Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 539–548
DOI: 10.24193/subbmath.2018.4.09

Partial averaging of discrete-time set-valued systems

Tatyana A. Komleva, Liliya I. Plotnikova and Andrej V. Plotnikov

Abstract. In the introduction of the article we given an overview of the results for set-valued equations. Further we considered the set-valued discrete-time dynamical systems and substantiates the averaging method for nonlinear set-valued discrete-time systems with a small parameter.

Mathematics Subject Classification (2010): 49M25, 34C29, 49J53.

Keywords: Averaging method, discrete-time system, set-valued mapping, Hukuhara difference.

1. Introduction

As it is well known, there are two main types of dynamical systems: differential equations and discrete-time equations. Differential equation describes the continuous time evaluation of the system, whereas discrete-time equation describes the discrete time evaluation of the system. The theory of discrete dynamical systems and difference equations developed greatly during the last decades (see [8, 18, 34] and references cited there).

In 1969, F.S. de Blasi and F. Iervolino [5] begun studying of set-valued differential equations in semilinear metric spaces. Later, the development of calculus in metric spaces became an object of attention of many researchers (see [7, 19, 20, 22, 30, 31, 27, 32, 40] and the references therein) and transformed into the theory of set-valued equations as an independent discipline. Set-valued equations are useful in other areas of mathematics. For example, set-valued differential equations are used as an auxiliary tool to prove the existence results for differential inclusions [19, 22, 27, 40]. Also, one can employ set-valued differential equations in the investigation of fuzzy differential equations [20, 30]. Moreover, set-valued differential equations are a natural generalization of usual ordinary differential equations in finite (or infinite) dimensional Banach spaces [40]. Clearly, in many cases, when modeling real-world phenomena, information about the behavior of a dynamical system is uncertain and one has to consider these uncertainties to gain better understanding of the full models. The set-valued equations can be used to model dynamical systems subjected to uncertainties.

This article deals with discrete set-valued dynamical systems, where time is measured by the number of iterations carried out, the dynamics are not continuous and values at each iteration is a set. In applications this would imply that the solutions are observed at discrete time intervals and also under uncertainty or interference effects [9, 13, 24, 35, 36, 38, 41]. Recurrence relations can be used to construct mathematical models of discrete systems under uncertainty. They are also used extensively to solve many differential equations with set-valued right-hand side which do not have an analytic solution; the set-valued differential equations are represented by recurrence relations (or difference equations) that can be solved numerically on a computer [1, 4, 24, 41].

Averaging theory for ordinary differential equations has a rich history, dating to back to the work of N.M. Krylov and N.N. Bogoliubov [17]. Also is well known, the averaging methods combined with the asymptotic representations began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations [3, 27, 37] and the references therein. The possibility of using some averaging schemes for set-valued equations was studied in [11, 12, 14, 15, 16, 22, 23, 25, 30, 26, 29, 27, 39]. Throughout the years, many authors have published papers on averaging methods for different kinds of differential systems and discrete-time system [2, 21, 28]. The bulk of this article is concerned with the averaging method for nonlinear discrete-time set-valued systems.

2. Preliminaries

Let $conv(R^n)$ be a space of all nonempty convex compact subsets of R^n with the Hausdorff metric

$$h(A, B) = \min_{r \geq 0} \{B \subset S_r(A), \;\; A \subset S_r(B)\}$$

where $A, B \in conv(R^n)$, $S_r(A)$ be a r-neighborhood of the set A.

The usual set operations, i.e., well-known as Minkowski addition and scalar multiplication, are defined as follows

$$A + B = \{a + b \, : \, a \in A, \, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a \, : \, a \in A, \, \lambda \in R\}.$$

Lemma 2.1. [32] *The following properties hold:*

1. $(conv(R^n), h)$ *is a complete metric space,*
2. $h(A + C, B + C) = h(A, B)$,
3. $h(\lambda A, \lambda B) = |\lambda| h(A, B)$ *for all* $A, B, C \in conv(R^n)$ *and* $\lambda \in R$.

For any $A \in conv(R^n)$, it can be seen $A + (-1)A \neq \{0\}$ in general, thus the opposite of A is not the inverse of A with respect to the Minkowski addition unless $A = \{a\}$ is a singleton. To partially overcome this situation, the Hukuhara difference has been introduced [10].

Definition 2.2. [10] Let $X, Y \in conv(R^n)$. A set $Z \in conv(R^n)$ such that $X = Y + Z$ is called a Hukuhara difference of the sets X and Y and is denoted by $X \overset{h}{-} Y$.

An important property of Hukuhara difference is that $A \overset{h}{-} A = \{0\}, \forall A \in conv(R^n)$ and $(A+B) \overset{h}{-} B = A, \forall A, B \in conv(R^n)$; Hukuhara difference is unique, but a necessary condition for $A \overset{h}{-} B$ to exist is that A contains a translate $\{c\}+B$ of B.

Now consider the non-autonomous set-valued discrete-time equations

$$X_{i+1} = X_i + F(i, X_i), \tag{2.1}$$

and

$$X_{i+1} = X_i \overset{h}{-} F(i, X_i), \tag{2.2}$$

where $i \in I = \{0, 1, ..., N\}$, $X_i \in conv(R^n)$, $F : I \times conv(R^n) \to conv(R^n)$. If one starts with an initial value, say, X_0, then iteration of (2.1) (or (2.2)) leads to a sequence of the form

$$\{X_i : i = 0 \,\text{to}\, N\} = \{X_0, X_1, ..., X_N\}.$$

Definition 2.3. A solution to the set-valued discrete-time equation (2.1) (or (2.2)) is a discrete-time set-valued trajectory, $\{X_i\}_{i=0}^N$, that satisfies this equation at any point $i \in I$.

Remark 2.4. It is obvious that the solution of (2.1) exists for any $X_0 \in conv(R^n)$ and I.

Remark 2.5. Obviously, the differences in (2.2) may not always exist. For example,

1) let $n \geq 1$, $X_0 = \{a \in R^n : \|a\| \leq 1\}$, $F(i, X_i) = (i+2)\, X_i$, i.e. $F(0, X_0) = \{b \in R^n : \|b\| \leq 2\}$. In this case, the difference in (2.2) does not exist for $i = 0$;
2) let $n = 2$, $X_0 = \{a \in R^2 : |a_k| \leq 1,\ k = 1, 2\}$,

$$K(i) = \begin{pmatrix} cos(i+1) & sin(i+1) \\ -sin(i+1) & cos(i+1) \end{pmatrix},$$

$F(i, X_i) = K(i)\, X_i$. Also, the difference in (2.2) does not exist for $i = 0$.

Let $CC(R^n)$ $(n \geq 2)$ be a space of all nonempty strictly convex closed sets of R^n and all elements of R^n [33].

Remark 2.6. If $A, B \in CC(R^n)$ and $A + C = B$ then $C \in CC(R^n)$ [33].

Remark 2.7. If $A, B \in CC(R^n)$ and there exists $c \in R^n$ such that $A + c \subset B$, then there exists $C \in CC(R^n)$ such that $A + C = B$, i.e. $C = B \overset{h}{-} A$ [33].

Then the following theorem holds.

Theorem 2.8. *Let the following conditions hold:*

1) $F(i, X) \in CC(R^n)$ for any $i \in I$ and $X \in CC(R^n)$;

2) the following inequality

$$|C(X, \psi) + C(X, -\psi)| \geq |C(F(i, X), \psi) + C(F(i, X), -\psi)|$$

holds for all $\psi \in R^n$ $(\|\psi\| = 1)$, $i \in I$ and $X \in CC(R^n)$, where

$$C(A, \psi) = \max_{a \in A}(a_1\psi_1 + ... + a_n\psi_n),\ A \in CC(R^n).$$

Then the solution of (2.2) exists for any $X_0 \in CC(R^n)$ and I.

Proof. We put any set $X_0 \in CC(R^n)$. By condition 1) of the theorem, we have $F(0,X_0) \in CC(R^n)$. By condition 2) of the theorem, we obtain

$$|C(X_0,\psi)+C(X_0,-\psi)| \geq |C(F(0,X_0),\psi)+C(F(0,X_0),-\psi)|$$

for all $\psi \in R^n$, $\|\psi\| = 1$. Then, there exists $c \in R^n$ such that $F(0,X_0)+c \subset X_0$ [33]. By remark 2.7, we have the set $C \in CC(R^n)$ such that $F(0,X_0)+C = X_0$. Therefore, $X_1 = C = X_0 \overset{h}{-} F(0,X_0)$ and $X_1 \in CC(R^n)$. Further, applying the method of mathematical induction, we obtain $X_{i+1} = X_i \overset{h}{-} F(i,X_i)$ and $X_{i+1} \in CC(R^n)$ for all $i \in I$. The theorem is proved. □

3. The method of averaging

Now consider the non-autonomous set-valued discrete-time equations with a small parameter

$$X_{i+1} = X_i + \varepsilon F(i,X_i), \tag{3.1}$$

and

$$X_{i+1} = X_i \overset{h}{-} \varepsilon F(i,X_i), \tag{3.2}$$

where $\varepsilon > 0$ be a small parameter, $L > 0$ is any real number, $N = [L\varepsilon^{-1}]$, $[\cdot]$ is floor function.

3.1. Case (3.1).

In the beginning we consider the equation (3.1). We associate with the equation (3.1) the following averaged set-valued discrete-time equation with a small parameter

$$X_{i+1} = X_i + \varepsilon \overline{F}(i,X_i), \tag{3.3}$$

where $\overline{F}(i,X)$ such that

$$\lim_{n\to\infty} h\left(\frac{1}{n}\sum_{i=0}^{n-1} F(i,X), \quad \frac{1}{n}\sum_{i=0}^{n-1} \overline{F}(i,X)\right) = 0. \tag{3.4}$$

The main theorem of this subsection is on averaging for set-valued discrete-time equation with a small parameter. It establishes nearness of solutions of (3.1) and (3.3), and reads as follows.

Theorem 3.1. *Let in the domain* $Q = \{\,(i,X) \,:\, i \in I, X \subset B \subset R^n\,\}$ *the following conditions hold:*

1) mappings $F(i,X)$ *and* $\overline{F}(i,X)$ *satisfy a Lipschitz condition, i.e. there is a constant* $\lambda > 0$ *such that*

$$h(F(i,X'),F(i,X")) \leq \lambda h(X',X"), \quad h(\overline{F}(i,X'),\overline{F}(i,X")) \leq \lambda h(X',X"),$$

whenever $(i,X'),(i,X") \in Q$;

3) there exists $\gamma > 0$ *such that* $h(F(i,X),\{0\}) \leq \gamma$, $h(\overline{F}(i,X),\{0\}) \leq \gamma$ *for every* $(i,X) \in Q$*;*

4) limit (3.4) *exists uniformly with respect to* X *in the domain* B;

5) the solution of the problem (3.3) *together with a* $\rho-$*neighborhood belong to the domain* B *for* $\varepsilon \in (0,\bar{\varepsilon}]$.

Then for any $\eta \in (0, \rho]$ *and* $L > 0$ *there exists* $\varepsilon_0(\eta, L) \in (0, \bar{\varepsilon}]$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $i \in I$ *the following inequality holds*

$$h(X_i, \overline{X}_i) < \eta \tag{3.5}$$

where $\{X_i\}_{i=0}^N$, $\{\overline{X}_i\}_{i=0}^N$ *are the solutions of initial and averaged problems.*

Proof. We write the equations (3.1) and (3.3) in the form

$$X_{i+1} = X_0 + \varepsilon \sum_{j=0}^{i} F(j, X_j), \tag{3.6}$$

$$\overline{X}_{i+1} = X_0 + \varepsilon \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j). \tag{3.7}$$

By (3.6) and (3.7), we have

$$h(X_{i+1},\ \overline{X}_{i+1}) = h\left(\varepsilon \sum_{j=0}^{i} F(j, X_j), \varepsilon \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\leq \varepsilon \sum_{j=0}^{i} h(F(j, X_j), F(j, \overline{X}_j)) + \varepsilon h\left(\sum_{j=0}^{i} F(j, \overline{X}_j), \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\leq \lambda\varepsilon \sum_{j=0}^{i} h(X_j, \overline{X}_j) + \phi, \tag{3.8}$$

where

$$\phi = \varepsilon h\left(\sum_{j=0}^{i} F(j, \overline{X}_j), \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right).$$

Now we will estimate ϕ on I. Divide the interval I into partial intervals by the points $t_k = kl(\varepsilon),\ k = \overline{0, m}$, $t_{m-1} < L\varepsilon^{-1} \leq t_m$, where $l(\varepsilon)$ is integer and

$$\lim_{\varepsilon \to 0} l(\varepsilon) = \infty,\ \lim_{\varepsilon \to 0} \varepsilon l(\varepsilon) = 0. \tag{3.9}$$

Let $kl(\varepsilon) < i \leq (k+1)l(\varepsilon)$. Then we have

$$\phi = \varepsilon h\left(\sum_{j=0}^{i} F(i, \overline{X}_i), \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\leq \varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} F(j, \overline{X}_j), \sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j, \overline{X}_j)\right)$$

$$+\varepsilon h\left(\sum_{j=kl(\varepsilon)}^{i} F(j, \overline{X}_j), \quad \sum_{j=kl(\varepsilon)}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\leq \varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_j), \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)})\right)$$

$$+\varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)}), \sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)})\right)$$

$$+\varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=h\zeta+1}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)}), \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_j)\right)$$

$$+\varepsilon \sum_{j=kl(\varepsilon)}^{i} h(F(j,\overline{X}_j),\ \ \overline{F}(j,\overline{X}_j)). \tag{3.10}$$

Now we will estimate terms in (3.10)

$$\varepsilon h\left(\sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_j), \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)})\right)$$

$$\leq \varepsilon \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} h(F(j,\overline{X}_j),\ \ F(j,\overline{X}_{\zeta l(\varepsilon)})) \leq \lambda \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} h(\overline{X}_j,\overline{X}_{\zeta l(\varepsilon)})$$

$$\leq \varepsilon^2\,\lambda \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} \sum_{r=k\zeta}^{j-1} \left\|\overline{F}(\overline{X}_j)\right\| \ \leq\ \varepsilon^2\lambda\gamma l(\varepsilon)^2/2. \tag{3.11}$$

Also, we obtain

$$\varepsilon h\left(\sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_j), \sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)})\ \right) \leq\ \varepsilon^2\lambda\gamma l(\varepsilon)^2/2. \tag{3.12}$$

Obviously,

$$\varepsilon \sum_{j=kl(\varepsilon)}^{i} \delta(F(j,\overline{X}_{kl(\varepsilon)}),\overline{F}(j,\overline{X}_{kl(\varepsilon)}))\ \leq 2\varepsilon\gamma l(\varepsilon). \tag{3.13}$$

From the condition 4) of the theorem there exists an increasing function $\mu(l)$, such that

1) $\lim\limits_{t\to\infty} \mu(t) = 0$;

2) $\varepsilon \sum\limits_{\zeta=0}^{k-1} h\left(\sum\limits_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)}), \sum\limits_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)})\right)$

$$\leq m\varepsilon l(\varepsilon)\mu(l(\varepsilon)) \leq\ \ L\,\mu(l(\varepsilon)). \tag{3.14}$$

Combining (3.10) – (3.14), we obtain

$$\phi \leq \varepsilon l(\varepsilon)\gamma(\lambda L + 2) + L\mu(l(\varepsilon)). \tag{3.15}$$

By (3.9), we take $\varepsilon^0 \in (0,\rho]$ such that

$$e^{\lambda L}[\varepsilon l(\varepsilon)\gamma(\lambda L + 2) + L\phi(l(\varepsilon))] < \eta \tag{3.16}$$

for all $\varepsilon \in (0, \varepsilon^0]$.
From (3.8), (3.15), (3.16) we obtain (3.5). The theorem is proved. □

Remark 3.2. If $F(i, X_i) = \Delta \cdot G(t_0 + i\Delta, X_i)$, $G : R \times conv(R^n) \to conv(R^n)$, $X_i = X(t_0 + i\Delta)$, discrete-time equation (2.1) is a Euler polygonal curve for the differential equation with Hukuhara derivative [6]

$$D_h X(t) = G(t, X(t)), \quad X(t_0) = X_0,$$

where $X : R \to conv(R^n)$ is set-valued mapping, $D_h X(t)$ is Hukuhara derivative [6, 10]. Thus, Theorem 3.1 is a discrete analogue of the first Bogolyubov theorem for a differential equation with derivative Hukuhara [15, 16, 25, 30, 27].

3.2. Case (3.2)**.**

We associate with the equation (3.2) the following averaged set-valued discrete-time equation with a small parameter

$$X_{i+1} = X_i \overset{h}{-} \varepsilon \overline{F}(i, X_i), \tag{3.17}$$

where $\overline{F}(i, X)$ such that limit (3.4) exists.

Theorem 3.3. *Let in the domain $Q = \{\, (i, X) \,:\, i \in I, X \in CC(R^n),\, X \subset B \subset R^n \,\}$ the following conditions hold:*

1) mappings $F(i, X), \overline{F}(i, X) \in CC(R^n)$ for any $(i, X) \in Q$;

2) the inequality

$$|C(X, \psi) + C(X, -\psi)| \geq |C(\varepsilon F(i, X), \psi) + C(\varepsilon F(i, X), -\psi)|,$$

$$|C(X, \psi) + C(X, -\psi)| \geq |C(\varepsilon \overline{F}(i, X), \psi) + C(\varepsilon \overline{F}(i, X), -\psi)|$$

are true for all $\psi \in R^n$ ($\|\psi\| = 1$), $\varepsilon \in (0, \bar{\varepsilon}]$, $i \in I$ and $X \subset B$;

3) mappings $F(i, X)$ and $\overline{F}(i, X)$ satisfy a Lipschitz condition

$$h(F(i, X'), F(i, X")) \leq \lambda h(X', X"), \quad h(\overline{F}(i, X'), \overline{F}(i, X")) \leq \lambda h(X', X"),$$

with a Lipschitz constant $\lambda > 0$;

4) there exists $\gamma > 0$ such that $h(F(i, X), \{0\}) \leq \gamma$, $h(\overline{F}(i, X), \{0\}) \leq \gamma$ for every $(i, X) \in Q$;

*5) limit (*3.4*) exists uniformly with respect to X in the domain B;*

*6) the solution of the problem (*3.17*) together with a $\rho-$neighborhood belong to the domain B for $\varepsilon \in (0, \bar{\varepsilon}]$.*

*Then for any $\eta \in (0, \rho]$ and $L >$ 0 there exists $\varepsilon_0(\eta, L) \in (0, \bar{\varepsilon}]$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and $i \in I$ the inequality (*3.5*) holds.*

Proof. We write the equations (3.2) and (3.17) in the form

$$X_{i+1} = X_0 \overset{h}{-} \varepsilon \sum_{j=0}^{i} F(j, X_j), \quad \text{and} \quad \overline{X}_{i+1} = X_0 \overset{h}{-} \varepsilon \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j). \tag{3.18}$$

By (3.18), we have

$$h(X_{i+1},\ \overline{X}_{i+1}) = h\left(\varepsilon\sum_{j=0}^{i} F(j, X_j), \varepsilon\sum_{j=0}^{i}\overline{F}(j, \overline{X}_j)\right).$$

Further, Theorem 3.3 is proved similarly to Theorem 3.1. This concludes the proof. □

Remark 3.4. If $\overline{F}(i, X) \equiv \overline{F}(X)$, i.e.

$$\lim_{n\to\infty} h\left(\frac{1}{n}\sum_{i=0}^{n-1} F(i, X), \frac{1}{n}\sum_{i=0}^{n-1}\overline{F}(X)\right) = 0,$$

then the validity of the full averaging scheme for (3.1) and (3.2) follows from the theorems 3.1 and 3.3.

References

[1] Baier, R., Donchev, T., *Discrete approximation of impulsive differential inclusions*, Numer. Funct. Anal. Optim., **31**(2010), no. 6, 653–678.

[2] Belan, E.P., *Averaging in the theory of finite-difference equations*, Ukr. Math. J., **19**(1967), no. 3, 319–323.

[3] Burd, V., *Method of Averaging for Differential Equations on an Infinite Interval. Theory and Applications*, Lect. Pure Appl. Math., **255**, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[4] Chahma, I.A., *Set-valued discrete approximation of state-constrained differential inclusions*, Bayreuth. Math. Schr., **67**(2003), 3–162.

[5] de Blasi, F.S., Iervolino, F., *Equazioni differentiali con soluzioni a valore compatto convesso*, Boll. Unione Mat. Ital., **2**(4–5)(1969), 491–501.

[6] de Blasi, F., Iervolino, F., *Euler method for differential equations with set-valued solutions*, Boll. Unione Mat. Ital., **4**(1971), no. 4, 941–949.

[7] Deimling, K., *Multivalued Differential Equations*, De Gruyter series in nonlinear analysis and applications, **1**, Walter de Gruyter, 1992.

[8] Elaydi, S.N., *Discrete Chaos: With Applications in Science and Engineering, Second Edition*, Chapman and Hall/CRC, Boca Raton, FL, 2008.

[9] Gu, R.B., Guo, W.J., *On mixing properties in set valued discrete system*, Chaos Solitons Fractals, **28**(2006), no. 3, 747–754.

[10] Hukuhara, M., *Integration des applications mesurables dont la valeur est un compact convexe*, Funkc. Ekvacioj, Ser. Int., **10**(1967), 205–223.

[11] Janiak, T., Łuczak-Kumorek, E., *Method on partial averaging for functional-differential equations with Hukuhara's derivative*, Stud. Univ. Babeş-Bolyai, Math., **48**(2003), no. 2, 65–72.

[12] Janiak, T., Łuczak-Kumorek, E., *Bogolubov's type theorem for functional-differential inclusions with Hukuhara's derivative*, Stud. Univ. Babeş-Bolyai, Math., **36**(1991), no. 1, 41–55.

[13] Khan, A., Kumar, P., *Chaotic properties on time varying map and its set valued extension*, Adv. Pure Math., **3**(2013), 359–364.

[14] Kichmarenko, O.D., *Averaging of differential equations with Hukuhara derivative with maxima*, Int. J. Pure Appl. Math., **57**(2009), no. 3, 447–457.

[15] Kisielewicz, M., *Method of averaging for differential equations with compact convex valued solutions*, Rend. Mat., VI, Ser., **9**(1976), no. 3, 397–408.

[16] Komleva, T.A., Plotnikova L.I., Plotnikov A.V., Skripnik, N.V., *Averaging in fuzzy controlled systems*, Nonlinear Oscil., **14**(2012), no. 3, 342–349.

[17] Krylov, N.M., Bogoliubov, N.N., *Introduction to Nonlinear Mechanics*, Princeton University Press, Princeton, 1947.

[18] Kulenovic, M.R.s., Merino, O., *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall / CRC, 2002.

[19] Lakshmikantham, V., Granna Bhaskar, T., Vasundhara Devi, J., *Theory of Set Differential Equations in Metric Spaces*, Cambridge Scientific Publishers, Cambridge, 2006.

[20] Lakshmikantham, V., Mohapatra, R.N., *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis, London, 2003.

[21] Martynyuk, D.I., Danilov, Ya., Pan'kov, G., *Second Bogolyubov theorem for systems of difference equations*, Ukr. Math. J., **48**(1996), no. 4, 516–529.

[22] Perestyuk, N.A., Plotnikov, V.A., Samoilenko, A.M., Skripnik, N.V., *Differential equations with impulse effects: multivalued right-hand sides with discontinuities*, de Gruyter Stud. Math. **40**, Berlin/Boston, Walter De Gruyter GmbH & Co, 2011.

[23] Perestyuk, N.A., Skripnik, N.V., *Averaging of set-valued impulsive systems*, Ukr. Math. J., **65**(2013), no. 1, 140–157.

[24] Petersen, I.R., Savkin, A.V., *Discrete-Time Set-Valued State Estimation*, In: Robust Kalman Filtering for Signals and Systems with Large Uncertainties. Control Engineering. Birkhauser, Boston, MA, 1999.

[25] Plotnikov, A.V., *Averaging differential embeddings with Hukuhara derivative*, Ukr. Math. J., **41**(1989), no. 1, 112–115.

[26] Plotnikov, V.A., Kichmarenko, O.D., *Averaging of controlled equations with the Hukuhara derivative*, Nonlinear Oscil., **9**(2006), no. 3, 365–374.

[27] Plotnikov, V.A., Plotnikov, A.V., Vityuk, A.N., *Differential equations with a multivalued right-hand side. Asymptotic methods*, AstroPrint, Odessa, 1999.

[28] Plotnikov, V.A., Plotnikova, L.I., Yarovoi, A.T., *Averaging method for discrete systems and its application to control problems*, Nonlinear Oscil., **7**(2004), no. 2, 240–253.

[29] Plotnikov, V.A., Rashkov, P.I., *Averaging in differential equations with hukuhara derivative and delay*, Funct. Differ. Equ., **8**(2001), no. 3-4, 371–381.

[30] Plotnikov, A.V., Skripnik, N.V. *Differential equations with "clear" and fuzzy multivalued right-hand side. Asymptotics methods*, AstroPrint, Odessa, 2009.

[31] Plotnikov, A.V., Skripnik, N.V., *An existence and uniqueness theorem to the cauchy problem for generalised set differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **20**(2013), no. 4, 433–445.

[32] Polovinkin, E.S., *Multivalued analysis and differential inclusions*, FIZMATLIT, Moscow, 2014.

[33] Polovinkin, E.S., *Strongly convex analysis*, Sb. Math., **187**(1996), 259–286.

[34] Robinson, R.C., *An Introduction to Dynamical Systems: Continuous and Discrete,* Pearson Education, Inc., 2004.

[35] Roman-Flores, H., *A Note on Transitivity in Set-Valued Discrete Systems*, Chaos Solution Fractals, **17**(2003), no. 1, 99–104.

[36] Roman-Flores, H., Chalco-Cano, Y., *Robinsons Chaos in Set-Valued Discrete Systems*, Chaos Solitons Fractals, **25**(2005), no. 1, 33–42.

[37] Sanders, J.A., Verhulst, F., *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sciences, **59**, Springer-Verlag, New York, 1985.

[38] Shi, Y.M., Chen, G.R., *Chaos of time-varying discrete dynamical systems*, J. Difference Equ. Appl., **15**(2009), no. 5, 429–449.

[39] Skripnik, N.V., *Averaging of impulsive differential inclusions with Hukuhara derivative*, Nonlinear Oscil., **10**(2007), no. 3, 422–438.

[40] Tolstonogov, A., *Differential Inclusions in a Banach Space*, Kluwer Academic Publishers, Dordrecht, 2000.

[41] Ungureanu, V., Lozan, V., *Linear discrete-time set-valued Pareto-Nash-Stackelberg control processes and their principles*, ROMAI J., **9**(2013), no. 1, 185–198.

Tatyana A. Komleva
Department of Mathematics
Odessa State Academy Civil Engineering and Architecture
4, Didrihsona Street, 65029 Odessa, Ukraine
e-mail: `t-komleva@ukr.net`

Liliya I. Plotnikova
Department of Mathematics
Odessa National Polytechnic University
1, Shevchenko Avenue, 65044 Odessa, Ukraine
e-mail: `liplotnikova@ukr.net`

Andrej V. Plotnikov
Department of Information Technology and Applied Mathematics
Odessa State Academy Civil Engineering and Architecture
4, Didrihsona Street, 65029 Odessa, Ukraine
e-mail: `a-plotnikov@ukr.net`