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Some results on a question of Li, Yi and Li

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Abstract. The purpose of this paper is to study the uniqueness problems of certain difference polynomials of meromorphic functions sharing a nonzero polynomial. The results of this paper improve and generalize some recent results due to Li, Yi and Li [11]. Some examples have been exhibited to show that some conditions used in the paper are sharp.

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1. Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

We adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$. The order of f is defined by

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log T(r,f)}{\log r}.$$

For $a \in \mathbb{C} \cup \{\infty\}$, we define

$$\Theta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)}$$

and

$$\delta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r,a;f)}{T(r,f)}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities, we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

We say that a finite value z_0 is called a fixed point of f if $f(z_0)=z_0$ or z_0 is a zero of $f(z)-z$.

Let $f(z)$ be a transcendental meromorphic function and $n\in\mathbb{N}$. Many authors have investigated the value distributions of f^nf'. At the starting point, we recall the result of Hayman (see [5], Corollary of Theorem 9). In 1959, Hayman proved the following theorem.

Theorem A. [5] *Let $f(z)$ be a transcendental meromorphic function and $n\in\mathbb{N}$ such that $n\geq 3$. Then $f^n(z)f'(z)=1$ has infinitely many solutions.*

The case $n=2$ was settled by Mues [15] in 1979. Bergweiler and Eremenko [1] showed that $f(z)f'(z)-1$ has infinitely many zeros.

For an analogue of the above results Laine and Yang [10] investigated the value distribution of difference products of entire functions in the following manner.

Theorem B. [10] *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n\geq 2$, $f^n(z)f(z+c)$ assumes every non-zero value $a\in\mathbb{C}$ infinitely often.*

The following example shows that Theorem B does not remain valid if $n=1$.

Example 1.1. [10] Let $f(z)=1+e^z$. Then $f(z)f(z+\pi i)-1=-e^{2z}$ has no zeros.

The following example shows that Theorem B does not remain valid if $f(z)$ is of infinite order.

Example 1.2. [13] Let $f(z)=e^{-e^z}$. Then $f^2(z)f(z+c)-2=-1$ and $\rho(f)=\infty$, where c is a non-zero constant satisfying $e^c=-2$. Clearly $f^2(z)f(z+c)-2$ has no zeros.

It is to be mentioned that in the meantime Chen, Huan and Zheng [2] obtained some results a part of which related to the content of the present paper.

In 2009, Liu and Yang [13] further improved Theorem B and obtained the next result.

Theorem C. [13] *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n\geq 2$, $f^n(z)f(z+c)-p(z)$ has infinitely many zeros, where $p(z)$ is a non-zero polynomial.*

The following example shows that the condition "$\rho(f)<\infty$" in Theorem C is necessary.

Example 1.3. [13] Let $f(z)=e^{-e^z}$. Then $f^n(z)f(z+c)-P(z)=1-P(z)$ and $\rho(f)=\infty$, where c is a non-zero constant satisfying $e^c=-n$, $P(z)$ is a non-constant polynomial, n is a positive integer. Clearly $f^n(z)f(z+c)-P(z)$ has finitely many zeros.

In 2010, Qi, Yang and Liu [16] studied the uniqueness of the difference monomials and obtained the following result.

Theorem D. [16] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $c \in \mathbb{C} \setminus \{0\}$; let $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share z CM, then $f(z) \equiv t_1 g(z)$ for a constant t_1 that satisfies $t_1^{n+1} = 1$.*

Theorem E. [16] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $c \in \mathbb{C} \setminus \{0\}$; let $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f(z)g(z) \equiv t_2$ or $f(z) \equiv t_3 g(z)$ for some constants t_2 and t_3 that satisfy $t_2^{n+1} = 1$ and $t_3^{n+1} = 1$.*

In 2014, Li, Yi and Li [11] improved Theorems C, D and E to meromorphic functions and obtained a number of results as follows.

Theorem F. [11] *Let $f(z)$ be a transcendental meromorphic function such that its order $\rho(f) < \infty$, let c be a non-zero complex number, and let $n \geq 6$ be an integer. Suppose that $P(z) \not\equiv 0$ is a polynomial. Then $f^n(z)f(z+c) - P(z)$ has infinitely many zeros.*

Theorem G. [11] *Let $f(z)$ be a transcendental meromorphic function such that its order $\rho(f) < \infty$ and $\delta(\infty; f(z)) > 0$, let c be a non-zero complex number, and let $n \geq 5$ be an integer. Suppose that $P(z) \not\equiv 0$ is a polynomial. Then $f^n(z)f(z+c) - P(z)$ has infinitely many zeros.*

Theorem H. [11] *Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions of finite order, let c be a non-zero complex number, let $n \geq 14$ be an integer and let $P(z) \not\equiv 0$ be a polynomial such that $2\deg(P) < n - 1$. Suppose that $f^n(z)f(z+c) - P(z)$ and $g^n(z)g(z+c) - P(z)$ share 0 CM. Then*

(I) *if $n \geq 10$ and if $f^n(z)f(z+c)/P(z)$ is a Möbius transformation of $g^n(z)g(z+c)/P(z)$, then one of the following two cases will hold:*
  (i) *$f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*
  (ii) *$f(z)g(z) = t$, where $P(z)$ reduces to a non-zero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$.*

(II) *if $n \geq 14$, then one of the two cases (I) (i) and (I) (ii) will hold.*

Theorem I. [11] *Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions of finite order, let c be a non-zero complex number, let $n \geq 12$ be an integer and let $P(z) \not\equiv 0$ be a polynomial such that $2\deg(P) < n + 1$. Suppose that $f(z)$ and $g(z)$ share ∞ IM, $f^n(z)f(z+c) - P(z)$ and $g^n(z)g(z+c) - P(z)$ share $0, \infty$ CM. Then*

(I) *if $n \geq 10$ and if $f^n(z)f(z+c)/P(z)$ is a Möbius transformation of $g^n(z)g(z+c)/P(z)$, then one of the following two cases will hold:*
  (i) *$f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*
  (ii) *$f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, where $P(z)$ reduces to a non-zero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$, $Q(z)$ is a non-constant polynomial.*

(II) *if $n \geq 12$, then one of the two cases (I) (i) and (I) (ii) will hold.*

Theorem J. [11] *Let $f(z)$ and $g(z)$ be two distinct non-constant meromorphic functions of finite order. Suppose that c is a non-zero complex number and $n \geq 17$ is an integer. If $f^n(z)f(z+c)-z$ and $g^n(z)g(z+c)-z$ share 0 CM, then $f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1}=1$.*

Theorem K. [11] *Let $f(z)$ and $g(z)$ be two distinct non-constant meromorphic functions of finite order, c be a non-zero complex number and $n \geq 13$ be an integer. Suppose that $f(z)$ and $g(z)$ share ∞ IM, $f^n(z)f(z+c)-z$ and $g^n(z)g(z+c)-z$ share $0, \infty$ CM. Then $f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1}=1$.*

At the end of [11] the following open problem was posed by the authors.

Open problem. What can be said about the conclusion of Corollary 1.1 [11] if we replace the condition "$n \geq 6$" with "$2 \leq n \leq 5$"?

One of our objective to write this paper is to solve this open problem.

Next we recall the notion of weighted sharing [9] as it will render an useful tool to relax the nature of sharing.

Definition 1.1. [9] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Next observing the above results the following questions are inevitable.

Question 1. Can the lower bound of n be further reduced in Theorem I?

Question 2. Can one replaced the condition $\delta(\infty;f) > 0$ of Theorem G by weaker one?

Question 3. Can"CM" sharing in Theorems H, I, J, K be reduced to finite weight sharing?

In this paper we want to investigate the above situations. We now present the following theorems which are the main results of the paper.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed, $n \in \mathbb{N}$ such that $n > 1$ and let $a(z) (\not\equiv 0, \infty)$ be a small function of $f(z)$. If*

$$\Theta(0;f) + \Theta(\infty;f) > \frac{5-n}{2},$$

then $f^n(z)f(z+c)-a(z)$ has infinitely many zeros.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions of finite order, c be a non-zero complex number, $n \geq 14$ be an integer and $p(z) \not\equiv 0$ be a polynomial such that $2\deg(p) < n-1$. Suppose that $f^n(z)f(z+c)-p(z)$ and $g^n(z)g(z+c)-p(z)$ share $(0,2)$. Then*

(I) *if $n \geq 10$ and if $f^n(z)f(z+c)/p(z)$ is a Möbius transformation of $g^n(z)g(z+c)/p(z)$, then one of the following two cases will hold:*

(i) *$f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*

(ii) *$f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$.*

(II) *if $n \geq 14$, then one of the two cases (I) (i) or (I) (ii) will hold.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions of finite order, c be a non-zero complex number, $n \geq 12$ be an integer and $p(z) \not\equiv 0$ be a polynomial such that $2\deg(p) < n+1$. Suppose that f and g share $(\infty, 0)$, $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 2)$ and (∞, ∞). Then*

(I) *if $n \geq 8$ and if $f^n(z)f(z+c)/p(z)$ is a Möbius transformation of $g^n(z)g(z+c)/p(z)$, then one of the following two cases will hold:*

(i) *$f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*

(ii) *$f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, where $p(z)$ reduces to a non-zero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$, $Q(z)$ is a non-constant polynomial.*

(II) *if $n \geq 12$, then one of the two cases (I) (i) or (I) (ii) will hold.*

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two distinct non-constant meromorphic functions of finite order and let $p(z)$ be a non-constant polynomial such that $\deg(p) = l$. Suppose that c is a non-zero complex number and $n \geq 14 + 3l$ is an integer.*
If $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 2)$, then $f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.

Remark 1.1. It is easy to see that the conditions $f(z)$ and $g(z)$ as well as

$$f^n(z)f(z+c) - p(z) \text{ and } g^n(z)g(z+c) - p(z)$$

have common poles in Theorem 1.3 are sharp by the following examples.

Example 1.4. Let

$$P_1(z) = \frac{1}{e^z + 1} \text{ and } Q_1(z) = \frac{1}{e^z - 1}.$$

Let c be a non-zero constant satisfying $e^c = -1$. Clearly $P_1(z)$ and $Q_1(z)$ are transcendental meromorphic functions of finite order. Let t be a nonzero constant such that $t^{n+1} = 1$ and let

$$f(z) = \frac{P_1(z)}{Q_1(z)}, \quad g(z) = t\frac{Q_1(z)}{P_1(z)}.$$

Then $f(z)$ and $g(z)$ are transcendental meromorphic functions of finite order. Note that neither $f(z)$ and $g(z)$ nor $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ have common poles. Clearly $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ share $(0, \infty)$, but neither $f(z) \equiv tg(z)$ nor $f(z) = e^{Q(z)}$ and $g(z) = t_1e^{-Q(z)}$, where t_1 is a nonzero constant and $Q(z)$ is a non-constant polynomial.

Example 1.5. Let

$$f(z) = p(z)\frac{e^z - 1}{e^z + 1} \text{ and } g(z) = p(z)\frac{e^z + 1}{e^z - 1},$$

where $p(z)$ is a non-zero polynomial.
Let c be a non-zero constant satisfying $e^c = -1$. Clearly $f(z)$ and $g(z)$ are transcendental meromorphic functions of finite order. Note that neither $f(z)$ and $g(z)$ nor $f^n(z)f(z+c) - p^n(z)p(z+c)$ and $g^n(z)g(z+c) - p^n(z)p(z+c)$ have common poles. Clearly $f^n(z)f(z+c) - p^n(z)p(z+c)$ and $g^n(z)g(z+c) - p^n(z)p(z+c)$ share $(0, \infty)$, but neither $f(z) \equiv tg(z)$ nor $f(z) = e^{Q(z)}$ and $g(z) = t_1 e^{-Q(z)}$, where t_1 is a non-zero constant and $Q(z)$ is a non-constant polynomial.

Example 1.6. Let

$$P_1(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n + 2 \ \ and \ \ Q_1(z) = \sum_{n=0}^{\infty} e^{-n^3} z^{2n} + 3.$$

Clearly $P_1(z)$ and $Q_1(z)$ are transcendental entire functions with zero order. Let t be a non-zero constant such that $t^{n+1} = 1$ and let

$$f(z) = \frac{P_1(z)}{Q_1(z)}, \quad g(z) = t\frac{Q_1(z)}{P_1(z)}.$$

Then $f(z)$ and $g(z)$ are transcendental meromorphic functions with zero order. Note that neither $f(z)$ and $g(z)$ nor $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ have common poles. Clearly $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ share $(0, \infty)$, but neither $f(z) \equiv tg(z)$ nor $f(z) = e^{Q(z)}$ and $g(z) = t_1 e^{-Q(z)}$, where t_1 is a non-zero constant and $Q(z)$ is a non-constant polynomial.

We now explain following definitions and notations which are used in the paper.

Definition 1.2. [8] Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid \leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid \leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r, a; f \mid \geq p)$ and $\overline{N}(r, a; f \mid \geq p)$.

Definition 1.3. [9] Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then $N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \ldots + \overline{N}(r, a; f \mid \geq k)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

2. Lemmas

Let F and G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [18] *Let $f(z)$ be a non-constant meromorphic function and let $a_n(z)$ $(\not\equiv 0)$, $a_{n-1}(z)$, ..., $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. [3] *Let $f(z)$ be a meromorphic function of finite order σ, and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.3. [4] *Let $f(z)$ be a meromorphic function of finite order σ, and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r)$$

and

$$\sigma(f(z+c)) = \sigma(f(z)).$$

The following lemma has little modifications of the original version (Theorem 2.1 of [3]).

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.5. [7] *Let $f(z)$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, 0; f(z+c)) \leq N(r, 0; f(z)) + S(r, f), \quad N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f),$$
$$\overline{N}(r, 0; f(z+c)) \leq \overline{N}(r, 0; f(z)) + S(r, f), \quad \overline{N}(r, \infty; f(z+c)) \leq \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.6. *Let $f(z)$ be a non-constant meromorphic function of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ be fixed and let $\Phi(z) = f^n(z)f(z+c)$, where $n \in \mathbb{N}$ such that $n > 1$. Then for each $\varepsilon > 0$, we have*

$$(n-1)\, T(r, f) \leq T(r, \Phi) + O(r^{\sigma-1+\varepsilon}) + S(r, f).$$

Proof. The proof of lemma follows from Lemmas 2.6 [14] and 2.2. □

Lemma 2.7. *Let $f(z)$ be a non-constant meromorphic function of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ be fixed and let $n \in \mathbb{N}$ with $n > 1$. Then $S(r, f^n(z)f(z+c)) = S(r, f)$.*

Proof. By Lemmas 2.1 and 2.3 we have

$$\begin{aligned} T(r, f^n(z)f(z+c)) &\leq T(r, f^n) + T(r, f(z+c)) \\ &\leq T(r, f^n) + T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r, f) \\ &\leq (n+1)\, T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r, f), \end{aligned}$$

for all $\varepsilon > 0$. This shows that $T(r, f^n(z)f(z+c)) = O(T(r, f))$.
Also by Lemma 2.6 we have $T(r, f) = O(T(r, f^n(z)f(z+c)))$. Thus we have

$$S(r, f^n(z)f(z+c)) = S(r, f).$$

This completes the proof. □

Lemma 2.8. [9] *Let f and g be two non-constant meromorphic functions sharing $(1, 2)$. Then one of the following holds:*

(i) $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r, f) + S(r, g),$

(ii) $fg \equiv 1$,
(iii) $f \equiv g$.

Lemma 2.9. [20] *Let H be defined as in (2.1). If $H \equiv 0$ and*

$$\limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)}{T(r)} < 1, \quad r \in I,$$

where I is a set of infinite linear measure, then $F \equiv G$ or $F \cdot G \equiv 1$.

Lemma 2.10. [[19], Lemma 7.1] *Let F and G be two non-constant meromorphic functions such that G is a Möbius transformation of F. Suppose that there exists a subset $I \subset \mathbb{R}^+$ with its measure $mesI = +\infty$ such that*

$$\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) < (\lambda + o(1))T(r,F),$$

as $r \in I$ and $r \to \infty$, where $\lambda < 1$, then $F \equiv G$ or $F \cdot G \equiv 1$.

Lemma 2.11. [Hadamard Factorization Theorem] *Let f be an entire function of finite order σ with zeros $a_1, a_2, \ldots$, each zeros is counted as often as its multiplicity. Then f can be expressed in the form*

$$f(z) = \beta(z)e^{\alpha(z)},$$

where $\alpha(z)$ is a polynomial of degree not exceeding $[\sigma]$ and $\beta(z)$ is the canonical product formed with the zeros of f.

Lemma 2.12. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n \geq 2$. If*

$$f^n(z)f(z+c) \equiv g^n(z)g(z+c),$$

then $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$.

Proof. Suppose

$$f^n(z)f(z+c) \equiv g^n(z)g(z+c). \tag{2.2}$$

Let $h = \frac{f}{g}$. Then from (2.2) we have

$$h^n(z) \equiv \frac{1}{h(z+c)}. \tag{2.3}$$

Now by Lemmas 2.1, 2.2 and 2.5 we get

$$\begin{aligned} nT(r,h) = T(r,h^n) + S(r,h) &= T\left(r, \frac{1}{h(z+c)}\right) + S(r,h) \\ &\leq N(r,0;h(z+c)) + m\left(r, \frac{1}{h(z+c)}\right) + S(r,h) \\ &\leq N(r,0;h(z)) + m\left(r, \frac{h(z)}{h(z+c)}\right) + m\left(r, \frac{1}{h(z)}\right) + S(r,h) \\ &\leq T(r,h) + O(r^{\sigma-1+\varepsilon}) + S(r,h), \end{aligned}$$

which is a contradiction since $n \geq 2$. Hence h must be a constant, which implies that $h^{n+1} = 1$, where $h \neq 1$, thus $f(z) = tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$. This completes the the proof. $\square$

Lemma 2.13. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order σ, $c \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}$ such that $n \geq 2$. Let $p(z)$ be a nonzero polynomial such that $2\deg(p) < n-1$. Suppose*

$$f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z).$$

Then $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$.
In particular when f and g share $(\infty, 0)$ and $2\deg(p) < n+1$, then

$$f(z) = e^{Q(z)} \text{ and } g(z) = te^{-Q(z)},$$

where $p(z)$ reduces to a nonzero constant c_1, say, and t is a constant such that

$$t^{n+1} = c_1^2,$$

$Q(z)$ is a non-constant polynomial.

Proof. Suppose

$$f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z). \tag{2.4}$$

Let $h_1 = fg$. Then from (2.4) we have

$$h_1^n(z) \equiv \frac{p^2(z)}{h_1(z+c)}. \tag{2.5}$$

First we suppose that $h_1(z)$ is a non-constant meromorphic function. We now consider following two cases.
Case 1. Let $h_1(z)$ be a transcendental meromorphic function.
Now by Lemmas 2.1, 2.2 and 2.5 we get

$$\begin{aligned} nT(r,h_1) &= T(r,h_1^n) + S(r,h_1) = T\left(r, \frac{p^2}{h_1(z+c)}\right) + S(r,h_1) \\ &\leq N(r,0;h_1(z+c)) + m\left(r, \frac{1}{h_1(z+c)}\right) + S(r,h_1) \\ &\leq N(r,0;h_1(z)) + m\left(r, \frac{1}{h_1(z)}\right) + O(r^{\sigma-1+\varepsilon}) + S(r,h_1) \\ &\leq T(r,h_1) + O(r^{\sigma-1+\varepsilon}) + S(r,h_1), \end{aligned}$$

which is a contradiction.
Case 2. Let $h_1(z)$ be a rational function.
Let

$$h_1 = \frac{h_2}{h_3}, \tag{2.6}$$

where h_2 and h_3 are two nonzero relatively prime polynomials. From (2.6) we have

$$T(r,h_1) = \max\{\deg(h_2), \deg(h_3)\}\log r + O(1). \tag{2.7}$$

Now from (2.5), (2.6) and (2.7) we have

$$\begin{aligned} & n\ \max\{\deg(h_2),\deg(h_3)\}\ \log r \\ = \ & T(r,h_1^n)+O(1) \\ \leq \ & T(r,h_1(z+c))+2\,T(r,p)+O(1) \\ = \ & \max\{\deg(h_2),\deg(h_3)\}\ \log r+2\ \deg(p)\ \log r+O(1). \end{aligned} \tag{2.8}$$

We see that

$$\max\{\deg(h_2),\deg(h_3)\}\geq 1.$$

Now from (2.8) we deduce that

$$n-1\leq 2\deg(p),$$

which contradicts our assumption that $2\deg(p)<n-1$.
Hence $h_1(z)$ is a non-zero constant. Let $h_1=t\in\mathbb{C}\setminus\{0\}$. Therefore in this case $p(z)$ reduces to a non-zero constant. Let $p(z)=c_1\in\mathbb{C}\setminus\{0\}$. So from (2.5) we see that

$$h_1^{n+1}\equiv c_1^2,\ \text{ i.e., }\ t^{n+1}\equiv c_1^2.$$

Therefore

$$f(z)g(z)\equiv t,$$

where t is a constant such that $t^{n+1}=c_1^2$.

In particular, suppose $f(z)$ and $g(z)$ share $(\infty,0)$. Now from (2.4) one can easily say that $f(z)$ and $g(z)$ are non-constant entire functions.
Let $h_1=fg$. First we suppose that h_1 is non-constant.

Now from Case 1, one can easily say that h_1 can not be a transcendental entire function. Hence h_1 is a non-constant polynomial. Since $2\deg(p)<n+1$, from (2.4), we arrive at a contradiction. Hence h_1 is a nonzero constant, say t. Therefore in this case $p(z)$ reduces to a non-zero constant. Let $p(z)=c_1\in\mathbb{C}\setminus\{0\}$.

Clearly 0 is a Picard exceptional value of both $f(z)$ and $g(z)$. Consequently both $f(z)$ and $g(z)$ are transcendental entire functions.

Now by Lemma 2.11, $f(z)$ and $g(z)$ take the forms

$$f(z)=e^{Q(z)}\text{ and }g(z)=te^{-Q(z)},$$

where t is a constant such that $t^{n+1}=c_1^2$ and $Q(z)$ is a non-constant polynomial. This completes the proof. □

Lemma 2.14. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order σ, $c\in\mathbb{C}\setminus\{0\}$ and $n\in\mathbb{N}$ such that $n\geq 2$. Let $p(z)$ be a non-constant polynomial such that $2\deg(p)<n-1$. Then*

$$f^n(z)f(z+c)g^n(z)g(z+c)\not\equiv p^2(z).$$

Proof. The proof of lemma follows from Lemma 2.13. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let $\Phi(z) = f^n(z)f(z+c)$. Now in view of Lemmas 2.1, 2.6 and the second theorem for small functions (see [17]), we get

$$\begin{aligned}
&(n-1)T(r,f)\\
&\leq T(r,\Phi) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq \overline{N}(r,0;\Phi) + \overline{N}(r,\infty;\Phi) + \overline{N}(r,a(z);\Phi) + O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq \overline{N}(r,0;f^n) + \overline{N}(r,0;f(z+c)) + \overline{N}(r,\infty;f^n) + \overline{N}(r,\infty;f(z+c)) + \overline{N}(r,a(z);\Phi)\\
&+ O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + \overline{N}(r,a(z);\Phi) + O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq \left(4 - 2\Theta(0;f) - 2\Theta(\infty;f) + \frac{2\varepsilon}{3}\right)T(r,f) + \overline{N}(r,a(z);\Phi)\\
&+ O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq (4 - 2\Theta(0;f) - 2\Theta(\infty;f) + \varepsilon)T(r,f) + \overline{N}(r,a(z);\Phi) + O(r^{\sigma-1+\varepsilon}) + o(T(r,f)),
\end{aligned}$$

for all $\varepsilon > 0$. Take $\varepsilon < 2\Theta(0;f) + 2\Theta(\infty;f)$. Since $\Theta(0;f) + \Theta(\infty;f) > \frac{5-n}{2}$, from above one can easily say that $\Phi(z) - a(z)$ has infinitely many zeros. This completes the proof. □

Proof of Theorem 1.2. Let

$$F(z) = \frac{f^n(z)f(z+c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z+c)}{p(z)}.$$

Then F and G share $(1,2)$. We now consider following two cases.

Case 1. Suppose F is a Möbius transformation of G.

By Valiron-Mokhon'ko Lemma, we see that $T(r,F) = T(r,G) + O(1)$. Clearly $S(r,F) = S(r,G)$. Now in view of Lemmas 2.5 and 2.6, we get

$$\begin{aligned}
&\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\\
&= \overline{N}(r,0;f) + \overline{N}(r,0;f(z+c)) + \overline{N}(r,0;g) + \overline{N}(r,0;g(z+c))\\
&+ \overline{N}(r,\infty;f) + \overline{N}(r,\infty;f(z+c)) + \overline{N}(r,\infty;g) + \overline{N}(r,\infty;g(z+c))\\
&+ S(r,f) + S(r,g)\\
&= 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;g) + 2\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\
&\leq 4T(r,f) + 4\,T(r,g) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g)\\
&\leq \frac{4}{n-1}T(r,F) + \frac{4}{n-1}T(r,G) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,F) + S(r,G)\\
&\leq \frac{8}{n-1}T(r,F) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,F),
\end{aligned}$$

for all $\varepsilon > 0$. Since $n \geq 10$, we must have

$$\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) < (\lambda+o(1))T(r,F),$$

where $\lambda < 1$ and so by Lemma 2.10, we have either $F \equiv G$ or $F \cdot G \equiv 1$.
We now consider following two sub-cases.
Sub-case 1.1. $F \equiv G$.
Then by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that

$$t^{n+1} = 1.$$

Sub-case 1.2. $F \cdot G \equiv 1$.
Then

$$f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$$

and so by Lemma 2.13, we have $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that

$$t^{n+1} = c_1^2.$$

Case 2. Suppose $n \geq 14$.
Now applying Lemma 2.8, we see that one of the following three sub-cases holds.
Sub-case 2.1. Suppose

$$T(r,F) \leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+S(r,F)+S(r,G).$$

Now by applying Lemmas 2.1 and 2.5, we have

$$\begin{aligned}
&T(r,F)\\
&\leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+S(r,f)+S(r,g)\\
&= N_2(r,0;f^nf(z+c))+N_2(r,0;g^ng(z+c))\\
&+ N_2(r,\infty;f^nf(z+c))+N_2(r,\infty;g^ng(z+c))+S(r,f)+S(r,g)\\
&\leq N_2(r,0;f^n)+N_2(r,0;f(z+c))+N_2(r,0;g^n)+N_2(r,0;g(z+c))+N_2(r,\infty;f^n)\\
&+ N_2(r,\infty;f(z+c))+N_2(r,\infty;g^n)+N_2(r,\infty;g(z+c))+S(r,f)+S(r,g)\\
&\leq 2N(r,0;f)+N(r,0;f(z+c))+2N(r,0;g)+N(r,0;g(z+c))+2N(r,\infty;f)\\
&+ N(r,\infty;f(z+c))+2N(r,\infty;g)+N(r,\infty;g(z+c))+S(r,f)+S(r,g)\\
&\leq 4T(r,f)+N(r,0;f)+N(r,\infty;f)+4T(r,g)\\
&+ N(r,0;g)+N(r,\infty;g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g)\\
&\leq 6T(r,f)+6T(r,g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$. From Lemma 2.6, we have

$$\begin{aligned}
(n-1)T(r,f) &\leq 6T(r,f)+6T(r,g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g)\\
&\leq 12T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r). \qquad (3.1)
\end{aligned}$$

Similarly we have

$$(n-1)\,T(r,g) \leq 12\,T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r). \qquad (3.2)$$

Combining (3.1) and (3.2), we get

$$(n-1)\,T(r) \leq 12\,T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r),$$

which contradicts with $n \geq 14$.
Sub-case 2.2. $F \equiv G$.
Then by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that

$$t^{n+1} = 1.$$

Sub-case 2.3. $F \cdot G \equiv 1$.
Then by Lemma 2.13, we have $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that $t^{n+1} = c_1^2$. This completes the proof. □

Proof of Theorem 1.3. Let

$$F(z) = \frac{f^n(z)f(z+c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z+c)}{p(z)}.$$

Also F, G share $(1,2)$ and (∞,∞) except for zeros of $p(z)$. We now consider following two cases.
Case 1. Suppose F is a Möbius transformation of G.
Let

$$F \equiv \frac{AG+B}{CG+D}, \tag{3.3}$$

where A, B, C, D are constants and $AD - BC \neq 0$. Again

$$T(r,F) = T(r,G) + O(1). \tag{3.4}$$

Clearly $S(r,F) = S(r,G)$. We now consider the following sub-cases:
Sub-case 1.1. Let $AC \neq 0$. Since F, G share (∞,∞), it follows from (3.3) that

$$N(r,\infty;F) = S(r,F) \text{ and } N(r,\infty;G) = S(r,F).$$

Again since

$$F \equiv \frac{A+\frac{B}{G}}{C+\frac{D}{G}},$$

it follows that

$$N(r,\frac{A}{C};F) = S(r,F).$$

So in view of Lemma 2.6 and using the second fundamental theorem, we get

$$\begin{aligned}
(n-1)T(r,f) &\leq T(r,f^n(z)f(z+c)) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq T(r,F) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\left(r,\frac{A}{C};F\right) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq 4T(r,f) + O(r^{\sigma-1+\varepsilon}) + S(r,f),
\end{aligned}$$

for all $\varepsilon > 0$, which is impossible since $n \geq 6$.
Sub-case 1.2. Let $A \neq 0$ and $C = 0$. Then $F \equiv \alpha G + \beta$, where

$$\alpha = \frac{A}{D} \neq 0 \text{ and } \beta = \frac{B}{D}.$$

Sub-case 1.2.1. Let $\beta = 0$. Then we get $F \equiv \alpha G$. Since $n \geq 6$, it follows that $F-1$ and $G-1$ have infinitely many zeros. Clearly 1 can not be a Picard exceptional value of F and G. Since F, G share $(1, \infty)$, it follows that $\alpha = 1$ and so $F \equiv G$, i.e.,

$$f^n(z)f(z+c) \equiv g^n(z)g(z+c).$$

Now by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$.
Sub-case 1.2.2. Let $\beta \neq 0$. Clearly $\alpha \neq 1$, as F, G share $(1, \infty)$. So in view of Lemmas 2.5 and 2.6 and using the second fundamental theorem, we get

$$\begin{aligned}
&(n-1)T(r,f)\\
&\leq T(r, f^n(z)f(z+c)) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq T(r,F) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\beta;F) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,f)\\
&\leq 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + O(r^{\sigma-1+\varepsilon}) + S(r,f) + S(r,g)\\
&\leq 4T(r,g) + 2T(r,f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$. Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.
So for $r \in I$, we have

$$(n-7)\,T(r,g) \leq O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,g),$$

for all $\varepsilon > 0$, which is a contradiction since $n \geq 8$.
Case 1.3. Let $A = 0$ and $C \neq 0$. Then $F \equiv \frac{1}{\gamma G+\delta}$, where $\gamma = \frac{C}{B} \neq 0$ and $\delta = \frac{D}{B}$.
Sub-case 1.3.1. Let $\delta = 0$. Then $F \equiv \frac{1}{\gamma G}$. Since F, G share $(1, \infty)$, it follows that $\gamma = 1$ and then $FG \equiv 1$, i.e., $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$. Now by Lemma 2.13, we have $f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that $t^{n+1} = c_1^2$ and $Q(z)$ is a non-constant polynomial.
Sub-case 1.3.2. Let $\delta \neq 0$. Clearly $\gamma \neq 1$, as F, G share $(1, \infty)$. Since F, G share (∞, ∞), it follows that $N(r,\infty;F) = S(r,F)$ and $N(r,\infty;G) = S(r,F)$. Consequently

$$N(r, -\frac{\delta}{\gamma}; G) = S(r,F).$$

So in view of Lemma 2.6 and using the second fundamental theorem, we get

$$\begin{aligned}
&(n-1)T(r,g)\\
&\leq T(r, g^n(z)g(z+c)) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq T(r,G) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}\left(r, \frac{-\delta}{\gamma}; G\right) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq 2\overline{N}(r,0;g) + 2\overline{N}(r,\infty;g) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq 4T(r,g) + O(r^{\sigma-1+\varepsilon}) + S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$, which is impossible since $n \geq 6$.

Case 2. Suppose $n \geq 12$.
We now consider following two sub-cases.

Sub-case 2.1. Let $H \not\equiv 0$.
From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.
Since H has only simple poles we get

$$\begin{aligned} & N(r,\infty;H) \\ \leq\ & \overline{N}_*(r,\infty;F,G)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2) \\ & +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g), \end{aligned} \tag{3.5}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.
Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{3.6}$$

Note that

$$\overline{N}_*(r,\infty;F,G)=S(r,f).$$

Now using (3.5) and (3.6) we get

$$\begin{aligned} & \overline{N}(r,1;F) \\ \leq\ & N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2) \\ \leq\ & \overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G\ |\geq 2) \\ & +\overline{N}(r,1;F\ |\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g). \end{aligned} \tag{3.7}$$

Now in view of Lemma 2.3 we get

$$\begin{aligned} & \overline{N}_0(r,0;G')+\overline{N}(r,1;F\ |\geq 2)+\overline{N}_*(r,1;F,G) \\ \leq\ & \overline{N}_0(r,0;G')+\overline{N}(r,1;F\ |\geq 2)+\overline{N}(r,1;F\ |\geq 3) \\ =\ & \overline{N}_0(r,0;G')+\overline{N}(r,1;G\ |\geq 2)+\overline{N}(r,1;G\ |\geq 3) \\ \leq\ & \overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G) \\ \leq\ & N(r,0;G'\mid G\neq 0)\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,g). \end{aligned} \tag{3.8}$$

Hence using (3.7), (3.8) and Lemma 2.5, we get from the second fundamental theorem that

$$\begin{aligned}
&T(r,F)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F')+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,1;F,G)\\
&+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\
&\leq N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\
&= N_2(r,0;f^nf(z+c))+N_2(r,0;g^ng(z+c))\\
&+\overline{N}(r,\infty;f^nf(z+c))+\overline{N}(r,\infty;g^ng(z+c))+S(r,f)+S(r,g)\\
&\leq N_2(r,0;f^n)+N_2(r,0;f(z+c))+N_2(r,0;g^n)+N_2(r,0;g(z+c))+\overline{N}(r,\infty;f^n)\\
&+\overline{N}(r,\infty;f(z+c))+\overline{N}(r,\infty;g^n)+\overline{N}(r,\infty;g(z+c))+S(r,f)+S(r,g)\\
&\leq 2\ N(r,0;f)+N(r,0;f(z+c))+2\ N(r,0;g)+N(r,0;g(z+c))+N(r,\infty;f)\\
&+N(r,\infty;f(z+c))+N(r,\infty;g)+N(r,\infty;g(z+c))+S(r,f)+S(r,g)\\
&\leq 3T(r,f)+N(r,0;f)+N(r,\infty;f)+3T(r,g)\\
&+N(r,0;g)+N(r,\infty;g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g)\\
&\leq 5T(r,f)+5T(r,g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g),
\end{aligned}$$

for all $\varepsilon>0$. From Lemma 2.6, we have

$$\begin{aligned}
(n-1)T(r,f) &\leq 5T(r,f)+5T(r,g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g)\\
&\leq 10T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r). \qquad (3.9)
\end{aligned}$$

Similarly we have

$$(n-1)\ T(r,g)\leq 10\ T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r). \qquad (3.10)$$

Combining (3.9) and (3.10), we get

$$(n-1)\ T(r)\leq 10\ T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r),$$

which contradicts with $n\geq 12$.

Case 2.2. Let $H\equiv 0$.

Here in view of Lemmas 2.5, 2.6 and proceeding in the same way as done in Theorem 1.2, we get

$$\begin{aligned}
&\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
\leq\ &\frac{8}{n-1}\ T(r,F)+O(r^{\sigma-1+\varepsilon})+S(r,F),
\end{aligned}$$

for all $\varepsilon>0$. Since $n\geq 10$, we must have

$$\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)<(\lambda+o(1))T(r,F),$$

where $\lambda<1$ and so by Lemma 2.9, we have either $F\equiv G$ or $F\cdot G\equiv 1$.

We now consider following two sub-cases.

Sub-case 2.2.1. $F \equiv G$.
Then by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$.
Sub-case 2.2.2. $F \cdot G \equiv 1$.
Then $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$ and so by Lemma 2.13, we have $f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that $t^{n+1} = c_1^2$, $Q(z)$ is a non-constant polynomial. This completes the proof. $\square$

Proof of Theorem 1.4. Let

$$F(z) = \frac{f^n(z)f(z+c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z+c)}{p(z)}.$$

Then F and G share $(1,2)$ except for zeros of $p(z)$. Note that

$$T(r, f^n(z)f(z+c)) \leq T(r,F) + l\log r \text{ and } T(r, g^n(z)g(z+c)) \leq T(r,G) + l\log r.$$

Also we see that $T(r,f) \geq \log r + O(1)$ and $T(r,g) \geq \log r + O(1)$.
Now applying Lemma 2.8, we see that one of the following three cases holds.
Case 1. Suppose

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G).$$

Now by applying Lemmas 2.1 and 2.5, we have

$$\begin{aligned}
&T(r,F)\\
&\leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,f) + S(r,g)\\
&= N_2(r,0;f^nf(z+c)) + N_2(r,0;g^ng(z+c))\\
&+ N_2(r,\infty;f^nf(z+c)) + N_2(r,\infty;g^ng(z+c)) + 2l\log r + S(r,f) + S(r,g)\\
&\leq N_2(r,0;f^n) + N_2(r,0;f(z+c)) + N_2(r,0;g^n) + N_2(r,0;g(z+c)) + N_2(r,\infty;f^n)\\
&+ N_2(r,\infty;f(z+c)) + N_2(r,\infty;g^n) + N_2(r,\infty;g(z+c)) + 2l\log r + S(r,f) + S(r,g)\\
&\leq 2N(r,0;f) + N(r,0;f(z+c)) + 2N(r,0;g) + N(r,0;g(z+c)) + 2N(r,\infty;f)\\
&+ N(r,\infty;f(z+c)) + 2N(r,\infty;g) + N(r,\infty;g(z+c)) + 2l\log r + S(r,f) + S(r,g)\\
&\leq 4T(r,f) + N(r,0;f) + N(r,\infty;f) + 4T(r,g)\\
&+ N(r,0;g) + N(r,\infty;g) + 2l\log r + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g)\\
&\leq 6T(r,f) + 6T(r,g) + 2l\log r + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$. From Lemma 2.6, we have

$$\begin{aligned}
&(n-1)T(r,f) \qquad (3.11)\\
&\leq T(r, f^n(z)f(z+c)) + O(r^{\sigma-1+\varepsilon})\\
&\leq T(r,F) + l\log r + O(r^{\sigma-1+\varepsilon})\\
&\leq 6T(r,f) + 6T(r,g) + 3l\log r + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g)\\
&\leq (12+3l)T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r).
\end{aligned}$$

Similarly we have

$$(n-1)\, T(r,g) \leq (12+3l)\, T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r). \qquad (3.12)$$

Combining (3.11) and (3.12), we get

$$(n-1)\,T(r) \leq (12+3l)\,T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r),$$

which contradicts with $n \geq 14+3l$.

Sub-case 2.2. $F \equiv G$.

Then by Lemma 2.12 we have $f(z) \equiv tg(z)$ for some constant $t \neq$ such that $t^{n+1} = 1$.

Sub-case 2.3. $F \cdot G \equiv 1$.

Then we have $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$. But this is impossible by Lemma 2.14. This completes the proof. □

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