begin{aligned}\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}}-|T^*|^2 &= \begin{pmatrix}|\lambda|^{2(k+1)} & O\\ O & A^*|A|^{2k}A\end{pmatrix}^{\frac{1}{k+1}}-\begin{pmatrix}|\lambda|^2 & O\\ O & |A^*|^2\end{pmatrix}\\ &= \begin{pmatrix}|\lambda|^2 & O\\ O & \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}}\end{pmatrix}-\begin{pmatrix}|\lambda|^2 & O\\ O & |A^*|^2\end{pmatrix}\\ &= \begin{pmatrix}O & O\\ O & \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}}-|A^*|^2\end{pmatrix}\end{aligned}$$

Since T is a class $*$-$\mathcal{A}(k)$ operator, then A is a class $*$-$\mathcal{A}(k)$ operator.
Let $x_2\in\ker(A-\lambda)$. Then

$$(T-\lambda)\begin{pmatrix}0\\ x_2\end{pmatrix}=\begin{pmatrix}O & O\\ O & A-\lambda\end{pmatrix}\begin{pmatrix}0\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Hence $x_2\in\ker(T-\lambda)$. Since $\ker(A-\lambda)\subseteq(\ker(T-\lambda))^{\perp}$, this implies $x_2=0$. Representation of T implies $A-\lambda$ is injective and by Theorem 2.8 A is $*$-$\mathcal{A}(k)$. □

3. Spectrum continuity on the set of class $*$-$\mathcal{A}(k)$ operator

Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}$. Let's define the inferior and superior limits of $\{E_n\}_{n\in\mathbb{N}}$, denoted respectively by $\liminf_{n\to\infty}\{E_n\}$ and $\limsup_{n\to\infty}\{E_n\}$ as it follows:

1) $\liminf_{n\to\infty}\{E_n\}=\{\lambda\in\mathbb{C}:\text{for every }\epsilon>0,\text{there exists }N\in\mathbb{N}\text{ such that }B(\lambda,\epsilon)\cap E_n\neq\emptyset\text{ for all }n>N\}$,

2) $\limsup_{n\to\infty}\{E_n\}=\{\lambda\in\mathbb{C}:\text{for every }\epsilon>0,\text{there exists }J\subseteq\mathbb{N}\text{ infinite such that }B(\lambda,\epsilon)\cap E_n\neq\emptyset\text{ for all }n\in J\}$.

If

$$\liminf_{n\to\infty}\{E_n\}=\limsup_{n\to\infty}\{E_n\},$$

then $\lim_{n\to\infty}\{E_n\}$ is said to exists and is equal to this common limit.

A mapping p, defined on $L(H)$, whose values are compact subsets on $\mathbb{C}$ is said to be upper semi-continuous at T, if $T_n\to T$ then $\limsup_{n\to\infty}p(T_n)\subset p(T)$, and lower semi-continuous at T, if $T_n\to T$ then $p(T)\subset\liminf_{n\to\infty}p(T_n)$. If p is both upper and lower semi-continuous at T, then it is said to be continuous at T and in this case $\lim_{n\to\infty}p(T_n)=p(T)$.

The spectrum $\sigma:T\to\sigma(T)$ is upper semi-continuous by [15, Problem 102], but it is not continuous in general, [25, Example 4.6]

We write $\alpha(T) = \dim\ker(T)$, $\beta(T) = \dim\ker(T^*)$. An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$ is said to be upper semi-Weyl operator if it is upper semi-Fredholm and $\mathrm{ind}(T) \leq 0$, while $T \in L(H)$ is said to be lower semi-Weyl operator if it is lower semi-Fredholm and $\mathrm{ind}(T) \geq 0$. An operator is said to be Weyl operator if it is Fredholm of index zero.

Lemma 3.1. [22] *If $\{T_n\} \subset L(H)$ and $T \in L(H)$ are such that T_n converges, according to the operator norm topology, to T then*

$$\mathrm{iso}\sigma(T) \subseteq \liminf_{n\to\infty} \sigma(T_n).$$

Lemma 3.2. [2] *Let H be a complex Hilbert space. Then there exists a Hilbert space Y such that $H \subset Y$ and a map $\varphi : L(H) \to L(Y)$ with the following properties:*

1. *φ is a faithful $*$-representation of the algebra $L(H)$ on Y, so:*
$$\varphi(I_H) = I_Y \ ,\ \varphi(T^*) = (\varphi(T))^* \ ,\ \varphi(TS) = \varphi(T)\varphi(S)$$
$$\varphi(\alpha T + \beta S) = \alpha\varphi(T) + \beta\varphi(S) \textit{ for any } T, S \in L(H) \textit{ and } \alpha, \beta \in \mathbb{C},$$
2. *$\varphi(T) \geq 0$ for any $T \geq 0$ in $L(H)$,*
3. *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in L(H)$,*
4. *If T is a positive operator, then $\varphi(T^\alpha) = |\varphi(T)|^\alpha$, for $\alpha > 0$,*

Lemma 3.3. *If T is a class $*$-$\mathcal{A}(k)$ operator, then $\varphi(T)$ is a class $*$-$\mathcal{A}(k)$ operator.*

Proof. Let $\varphi : L(H) \to L(K)$ be Berberian's faithful $*$-representation and let T be a class $*$-$\mathcal{A}(k)$ operator. Then, we have

$$\begin{aligned}\left((\varphi(T))^*|\varphi(T)|^{2k}\varphi(T)\right)^{\frac{1}{k+1}} - |(\varphi(T))^*|^2 &= \left(\varphi(T^*)\varphi(|T|^{2k})\varphi(T)\right)^{\frac{1}{k+1}} - |\varphi(T^*)|^2 \\ &= \left(\varphi(T^*|T|^{2k}T)^{\frac{1}{k+1}} - \varphi(|T^*|^2)\right) \\ &= \varphi\left((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T^*|^2\right) \geq 0\end{aligned}$$

thus $\varphi(T)$ is a class $*$-$\mathcal{A}(k)$ operator. □

Theorem 3.4. *The spectrum σ is continuous on the set of class $*$-$\mathcal{A}(k)$ operator for $k > 0$.*

Proof of the theorem is based in idea's given in the paper [6].

Proof. Since the function σ is upper semi-continuous, if $\{T_n\} \subset L(H)$ is a sequence which converges, to $T \in L(H)$, by operator norm topology. Then $\limsup_{n\to\infty} \sigma(T_n) \subset \sigma(T)$. Thus, to prove the theorem it would suffice to prove that if $\{T_n\}$ is a sequence of operators so that it belongs to class $*$-$\mathcal{A}(k)$ operator and $\lim_{n\to\infty} \|T_n - T\| = 0$ for some class $*$-$\mathcal{A}(k)$ operator T, then $\sigma(T) \subset \liminf_{n\to\infty} \sigma(T_n)$. From [25, Proposition

4.9] it would suffice to prove $\sigma_a(T) \subset \liminf_n \sigma(T_n)$. Since $\sigma(T) = \sigma(\varphi(T))$, $\sigma(T_n) = \sigma(\varphi(T)_n)$ and $\sigma_a(T) = \sigma_a(\varphi(T))$ we have

$$\sigma_a(T) \subset \liminf_{n\to\infty} \sigma(T_n) \Longleftrightarrow \sigma_a(\varphi(T)) \subset \liminf_{n\to\infty} \sigma(\varphi(T)_n).$$

Let $\lambda \in \sigma_a(\varphi(T))$. Then $\lambda \in \sigma_p(\varphi(T))$.
By Theorem 2.9, $\varphi(T)$ has a representation

$$\varphi(T) = \lambda \oplus A \text{ on } H = \ker(\varphi(T) - \lambda) \oplus (\ker(\varphi(T) - \lambda))^{\perp} \text{ and } \ker(A - \lambda) = \{0\}.$$

Therefore $A - \lambda$ is upper semi-Fredholm operator and $\alpha(A - \lambda) = 0$. There exists a $\epsilon > 0$ such that $A-(\lambda-\mu_0)$ is upper semi-Fredholm operator with $\mathrm{ind}(A-(\lambda-\mu_0)) = \mathrm{ind}(A - \lambda)$ and $\alpha(A - (\lambda - \mu_0)) = 0$ for every μ_0 such that $0 < |\mu_0| < \epsilon$. Let's set $\mu = \lambda - \mu_0$, and we have $\varphi(T) - \mu = (\lambda - \mu) \oplus (A - \mu)$ is upper semi-Fredholm operator, $\mathrm{ind}(\varphi(T) - \mu) = \mathrm{ind}(A - \mu)$ and $\alpha(\varphi(T) - \mu) = 0$.

Suppose the contrary, $\lambda \notin \liminf_{n\to\infty} \sigma(\varphi(T)_n)$. Then, there exists a $\delta > 0$, a neighborhood $\mathcal{D}_\delta(\lambda)$ of λ and a subsequence $\{\varphi(T)_{n_l}\}$ of $\{\varphi(T)_n\}$ such that $\sigma(\varphi(T)_{n_l}) \cap \mathcal{D}_\delta(\lambda) = \emptyset$ for every $l \geq 1$. This implies that $\varphi(T)_{n_l} - \mu$ is a Fredholm operator and $\mathrm{ind}(\varphi(T)_{n_l} - \mu) = 0$ for every $\mu \in \mathcal{D}_\delta(\lambda)$ and

$$\lim_{n\to\infty} \|(\varphi(T)_{n_l} - \mu) - (\varphi(T) - \mu)\| = 0.$$

It follows from the continuity of the index that $\mathrm{ind}(\varphi(T) - \mu) = 0$ and $\varphi(T) - \mu$ is a Fredholm operator. Since $\alpha(\varphi(T) - \mu) = 0$, $\mu \notin \sigma(\varphi(T))$ for every μ in a ϵ-neighborhood of λ. This contradicts Lemma 3.1, therefore we must have $\lambda \in \liminf_{n\to\infty} \sigma(\varphi(T)_n)$. □

It is well known **Index Product Theorem**: "If S and T are Fredholm operators then ST is a Fredholm operator and $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$". The converse of this theorem is not true in general. To see this, we have operators on l_2:

$$T(x_1, x_2, x_3, ...) = (0, x_1, 0, x_2, 0, x_3, ...) \text{ and } S(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...).$$

We see $ST = I$, so ST is a Fredholm operator, but S and T are not Fredholm operators. However, if S and T are commuting operators and if ST is a Fredholm operator then S and T are Fredholm operators. This fact is not true in general if S and T are Weyl operators, see [19, Remark 1.5.3].

Theorem 3.5. *If S and T are commuting class $*$-$\mathcal{A}(k)$ operators for $0 < k \leq 1$, then*

$$S, T \text{ are Weyl operators} \Longleftrightarrow ST \text{ is Weyl operator.}$$

Proof. If S and T are Weyl operators, by Index Product Theorem, we have that ST is a Weyl operator.

The converse, since $ST = TS$ then

$$\ker S \cup \ker T \subseteq \ker(ST) \text{ and } \ker S^* \cup \ker T^* \subseteq \ker(ST)^*,$$

then S and T are Fredholm operators.

Since S and T are class $*$-$\mathcal{A}(k)$ operators, from [26, theorem 2] S and T are class absolute-k^*-paranormal operators and by [26, theorem 6] we have $\mathrm{ind}(S) \leq 0$ and $\mathrm{ind}(T) \leq 0$. From

$$\mathrm{ind}(S) + \mathrm{ind}(T) = \mathrm{ind}(ST) = 0,$$

we have $\operatorname{ind}(S) = 0$ and $\operatorname{ind}(T) = 0$, so S and T are Weyl operators. $\square$

4. Contractions of the class $*$-$\mathcal{A}(k)$ operator

Definition 4.1. If the contraction T is a direct sum of the unitary and $C_{\cdot 0}$ (c.n.u) contractions, then we say that T has a **Wold-type decomposition**.

Definition 4.2. [9] An operator $T \in L(H)$ is said to have the Fuglede-Putnam commutativity property (**PF property** for short) if $T^*X = XJ$ for any $X \in L(K, H)$ and any isometry $J \in L(K)$ such that $TX = XJ^*$.

Lemma 4.3. [8, 23] *Let T be a contraction. The following conditions are equivalent:*

1. *For any bounded sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset H$ such that $Tx_{n+1} = x_n$ the sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is constant,*
2. *T has a* **Wold-type decomposition***,*
3. *T has the* **PF property***.*

Fugen Gao and Xiaochun Li [14] have proved that if a contraction $T \in \mathcal{A}^*$ has no nontrivial invariant subspace, then (a) T is a proper contraction and (b) The nonnegative operator $D = |T^2| - |T^*|^2$ is a strongly stable contraction. In [17] the authors proved: if T belongs to k-quasi-$*$-class $\mathcal{A}$ and is a contraction, then T has a Wold-type decomposition and T has the PF property. In this section we extend these results to contractions in class $*$-$\mathcal{A}(k)$.

Lemma 4.4. [4, Hölder-McCarthy inequality] *Let T be a positive operator. Then, the following inequalities hold for all $x \in H$:*

1. *$\langle T^r x, x\rangle \le \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $0 < r < 1$,*
2. *$\langle T^r x, x\rangle \ge \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r \ge 1$.*

Proof of the theorems below is based in idea's given in the paper [7].

Theorem 4.5. *If T is a contraction of class $*$-$\mathcal{A}(k)$ operator, then the nonnegative operator*

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

is a contraction whose power sequence $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P and $T^*P = O$.

Proof. Suppose that T is a contraction of class $*$-$\mathcal{A}(k)$ operator. Then

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2 \ge O.$$

Let $R = D^{\frac{1}{2}}$ be the unique nonnegative square root of D. Then for every x in H and any nonnegative integer n, we have

$$\begin{aligned}
\langle D^{n+1}x, x\rangle &= \|R^{n+1}x\|^2 = \langle DR^n x, R^n x\rangle \\
&= \left\langle \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} R^n x, R^n x\right\rangle - \langle |T^*|^2 R^n x, R^n x\rangle \\
&\leq \langle T^*|T|^{2k}TR^n x, R^n x\rangle^{\frac{1}{k+1}} \|R^n x\|^{2(1-\frac{1}{k+1})} - \|T^* R^n x\|^2 \\
&= \||T|^k TR^n x\|^{\frac{2}{k+1}} \|R^n x\|^{2(1-\frac{1}{k+1})} - \|T^* R^n x\|^2 \\
&\leq \||T|^k T\|^{\frac{2}{k+1}} \|R^n x\|^2 - \|T^* R^n x\|^2 \\
&\leq \|R^n x\|^2 - \|T^* R^n x\|^2 \\
&\leq \|R^n x\|^2 = \langle D^n x, x\rangle
\end{aligned}$$

Thus R (and so D) is a contraction (set $n = 0$), and $\{D^n\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative contractions. Then, $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection, say P. Moreover

$$\sum_{n=0}^{m} \|T^* R^n x\|^2 \leq \sum_{n=0}^{m} \left(\|R^n x\|^2 - \|R^{n+1}x\|^2\right) = \|x\|^2 - \|R^{m+1}x\|^2 \leq \|x\|^2,$$

for all nonnegative integers m and for every $x \in H$. Therefore $\|T^* R^n x\| \to 0$ as $n \to \infty$. Then, we have

$$T^* P x = T^* \lim_{n\to\infty} D^n x = \lim_{n\to\infty} T^* R^{2n} x = 0,$$

for every $x \in H$. So that $T^* P = O$. □

Theorem 4.6. *Let T be a contraction of class $*$-$\mathcal{A}(k)$ operator. If T has no nontrivial invariant subspace, then*

1) T is a proper contraction;

2) The nonnegative operator

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

is a strongly stable contraction.

Proof. Suppose that T is a class $*$-$\mathcal{A}(k)$ operator.

1) From [18, Theorem 3.6] we have

$$T^*Tx = \|T\|^2 x \text{ if and only if } \|Tx\| = \|T\|\|x\| \text{ for every } x \in H.$$

Put $M = \{x \in H : \|Tx\| = \|T\|\|x\|\} = \ker(|T|^2 - \|T\|^2)$, which is a closed subspace of H. In the following, we shall show that M is a $T-$invariant subspace. For all $x \in M$, we have

$$\begin{aligned}
\|T(Tx)\|^2 &\leq \|T\|^2\|Tx\|^2 = \|T\|^4\|x\|^2 = \|\|T\|^2 x\|^2 = \|T^*Tx\|^2 \\
&\leq \|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|\|Tx\| \leq \|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|\|T\|\|x\|.
\end{aligned}$$

So,

$$\|T\|^4\|x\|^2 \leq \|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|\|T\|\|x\|,$$

thus,

$$\|T\|^3\|x\| \leq \| \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|$$

and

$$\begin{aligned}
\left\|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\right\| &= \left\langle \left(T^*|T|^{2k}T\right)^{\frac{2}{k+1}} Tx, Tx\right\rangle^{\frac{1}{2}} \\
&\leq \left\langle \left(T^*|T|^{2k}T\right)^{2} Tx, Tx\right\rangle^{\frac{1}{2(k+1)}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&= \left\|T^*|T|^{2k}TTx\right\|^{\frac{1}{k+1}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&\leq \|T\|^{\frac{2k+3}{k+1}} \|x\|^{\frac{1}{k+1}} \|T\|^{\frac{k}{k+1}} \|x\|^{\frac{k}{k+1}} \\
&= \|T\|^3\|x\|.
\end{aligned}$$

Hence,

$$\|T\|^3\|x\| = \| \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|. \tag{4.1}$$

From relation (4.1) we have

$$\begin{aligned}
\|T\|^3\|x\| &= \left\|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\right\| \\
&= \left\langle \left(T^*|T|^{2k}T\right)^{\frac{2}{k+1}} Tx, Tx\right\rangle^{\frac{1}{2}} \\
&\leq \left\langle \left(T^*|T|^{2k}T\right)^{2} Tx, Tx\right\rangle^{\frac{1}{2(k+1)}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&= \|T^*|T|^{2k}TTx\|^{\frac{1}{k+1}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&\leq \|T^*|T|^{2k}\|^{\frac{1}{k+1}} \|T(Tx)\|^{\frac{1}{k+1}} \|Tx\|^{(1-\frac{1}{k+1})}
\end{aligned}$$

Then,

$$\|T\|^2\|x\| \leq \|T(Tx)\| \Longrightarrow \|T\|^2\|x\| = \|T(Tx)\|,$$

respectively,

$$\|T(Tx)\| = \|T\|^2\|x\| = \|T\|\|Tx\|.$$

Thus, M is a T-invariant subspace.

Now, let T be a contraction, i.e., $\|Tx\| \leq x$, for every $x \in H$. If $\|T\| < 1$, thus T is a strict contraction, then it is trivially a proper contraction. If $\|T\| = 1$, thus T is nonstrict contraction, then $M = \{x \in H : \|Tx\| = \|x\|\}$. Since T has no nontrivial invariant subspace, then the invariant subspace M is trivial: either $M = \{0\}$ or $M = H$. If $M = H$ then T is an isometry, and isometries have invariant subspaces. Thus $M = \{0\}$ so that $\|Tx\| < \|x\|$ for every nonzero $x \in H$. So T is proper contraction.

2) Let T be a contraction of class $*$-$\mathcal{A}(k)$ operator. By the above theorem, we have D is a contraction, $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P, and $T^*P = O$. So, $PT = O$. Suppose T has no nontrivial invariant subspaces. Since $\ker P$ is a nonzero invariant subspace for T whenever $PT = O$ and $T \neq O$, it follows that $\ker P = H$. Hence $P = O$, and so $\{D^n\}_{n=1}^{\infty}$ converges strongly to null operator O, so D is a strongly stable contraction. Since D is self-adjoint, then $D \in C_{00}$. □

Corollary 4.7. *Let T be a contraction of the class $*$-$\mathcal{A}(k)$ operator. If T has no nontrivial invariant subspace, then both T and the nonnegative operator*

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

are proper contractions.

Proof. Since a self−adjoint operator T is a proper contraction if and only if T is a C_{00} contraction. □

Theorem 4.8. *If T is a contraction and class $*$-$\mathcal{A}(k)$ operator for $k > 0$, then T has a Wold-type decomposition.*

Proof. Since T is a contraction operator, the decreasing sequence $\{T^nT^{*n}\}_{n=1}^{\infty}$ converges strongly to a nonnegative contraction. We denote by

$$S = \left(\lim_{n\to\infty} T^nT^{*n}\right)^{\frac{1}{2}}.$$

The operators T and S are related by $T^*S^2T = S^2$, $O \leq S \leq I$ and S is self-adjoint operator. By [10] there exists an isometry $V : \overline{S(H)} \to \overline{S(H)}$ such that $VS = ST^*$, and thus $SV^* = TS$, and $\|SV^mx\| \to \|x\|$ for every $x \in \overline{S(H)}$. The isometry V can be extended to an isometry on H, which we still denote by V.

For an $x \in \overline{S(H)}$, we can define $x_n = SV^nx$ for $n \in \mathbb{N}\cup\{0\}$. Then for all nonnegative integers m we have

$$T^m x_{n+m} = T^mSV^{m+n}x = SV^{*m}V^{m+n}x = SV^nx = x_n,$$

and for all $m \leq n$ we have

$$T^m x_n = x_{n-m}.$$

Since T is class $*$-$\mathcal{A}(k)$ operator for $k > 0$ and nontrivial $x \in \overline{A(H)}$ we have

$$\begin{aligned}
\|x_n\|^4 &= \|Tx_{n+1}\|^4 \leq \|T^*Tx_{n+1}\|^2\|x_{n+1}\|^2 \\
&\leq \|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx_{n+1}\|\|Tx_{n+1}\|\|x_{n+1}\|^2 \\
&\leq \left\langle \left(T^*|T|^{2k}T\right)^2 Tx_{n+1}, Tx_{n+1}\right\rangle^{\frac{1}{2(k+1)}} \|Tx_{n+1}\|^{(1-\frac{1}{k+1})}\|Tx_{n+1}\|\|x_{n+1}\|^2 \\
&= \|T^*|T|^{2k}TTx_{n+1}\|^{\frac{1}{k+1}}\|x_n\|^{(1-\frac{1}{k+1})}\|x_n\|\|x_{n+1}\|^2 \\
&\leq \|TTx_{n+1}\|^{\frac{1}{k+1}}\|x_n\|^{\frac{2k+1}{k+1}}\|x_{n+1}\|^2 \\
&= \|x_{n-1}\|^{\frac{1}{k+1}}\|x_n\|^{\frac{2k+1}{k+1}}\|x_{n+1}\|^2
\end{aligned}$$

hence

$$\|x_n\| \leq \|x_{n-1}\|^{\frac{1}{2k+3}}\|x_{n+1}\|^{\frac{2k+2}{2k+3}} \leq \frac{1}{2k+3}\left(\|x_{n-1}\| + (2k+2)\|x_{n+1}\|\right).$$

Thus

$$(2k+2)\left(\|x_{n+1}\| - \|x_n\|\right) \geq \|x_n\| - \|x_{n-1}\|$$

Put, $b_n = \|x_n\| - \|x_{n-1}\|$, and we have

$$(2k+2)b_{n+1} \geq b_n. \tag{4.2}$$

Since $x_n = T(x_{n+1})$, then

$$\|x_n\| = \|Tx_{n+1}\| \leq \|x_{n+1}\| \text{ for every } n \in \mathbb{N},$$

then sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is increasing. From

$$SV^n = SV^*V^{n+1} = TSV^{n+1}$$

we have

$$\|x_n\| = \|SV^n x\| = \|TSV^{n+1}x\| \leq \|SV^{n+1}x\| \leq \|x\|,$$

for every $x \in \overline{S(H)}$ and $n \in \mathbb{N}\cup\{0\}$. Then $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded. From this we have $b_n \geq 0$ and $b_n \to 0$ as $n \to \infty$.

It remains to check that all b_n equal zero. Suppose that there exists an integer $i \geq 1$ such that $b_i > 0$. Using inequality (4.2) we get $b_{i+1} \geq \frac{b_i}{2k+2} > 0$, so $b_{i+1} > 0$. From that and using again inequality (4.2), we can show by induction that $b_n > 0$ for all $n > i$. This is contradictory with that $b_n \to 0$ as $n \to \infty$. So $b_n = 0$ for all $n \in \mathbb{N}$ and thus $\|x_{n-1}\| = \|x_n\|$ for all $n \geq 1$. Thus the sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is constant. From Lemma 4.3, T has a **Wold-type decomposition**. □

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