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Subclasses of analytic functions of complex order defined by $q-$derivative operator

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Abstract. Using the q-derivative operator in conjunction with the principle of subordination between analytic functions, we introduce two subclasses of analytic functions in the open unit disk $\mathbb{U}$. We investigate convolution properties and coefficient estimates for these subclasses.

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1. Introduction

Recently, the theory of q-analysis has attracted a considerable effort of researchers due to its application in many branches of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference, q-integral equations and in q-transform analysis (see for instance [1, 9, 11, 19]). The main purpose of this paper is to introduce and study two subclasses of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

by applying the q-derivative operator in conjunction with the principle of subordination between analytic functions.

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in $\mathbb{U}$. Also $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{S}$ consisting of starlike and convex functions of order α $(0 \leqq \alpha < 1)$. We note that

$$\mathcal{S}^*(0) = \mathcal{S}^* \qquad \text{and} \qquad \mathcal{K}(0) = \mathcal{K},$$

where $\mathcal{S}^*$ and $\mathcal{K}$ denote, respectively, the familiar subclasses of starlike and convex functions (see, for details, Srivastava and Owa [25]).

Let $\mathcal{K}[b; A, B]$ and $\mathcal{S}[b; A, B]$ ($b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ -1 \leqq B < A \leqq 1,\ z \in \mathbb{U}$) denote the subclasses of $\mathcal{A}$ and satisfy the following conditions:

$$\mathcal{K}[b; A, B] = \left\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad 1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

and

$$\mathcal{S}[b; A, B] = \left\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad 1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \frac{1 + Az}{1 + Bz} \right\},$$

where the symbol $\prec$ stands for subordination between analytic functions (see [13]) (see also [5] and [23]). The class $\mathcal{K}[b; A, B]$ was introduced and studied by Aouf *et al.* [3] and the class $\mathcal{S}[b; A, B]$ was introduced and studied by Sohi and Singh [21] (see also Aouf *et al.* [3] and [4]).

We note that

(i) $\mathcal{K}[b; 1, -1] = \mathcal{C}(b)$ (see Nasr and Aouf [15]).

(ii) $\mathcal{S}[b; 1, -1] = \mathcal{S}(b)$ (see Nasr and Aouf [18]).

For $f(z) \in \mathcal{A}$, the q-derivative ($0 < q < 1$) of $f(z)$ is defined by (see Gasper and Rahman [9])

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists. From (1.2), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \ a_k z^{k-1} \quad (z \neq 0),$$

where

$$[k]_q = \frac{1 - q^k}{1 - q}.$$

As $q \to 1^-,\ [k]_q \to k$ and

$$\lim_{q \to 1^-} D_q f(z) = f'(z).$$

Also, the q-integral of a function $f(z)$ is defined by (see Gasper and Rahman [9])

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{1.3}$$

It should be observed here that, as already pointed out by Srivastava and Bansal [24, p. 62], although the q-derivative operator in (1.2) was first applied to study a q-extension of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [22, pp. 347 *et seq.*]).

Making use of the q-derivative D_q given by (1.2), we introduce $\mathcal{K}_q[b; A, B]$ and $\mathcal{S}_q[b; A, B]$ of $\mathcal{A}$ for $b \in \mathbb{C}^*$, $0 < q < 1$ and $-1 \leqq B < A \leqq 1$ as follows:

$$\mathcal{K}_q[b; A, B] = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b}\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - 1\right] \prec \frac{1+Az}{1+Bz} \right\}, \tag{1.4}$$

and

$$\mathcal{S}_q[b; A, B] = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b}\left[\frac{zD_qf(z)}{f(z)} - 1\right] \prec \frac{1+Az}{1+Bz} \right\}. \tag{1.5}$$

From (1.4) and (1.5), we find that

$$f(z) \in \mathcal{S}_q[b; A, B] \Longleftrightarrow \int_0^z \frac{f(\zeta)}{\zeta} d_q\zeta \in \mathcal{K}_q[b; A, B]. \tag{1.6}$$

We also note that
(i) $\mathcal{K}_q[1; A, B] = \mathcal{K}_q[A, B]$ and $\mathcal{S}_q[1; A, B] = \mathcal{S}_q[A, B]$ (see Seoudy and Aouf [20]);
(ii) $\lim\limits_{q\to 1^-} \mathcal{K}_q[b; A, B] = \mathcal{K}[b; A, B]$ (see Aouf et al. [3]) and
$\lim\limits_{q\to 1^-} \mathcal{S}_q[b; A, B] = \mathcal{S}[b; A, B]$ (see Sohi and Singh [21]) (see also Aouf *et al.* [3] and [4]);
(iii) $\mathcal{K}_q\left[b; 1, \frac{1-M}{M}\right] = \mathcal{G}_q(b, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{b - 1 + \frac{D_q(zD_qf(z))}{D_qf(z)}}{b} - M \right| < M \ \left(M > \frac{1}{2}\right) \right\};$$

and $\mathcal{S}_q\left[b; 1, \frac{1-M}{M}\right] = \mathcal{F}_q(b, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{b - 1 + \frac{zD_qf(z)}{f(z)}}{b} - M \right| < M \ \left(M > \frac{1}{2}\right) \right\};$$

(iv) $\lim\limits_{q\to 1^-} \mathcal{K}_q\left[b; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{G}_q(b, M) = \mathcal{G}(b, M)$ (see Nasr and Aouf [17]) and $\lim\limits_{q\to 1^-} \mathcal{S}_q\left[b; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{F}_q(b, M) = \mathcal{F}(b, M)$ (see Nasr and Aouf [16]);
(v) $\mathcal{G}_q\left(1 - m - M, \frac{M}{m+M-1}\right) = \mathcal{C}_q(m, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - m \right| < M \ \left(m = 1 - \frac{1}{M};\ M > \frac{1}{2}\right) \right\};$$

and $\mathcal{F}_q\left(1 - m - M, \frac{M}{m+M-1}\right) = \mathfrak{B}_q(m, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{zD_qf(z)}{f(z)} - m \right| < M \ \left(m = 1 - \frac{1}{M};\ M > \frac{1}{2}\right) \right\};$$

(vi) $\lim\limits_{q\to 1^-} \mathfrak{B}_q(m, M) = \mathfrak{B}(m, M)$ (see Jakubowski [10]);
(vii) $\mathcal{K}_q[b; 1, -1] = \mathcal{K}_q(b)$

$$= \left\{ f(z) \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{1}{b}\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - 1\right]\right) > 0 \ (z \in \mathbb{U}) \right\};$$

and $\mathcal{S}_q[b;1,-1] = \mathcal{S}_q(b)$

$$= \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{1}{b}\left[\frac{zD_q f(z)}{f(z)} - 1\right]\right) > 0 \ (z \in \mathbb{U}) \right\};$$

(viii) $\lim_{q\to 1^-} \mathcal{K}_q[b;1,-1] = \lim_{q\to 1^-} \mathcal{C}_q(b) = \mathcal{C}(b)$ (see Nasr and Aouf [15]) and $\lim_{q\to 1^-} \mathcal{S}_q[b;1,-1] = \lim_{q\to 1^-} \mathcal{S}_q(b) = \mathcal{S}(b)$ (see Nasr and Aouf [18]);

(ix) $\mathcal{K}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{K}_q^{\lambda}[A,B]$

$$= \left\{ f(z) \in \mathcal{A} : e^{i\lambda} \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} \prec \cos\lambda \frac{1+Az}{1+Bz} + i\sin\lambda \ \left(|\lambda| < \frac{\pi}{2}\right) \right\};$$

and $\mathcal{S}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{S}_q^{\lambda}[A,B]$

$$= \left\{ f(z) \in \mathcal{A} : e^{i\lambda} \frac{zD_q f(z)}{f(z)} \prec \cos\lambda \frac{1+Az}{1+Bz} + i\sin\lambda \ \left(|\lambda| < \frac{\pi}{2}\right) \right\};$$

(x) $\lim_{q\to 1^-} \mathcal{K}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{K}^{\lambda}[A,B]$ $\left(|\lambda| < \frac{\pi}{2}\right)$ (see Bhoosnurmath and Devadas [7]) and $\lim_{q\to 1^-} \mathcal{S}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{S}^{\lambda}[A,B]$ $\left(|\lambda| < \frac{\pi}{2}\right)$ (see Dashrath and Shukla [8]) (see Bhoosnurmath and Devadas [6]; see also the more recent work by Xu et al. [26]);

(xi) $\mathcal{K}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{G}_{q,\lambda,M}$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda}\frac{D_q(zD_q f(z))}{D_q f(z)} - i\sin\lambda}{\cos\lambda} - M \right| < M \ \left(|\lambda| < \frac{\pi}{2};\ M > \frac{1}{2}\right) \right\};$$

and $\mathcal{S}_q\left[e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \mathcal{F}_{q,\lambda,M}$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda}\frac{zD_q f(z)}{f(z)} - i\sin\lambda}{\cos\lambda} - M \right| < M \ \left(|\lambda| < \frac{\pi}{2};\ M > \frac{1}{2}\right) \right\};$$

(xii) $\lim_{q\to 1^-} \mathcal{K}_q\left[e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim_{q\to 1^-} \mathcal{G}_{q,\lambda,M} = \mathcal{G}_{\lambda,M}$ *and* $\lim_{q\to 1^-} \mathcal{S}_q\left[e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim_{q\to 1^-} \mathcal{F}_{q,\lambda,M} = \mathcal{F}_{\lambda,M}$ *(see Kulshrestha* [12]*);*

(xiii) $\mathcal{K}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \mathcal{G}_q[\lambda,\mu,M]$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda}\frac{D_q(zD_q f(z))}{D_q f(z)} - \mu\cos\lambda - i\sin\lambda}{(1-\mu)\cos\lambda} - M \right| < M \right.$$
$$\left. \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \mu < 1;\ M > \frac{1}{2}\right) \right\};$$

and $\mathcal{S}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \mathcal{F}_q[\lambda, \mu, M]$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda}\frac{zD_qf(z)}{f(z)} - \mu\cos\lambda - i\sin\lambda}{(1-\mu)\cos\lambda} - M \right| < M \right.$$

$$\left. \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \mu < 1;\ M > \frac{1}{2} \right) \right\};$$

(xiv) $\lim_{q\to 1^-} \mathcal{K}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim_{q\to 1^-} \mathcal{K}_q[\lambda, \mu, M] = \mathcal{K}[\lambda, \mu, M]$ *and* $\lim_{q\to 1^-} \mathcal{S}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim_{q\to 1^-} \mathcal{F}_q[\lambda, \mu, M] = \mathcal{F}[\lambda, \mu, M]$ (see Aouf [2]).

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $0 < q < 1$, $-1 \le B < A \le 1$, $b \in \mathbb{C}^*$, $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$.

Theorem 2.1. *If* $f(z) \in \mathcal{A}$, *then* $f(z) \in \mathcal{K}_q[b; A, B]$ *if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1 - (1 + M(\theta))(q+1)]qz^2}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0, \tag{2.1}$$

where the symbol $*$ *stands for the convolution between two power series and*

$$M(\theta) = M^{b;A,B}(\theta) = \frac{1}{b}\left(\frac{e^{-i\theta} + B}{A - B}\right). \tag{2.2}$$

Proof. It is easy to verify that

$$zD_qf(z) * \frac{z}{1-z} = zD_qf(z) \text{ and } zD_qf(z) * \frac{z}{(1-z)(1-qz)} = zD_q\left(zD_qf(z)\right). \tag{2.3}$$

In order to prove that (2.1) holds we will write (1.4) by using the definition of the subordination, that is

$$1 + \frac{1}{b}\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - 1\right] = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.4}$$

where $w(z)$ is Schwarz function, hence

$$\frac{1}{z}\left[zD_q\left(zD_qf(z)\right)\left(1 + Be^{i\theta}\right) - \left[1 + [B + b(A-B)]\,e^{i\theta}\right]zD_qf(z)\right] \neq 0. \tag{2.5}$$

Using (2.3), Eq. (2.5) may be written as

$$\frac{1}{z}\left[\left(1 + Be^{i\theta}\right)\left(zD_qf(z) * \frac{z}{(1-z)(1-qz)}\right)\right.$$

$$\left. -\left[1 + [B + b(A-B)]\,e^{i\theta}\right]\left(zD_qf(z) * \frac{z}{1-z}\right)\right] \neq 0,$$

which is equivalent to

$$\frac{1}{z}\left[zD_qf(z)*\frac{z-\left(1+\dfrac{e^{-i\theta}+B}{(A-B)b}\right)qz^2}{(1-z)(1-qz)}\cdot\Big[-(A-B)be^{i\theta}\Big]\right]\neq 0,$$

or

$$\frac{1}{z}\left[f(z)*zD_q\frac{z-\left(1+\dfrac{e^{-i\theta}+B}{(A-B)b}\right)qz^2}{(1-z)(1-qz)}\right]$$

$$=\frac{1}{z}\left[f(z)*\frac{z+\left[1-(q+1)\left(1+\dfrac{e^{-i\theta}+B}{(A-B)b}\right)\right]qz^2}{(1-z)(1-qz)(1-q^2z)}\right]\neq 0,$$

that is (2.1). Reversely, since, it was shown in the first part of the proof that the assumption (2.5) is equivalent to (2.1), we obtain that

$$\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\neq\frac{1+[B+(A-B)b]e^{i\theta}}{1+Be^{i\theta}}. \tag{2.6}$$

Suppose that

$$\varphi(z)=\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\quad\text{and}\quad\psi(z)=\frac{1+[B+(A-B)b]z}{1+Bz}.$$

The relation (2.6) means that

$$\varphi(\mathbb{U})\cap\psi(\partial\mathbb{U})=\emptyset.$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{U})$. From this, using the fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, this implies that $f(z)\in\mathcal{K}_q[b;A,B]$. Thus, the proof is completed. □

Theorem 2.2. *If $f(z)\in\mathcal{A}$, then $f(z)\in\mathcal{S}_q[b;A,B]$ if and only if*

$$\frac{1}{z}\left[f(z)*\frac{z-(1+M(\theta))\,qz^2}{(1-z)(1-qz)}\right]\neq 0, \tag{2.7}$$

where $M(\theta)$ is given by (2.2).

Proof. From (1.6), it follows that $f\in\mathcal{S}_q[b;A,B]$ if and only if

$$\Phi_q(z)=\int_0^z\frac{f(\zeta)}{\zeta}d_q\zeta\in\mathcal{K}_q[b;A,B].$$

Then, according to Theorem 2.1, the function Φ_q belongs to $\mathcal{S}_q[b;A,B]$ if and only if

$$\frac{1}{z}\left[\Phi_q(z)*g(z)\right]\neq 0,\ \text{for all } z\in\mathbb{U}\text{ and }\theta\in[0,2\pi), \tag{2.8}$$

where

$$g(z) = \frac{z + [1 - (1 + M(\theta))(q+1)]qz^2}{(1-z)(1-qz)(1-q^2z)}.$$

From (1.3), we have

$$\int_0^z \frac{g(\zeta)}{\zeta} d_q\zeta = \int_0^z \frac{1 + [1 - (1 + M(\theta))(q+1)]q\zeta}{(1-\zeta)(1-q\zeta)(1-q^2\zeta)} d_q\zeta$$
$$= z(1-q) \sum_{k=0}^{\infty} \frac{q^k + [1 - (1 + M(\theta))(q+1)]zq^{2k+1}}{(1-zq^k)(1-zq^{k+1})(1-zq^{k+2})},$$

and therefore

$$\int_0^z \frac{g(\zeta)}{\zeta} d_q\zeta = \frac{z - (1 + M(\theta))qz^2}{(1-z)(1-qz)}.$$

Using the above relation and the identity

$$\left[\int_0^z \frac{f(\zeta)}{\zeta} d_q\zeta\right] * g(z) = f(z) * \left[\int_0^z \frac{g(\zeta)}{\zeta} d_q\zeta\right],$$

it is easy to check that (2.8) is equivalent to (2.7). □

Theorem 2.3. *If $f(z) \in \mathcal{A}$, then $f(z) \in \mathcal{K}_q[b; A, B]$ if and only if*

$$1 - \sum_{k=2}^{\infty} [k]_q \frac{([k]_q - 1)(e^{-i\theta} + B) - (A - B)b}{(A-B)b} a_k z^{k-1} \neq 0 \text{ for all } \theta. \tag{2.9}$$

Proof. If $f(z) \in \mathcal{A}$, then from Theorem 2.1, we have $f(z) \in \mathcal{K}_q[b; A, B]$ if and only if (2.1) holds. Since

$$\frac{1}{(1-z)(1-qz)(1-q^2z)} = 1 + \left(1 + q + q^2\right)z + \left(1 + q + 2q^2 + q^3 + q^4\right)z^2$$
$$+ \left(1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6\right)z^3 + \ldots,$$

it follows that

$$\frac{z + [1 - (1 + M(\theta))(q+1)]qz^2}{(1-z)(1-qz)(1-q^2z)} = z + \sum_{k=2}^{\infty} [k]_q \Big(1 - qM(\theta)[k-1]_q\Big) a_k z^k,$$

where $M(\theta)$ is given by (2.2) and so (2.1) may be written as

$$1 - \sum_{k=2}^{\infty} [k]_q \frac{q[k-1]_q(e^{-i\theta} + B) - (A-B)b}{(A-B)b} a_k z^{k-1} \neq 0$$

that is (2.9). □

Theorem 2.4. *If $f(z) \in \mathcal{A}$, then $f(z) \in \mathcal{S}_q[b; A, B]$ if and only if*

$$1 - \sum_{k=2}^{\infty} \frac{([k]_q - 1)(e^{-i\theta} + B) - (A-B)b}{(A-B)b} a_k z^{k-1} \neq 0 \text{ for all } \theta. \tag{2.10}$$

Proof. If $f(z) \in \mathcal{A}$, then from Theorem 2.2, we have $f(z) \in \mathcal{S}_q[b; A, B]$ if and only if (2.7) holds. Since

$$\frac{1}{(1-z)(1-qz)} = 1 + (1+q)z + \left(1+q+q^2\right)z^2 + \left(1+q+q^2+q^3\right)z^3 + \ldots,$$

it follows that

$$\frac{z - (1 + M(\theta))\,qz^2}{(1-z)(1-qz)} = z + \sum_{k=2}^{\infty}\Big(1 - qM(\theta)\,[k-1]_q\Big)a_k z^k,$$

where $M(\theta)$ is given by (2.2). Now, we may express (2.7) as

$$1 - \sum_{k=2}^{\infty} \frac{q[k-1]_q(e^{-i\theta} + B) - (A-B)b}{(A-B)b} a_k z^{k-1} \neq 0,$$

or equivalently, (2.10). □

Theorem 2.5. *If $f(z) \in \mathcal{A}$ satisfies the inequality*

$$\sum_{k=2}^{\infty}[k]_q\Big[([k]_q - 1)(1 + |B|) + (A-B)\,|b|\Big]\,|a_k| \leq (A-B)\,|b|\,. \tag{2.11}$$

then $f(z) \in \mathcal{K}_q[b; A, B]$.

Proof. Since

$$\begin{aligned}
&\left|1 - \sum_{k=2}^{\infty}[k]_q\frac{([k]_q - 1)(e^{-i\theta} + B) - (A-B)b}{(A-B)b}a_k z^{k-1}\right| \\
&\geq 1 - \left|\sum_{k=2}^{\infty}[k]_q\frac{([k]_q - 1)(e^{-i\theta} + B) - (A-B)b}{(A-B)b}a_k z^{k-1}\right| \\
&\geq 1 - \sum_{k=2}^{\infty}[k]_q\frac{([k]_q - 1)(1 + |B|) + (A-B)\,|b|}{(A-B)\,|b|}\,|a_k| > 0.
\end{aligned}$$

Thus, the inequality (2.11) holds and our conclusion follows. □

By using arguments and analysis to those in the proof of Theorem 2.5, we can analogously derive Theorem 2.6.

Theorem 2.6. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}\Big[([k]_q - 1)(1 + |B|) + (A-B)\,|b|\Big]\,|a_k| \leq (A-B)\,|b|\,.$$

then $f(z) \in \mathcal{S}_q[b; A, B]$.

Remark 2.7. (i) For different choices of q, b, A and B in Theorems 2.1 and 2.2, the results of Seoudy and Aouf (see [20, Theorems 1 and 5]), Nasr and Aouf (see [14, Theorems 1 and 2]) and Bhoosnurmath and Devadas (see [6] and [7]) follow.

(ii) For $b = 1$ in Theorems from 2.3 to 2.6, the results of Seoudy and Aouf (see [20, Theorems 9, 13, 17 and 21]) will follow.

(iii) For different choices of q, b, A and B in our results, we will obtain new results for different classes mentioned in the introduction.

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