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Perturbations of local C-cosine functions

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Abstract. We show that $A+B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on a complex Banach space X defined by

$$T(t)x = \sum_{n=0}^{\infty} B^n \int_0^t j_{n-1}(s) j_n(t-s) C(|t-2s|)x ds$$

for all $x \in X$ and $0 \leq t < T_0$, if A is a closed subgenerator of a local C-cosine function $C(\cdot)$ on X and one of the following cases holds: (i) $C(\cdot)$ is exponentially bounded, and B is a bounded linear operator on $\overline{D(A)}$ so that $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$; (ii) B is a bounded linear operator on $\overline{D(A)}$ which commutes with $C(\cdot)$ on $\overline{D(A)}$ and $BA \subset AB$; (iii) B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Here $j_n(t) = \frac{t^n}{n!}$ for all $t \in \mathbb{R}$, and

$$\int_0^t j_{-1}(s) j_0(t-s) C(|t-2s|)x ds = C(t)x$$

for all $x \in X$ and $0 \leq t < T_0$.

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1. Introduction

Let X be a complex Banach space with norm $\|\cdot\|$, and let $L(X)$ denote the set of all bounded linear operators on X. For each $0 < T_0 \leq \infty$ and each injection $C \in L(X)$, a family $C(\cdot)$ $(= \{C(t) \,|\, 0 \leq t < T_0\})$ in $L(X)$ is called a local C-cosine function on X if it is strongly continuous, $C(0) = C$ on X and satisfies

$$2C(t)C(s) = C(t+s)C + C(|t-s|)C \tag{1.1}$$

on X for all $0 \leq t, s, t+s < T_0$ (see [5], [7], [14], [15], [21], [23], [25]). In this case, the generator of $C(\cdot)$ is a closed linear operator A in X defined by

$$D(A) = \{x \in X \mid \lim_{h \to 0^+} 2(C(h)x - Cx)/h^2 \in R(C)\}$$

and $Ax = C^{-1} \lim_{h \to 0^+} 2(C(h)x - Cx)/h^2$ for $x \in D(A)$. Moreover, we say that $C(\cdot)$ is locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that

$$\|C(t+h) - C(t)\| \leq K_{t_0} h \tag{1.2}$$

for all $0 \leq t, h, t+h \leq t_0$; exponentially bounded, if $T_0 = \infty$and there exist $K, \omega \geq 0$ such that

$$\|C(t)\| \leq K e^{\omega t} \tag{1.3}$$

for all $t \geq 0$; exponentially Lipschitz continuous, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that

$$\|C(t+h) - C(t)\| \leq K h e^{\omega(t+h)} \tag{1.4}$$

for all $t, h \geq 0$. In general, a local C-cosine function is also called a C-cosine function if $T_0 = \infty$ (see [2], [12], [14], [16]) or a cosine function if $C = I$ (identity operator on X) (see [1], [4], [6]), and a C-cosine function may not be exponentially bounded (see [16]). Moreover, a local C-cosine function is not necessarily extendable to the half line $[0, \infty)$ (see [21]) except for $C = I$ (see [1], [4], [6]) and the generator of a C-cosine function may not be densely defined (see [2]). Perturbations of local C-cosine functions have been extensively studied by many authors appearing in [1], [2], [4], [9], [11], [17], [18], [19]. Some interesting applications of this topic are also illustrated there. In particular, a classical perturbation result of cosine functions shows that if A is the generator of a C-cosine function $C(\cdot)$ on X, and B a bounded linear operator on X, then $A + B$ is the generator of a C-cosine function on X when $C = I$, but the conclusion may not be true when C is arbitrary, and is still unknown until now even though B and $C(\cdot)$ are commutable, which can be completely solved in this paper and several new additive perturbation theorems concerning local C-cosine functions are also established as results in [20] for the case of C-semigroup and in [8], [13] for the case of local C-semigroup. A new representation of the perturbation of a local C-cosine function is given in (1.5) below. We show that if $C(\cdot)$ is an exponentially bounded C-cosine function on X with closed subgenerator A and B a bounded linear operator on $\overline{D(A)}$ such that $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$, then $A + B$ is a closed subgenerator of an exponentially bounded C-cosine function $T(\cdot)$ on X defined by

$$T(t)x = \sum_{n=0}^{\infty} B^n \int_0^t j_{n-1}(s) j_n(t-s) C(|t-2s|) x ds \tag{1.5}$$

for all $x \in X$ and $0 \leq t < T_0$ (see Theorem 2.6 below). Here $j_n(t) = \frac{t^n}{n!}$ for all $t \in \mathbb{R}$, and

$$\int_0^t j_{-1}(s) j_0(t-s) C(|t-2s|) x ds = C(t)x$$

for all $x \in X$ and $0 \leq t < T_0$. Moreover, $T(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $C(\cdot)$ is. We then show that the exponential boundedness of $T(\cdot)$ can be deleted and C-cosine functions can be extended to the context of local C-cosine functions when the assumption of $BC(\cdot) = C(\cdot)B$ on $\overline{D(A)}$ is added (see Theorem 2.7 below). Moreover, $T(\cdot)$ is locally Lipschitz continuous or norm continuous if $C(\cdot)$ is. We also show that $A + B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on X if A is a closed subgenerator of a local C-cosine function $C(\cdot)$ on X and

B a bounded linear operator on X such that $BC(\cdot) = C(\cdot)B$ on X (see Theorem 2.8 below). A simple illustrative example of these results is presented in the final part of this paper.

2. Perturbation theorems

In this section, we first note some basic properties of a local C-cosine function with its subgenerator and generator.

Definition 2.1. (see [10], [14]) Let $C(\cdot)$ be a strongly continuous family in $\mathrm{L}(X)$. A linear operator A in X is called a subgenerator of $C(\cdot)$ if

$$C(t)x - Cx = \int_0^t \int_0^s C(r)Axdrds$$

for all $x \in \mathrm{D}(A)$ and $0 \leq t < T_0$, and

$$\int_0^t \int_0^s C(r)xdrds \in \mathrm{D}(A) \quad \text{ and } A\int_0^t \int_0^s C(r)xdrds = C(t)x - Cx$$

for all $x \in X$ and $0 \leq t < T_0$. A subgenerator A of $C(\cdot)$ is called the maximal subgenerator of $C(\cdot)$ if it is an extension of each subgenerator of $C(\cdot)$ to $D(A)$.

Proposition 2.2. (see [4], [5], [10], [14], [21]) *Let A be the generator of a local C-cosine function $C(\cdot)$ on X. Then*

$$C(t)x \in D(A) \quad \text{ and } C(t)Ax = AC(t)x \tag{2.1}$$

for all $x \in D(A)$ and $0 \leq t < T_0$;

$$C^{-1}AC = A \quad \text{ and } R(C(t)) \subset \overline{D(A)} \tag{2.2}$$

for all $0 \leq t < T_0$;

$$x \in D(A) \text{ and } Ax = y_x \text{ if and only if } C(t)x - Cx = \int_0^t \int_0^s C(r)y_xdrds \tag{2.3}$$

for all $0 \leq t < T_0$;

$$A_0 \text{ is closable and } C^{-1}\overline{A_0}C = A \tag{2.4}$$

for each subgenerator A_0 of $C(\cdot)$;

$$A \text{ is the maximal subgenerator of } C(\cdot). \tag{2.5}$$

From now on, we always assume that $A : D(A) \subset X \to X$ is a closed linear operator so that $CA \subset AC$.

Theorem 2.3. (see [10], [16]) *A strongly continuous family $C(\cdot)$ in $L(X)$ satisfying (1.3) is a C-cosine function on X with subgenerator A if and only if $CC(\cdot) = C(\cdot)C$, $\lambda^2 \in \rho_C(A)$, and $\lambda(\lambda^2 - A)^{-1}C = L_\lambda$ on X for all $\lambda > \omega$. Here*

$$L_\lambda x = \int_0^\infty e^{-\lambda t}C(t)xdt \text{ for } x \in X.$$

Lemma 2.4. (see [1]) *Let $C(\cdot)(=\{C(t)\,|\,0\le t<T_0\})$ be a strongly continuous family in $L(X)$. We set $C(-t)=C(t)$ for $0\le t<T_0$. Then $C(\cdot)$ is a local C-cosine function on X if and only if $2C(t)C(s)=C(t+s)C+C(t-s)C$ on X for all $|t|,|s|,|t-s|,|t+s|<T_0$. In this case,*

$$S(-t)=-S(t) \tag{2.6}$$

for all $0\le t<T_0$;

$$S(t+s)C=S(t)C(s)+C(t)S(s) \text{ on } X \tag{2.7}$$

for all $|t|,|s|,|t+s|<T_0$. Here $S(t)=j_0*C(t)$ for all $|t|<T_0$.

By slightly modifying the proof of [3, Lemma 2], the next lemma is also attained.

Lemma 2.5. *Let $C(\cdot)(=\{C(t)\,|\,0\le t<T_0\})$ be a local C-cosine function on X, and $C(-t)=C(t)$ for $0\le t<T_0$. Assume that S^{*n+1} denotes the $(n+1)$-fold convolution of S for $n\in\mathbb{N}\cup\{0\}$, that is*

$$S^{*2}(t)x=\int_0^t S(t-s)S(s)xds$$

and

$$S^{*n+1}(t)x=\int_0^t S^{*n}(t-s)S(s)xds.$$

Then

$$S^{*n+1}(t)=\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)C^n ds=\int_0^t j_n(s)j_n(t-s)C(t-2s)C^n ds$$

on X for all $|t|<T_0$. Here $S(t)=j_0*C(t)$ and

$$\int_0^t j_{-1}(s)j_0(t-s)S(t-2s)C^0 ds=S(t)=S^{*1}(t)$$

for all $|t|<T_0$.

Proof. It is easy to see that

$$\begin{aligned}S^{*n+1}(t)&=\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)C^n ds\\&=\int_0^t j_n(s)j_n(t-s)C(t-2s)C^n ds\end{aligned}$$

on X for $n = 0$. By induction, we have

$$\begin{aligned}
S^{*n+1}(t)x &= \int_0^t S^{*n}(s)S(t-s)xds \\
&= \int_0^t \int_0^s j_{n-1}(r)j_{n-1}(s-r)C(s-2r)C^{n-1}S(t-s)xdrds \\
&= \frac{1}{2}\int_0^t \int_0^s j_{n-1}(r)j_{n-1}(s-r)\big[S(t-2r)+S(t+2r-2s)\big]C^n xdrds \\
&= \int_0^t \int_0^s j_{n-1}(r)j_{n-1}(s-r)S(t-2r)C^n xdrds \\
&= \int_0^t \int_r^t j_{n-1}(r)j_{n-1}(s-r)S(t-2r)C^n xdsdr \\
&= \int_0^t j_{n-1}(r)j_n(t-r)S(t-2r)C^n xdr \\
&= \frac{1}{2}\int_0^t \big[j_{n-1}(r)j_n(t-r)-j_n(r)j_{n-1}(t-r)\big]S(t-2r)C^n xdr \\
&= \frac{1}{2}\int_0^t \frac{d}{dr}[j_n(r)j_n(t-r)]S(t-2r)C^n xdr \\
&= \int_0^t j_n(r)j_n(t-r)C(t-2r)C^n xdr
\end{aligned}$$

for all $n \in \mathbb{N}$, $x \in X$ and $|t| < T_0$. □

Applying Theorem 2.3 we can obtain the next perturbation theorem concerning exponentially bounded C-cosine functions just as a corollary of [11, Corollary 2.6.6].

Theorem 2.6. *Let A be a subgenerator of an exponentially bounded C-cosine function $C(\cdot)$ on X. Assume that $B \in L(\overline{D(A)}), BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A + B$ is a closed subgenerator of an exponentially bounded C-cosine function $T(\cdot)$ on X given as in (1.5). Moreover, $T(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $C(\cdot)$ is.*

Proof. It is easy to see that

$$(\lambda^2 - A - B)^{-1}C = \sum_{n=0}^{\infty} B^n(\lambda^2 - A)^{-n-1}C$$

for $\lambda > \omega$, and the boundedness of $\{\|C(t)\| \mid 0 \le t \le t_0\}$ for each $t_0 > 0$ and the strong continuity of $C(\cdot)$ imply that the right-hand side of (1.5) converges uniformly on compact subsets of $[0, \infty)$. In particular, $T(\cdot)$ is a strongly continuous family in $L(X)$. For simplicity, we may assume that $\|C(t)\| \le Ke^{\omega t}$ for all $t \ge 0$ and for some fixed $K, \omega \geq 0$. Then $\|T(t)\| \leq Ke^{(\omega+\sqrt{\|B\|})t}$ for all $t \geq 0$, and

$$\begin{aligned}(\lambda^2 - A - B)^{-1}Cx &= \sum_{n=0}^{\infty} B^n \int_0^{\infty} e^{-\lambda t} \int_0^t j_{n-1}(s) j_n(t-s) S(t-2s) x ds dt \\ &= \int_0^{\infty} \sum_{n=0}^{\infty} B^n e^{-\lambda t} \int_0^t j_{n-1}(s) j_n(t-s) S(t-2s) x ds dt \\ &= \int_0^{\infty} e^{-\lambda t} j_0 * T(t) x dt\end{aligned}$$

for $\lambda > \omega$ and $x \in X$ or equivalently,

$$\lambda(\lambda^2 - A - B)^{-1}Cx = \int_0^{\infty} e^{-\lambda t} T(t) x dt$$

for $\lambda > \omega$ and $x \in X$. Here

$$\int_0^t j_{-1}(s) j_0(t-s) S(t-2s) x ds = S(t)x \text{ for } t \geq 0.$$

Applying Theorem 2.3, we get that $T(\cdot)$ is an exponentially bounded C-cosine function on X with closed subgenerator $A + B$. Since

$$\begin{aligned}&\int_0^t j_{n-1}(r) j_n(t-r) C(t-2r) x dr \\ &- \int_0^s j_{n-1}(r) j_n(s-r) C(s-2r) x dr \\ &= \int_s^t j_{n-1}(r) j_n(t-r) C(t-2r) x dr \\ &+ \int_0^s j_{n-1}(r)[j_n(t-r) C(t-2r) - j_n(s-r) C(s-2r)] x dr\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\int_0^s j_{n-1}(r)[j_n(t-r) C(t-2r) - j_n(s-r) C(s-2r)] x dr \\ &= \int_0^s j_{n-1}(r) j_n(s-r)[C(t-2r) - C(s-2r)] x dr \\ &+ \int_0^s j_{n-1}(r)[j_n(t-r) - j_n(s-r)] C(t-2r) x dr \\ &= \int_0^s j_{n-1}(r) j_n(s-r)[C(|t-2r|) - C(|s-2r|)] x dr \\ &+ \int_0^s j_{n-1}(r)[j_n(t-r) - j_n(s-r)] C(|t-2r|) x dr\end{aligned} \tag{2.9}$$

for all $n \in \mathbb{N}$, $x \in X$ and $t \geq s \geq 0$, we observe from (1.5) that $T(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $C(\cdot)$ is. □

Next we deduce a new perturbation theorem concerning local C-cosine functions. In particular, the exponential boundedness of $T(\cdot)$ in Theorem 2.6 can be deleted when the assumption of $BC(\cdot) = C(\cdot)B$ on $\overline{D(A)}$ is added.

Theorem 2.7. *Let A be a subgenerator of a local C-cosine function $C(\cdot)$ on X. Assume that B is a bounded linear operator on $\overline{D(A)}$ such that $BC(\cdot) = C(\cdot)B$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A + B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on X given as in (1.5). Moreover, $T(\cdot)$ is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is.*

Proof. Just as in the proof of Theorem 2.6, we observe from (2.8)-(2.9) and (1.5) that $T(\cdot)$ is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is. Since

$$R(C(t)) \subset \overline{D(A)} \text{ and } BC(\cdot) = C(\cdot)B \text{ on } \overline{D(A)},$$

we have

$$CT(\cdot) = T(\cdot)C \text{ on } X.$$

Let $x \in X$ and $0 \le t \le r < T_0$ be fixed. Then

$$\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xds = \frac{1}{2}[j_1(t)\widetilde{S}(t) - \int_0^t \widetilde{S}(t-2s)xds]$$

for $n = 1$, and

$$\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xds$$
$$= \frac{1}{2}\int_0^t [j_{n-2}(s)j_n(t-s) - j_{n-1}(s)j_{n-1}(t-s)]\widetilde{S}(t-2s)xds$$

for all $n \ge 2$. Here

$$\widetilde{S}(\cdot) = j_0 * S(\cdot).$$

Since $BA \subset AB$ and

$$\widetilde{S}(r)x = \int_0^r \int_0^t C(s)xdsdt \in D(A),$$

we have

$$AB\int_0^r \big[j_1(t)\widetilde{S}(t)x - \int_0^t \widetilde{S}(t-2s)xds\big]dt$$
$$= BA\int_0^r \big[j_1(t)\widetilde{S}(t)x - \int_0^t \widetilde{S}(t-2s)xds\big]dt$$
$$= B\int_0^r \big(j_1(t)[C(t)x - Cx] - \int_0^t [C(t-2s)x - Cx]ds\big)dt$$
$$= B\int_0^r j_1(t)C(t)xdt - B\int_0^r \int_0^t C(t-2s)xdsdt$$
$$= B\int_0^r j_1(t)C(t)xdt - B\int_0^r S(t)xdt.$$

Since

$$\int_0^r j_1(t)C(t)xdt = xj_1(r)S(r)x - \widetilde{S}(r)x$$

and

$$j_1(r)S(r)x = 2\int_0^r j_1(r-s)C(r-2s)xds,$$

we also have

$$\begin{aligned} &AB\int_0^r \big[j_1(t)\widetilde{S}(t)x - \int_0^t \widetilde{S}(t-2s)xds\big]dt \\ &= 2B\int_0^r j_1(r-s)C(r-2s)xds - 2B\int_0^r\int_0^t C(s)xdsdt. \end{aligned} \tag{2.10}$$

Let $n \geq 2$ be fixed.
Using integration by parts, we have

$$\begin{aligned} &\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xds \\ &= \frac{1}{2}\int_0^t \big[j_{n-2}(s)j_n(t-s) - j_{n-1}(s)j_{n-1}(t-s)\big]\widetilde{S}(t-2s)xds. \end{aligned}$$

Since

$$\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)Cxdsdt = \int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)Cxdsdt,$$

we have

$$\begin{aligned} &A\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &= \frac{1}{2}\Big[\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)A\widetilde{S}(t-2s)xdsdt \\ &- \int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)A\widetilde{S}(t-2s)xdsdt\Big] \\ &= \frac{1}{2}\Big[\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)(C(t-2s)x - Cx)dsdt \\ &- \int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)(C(t-2s)x - Cx)dsdt\Big] \\ &= \frac{1}{2}\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)C(t-2s)xdsdt \\ &- \frac{1}{2}\int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)C(t-2s)xdsdt. \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}
&\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)C(t-2s)xdsdt\\
=&\int_0^r\int_s^r j_{n-2}(s)j_n(t-s)C(t-2s)xdtds\\
=&\int_0^r j_{n-2}(s)\big[j_n(r-s)S(r-2s)x\\
-&\int_s^r j_{n-1}(t-s)S(t-2s)xdt\big]ds\\
=&\int_0^r j_{n-2}(s)j_n(r-s)S(r-2s)xds\\
-&\int_0^r j_{n-2}(s)\int_s^r j_{n-1}(t-s)S(t-2s)xdtds,
\end{aligned}\tag{2.12}$$

$$\begin{aligned}
&\int_0^r j_{n-2}(s)j_n(r-s)S(r-2s)xds\\
=&\int_0^r j_{n-1}(s)j_{n-1}(r-s)S(r-2s)xds\\
&+2\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds\\
=&2\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
&\int_0^r\int_s^r j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdtds\\
=&\int_0^r\int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)C(t-2s)xdsdt\\
=&\ 2\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds\\
-&\int_0^r\int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt.
\end{aligned}\tag{2.14}$$

By Lemma 2.5, we have

$$\begin{aligned}
&\int_0^r\int_0^t j_n(s)j_n(t-s)C(t-2s)xdsdt\\
=&\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt.
\end{aligned}\tag{2.15}$$

Combining (1.11) with (2.14) and (2.15), we have

$$\begin{aligned} & A\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ = & \int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds \\ - & \int_0^r\int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt. \end{aligned} \tag{2.16}$$

It follows from (2.10)and (2.16) that we have

$$\begin{aligned} & A\int_0^r\sum_{n=0}^{\infty}B^n\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ = & A\sum_{n=0}^{\infty}B^n\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ = & \sum_{n=0}^{\infty}AB^n\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ = & A\int_0^r\int_0^t C(s)xdsdt + AB\int_0^r\int_0^t j_1(t-s)S(t-2s)xdsdt \\ + & \sum_{n=2}^{\infty}B^nA\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ = & \big[C(r)x-Cx\big] + B\left[\int_0^r j_1(r-s)C(r-2s)xds - \int_0^r\int_0^t C(s)xdsdt\right] \\ + & \sum_{n=2}^{\infty}B^n\left[\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds\right. \\ - & \left.\int_0^r\int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt\right] \\ = & \sum_{n=0}^{\infty}B^n\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds - Cx - B\int_0^r\int_0^t C(s)xdsdt \\ - & \int_0^r\sum_{n=1}^{\infty}B^{n+1}\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ = & \sum_{n=0}^{\infty}B^n\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds - Cx \\ - & B\int_0^r\sum_{n=0}^{\infty}B^n\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \end{aligned} \tag{2.17}$$

for all $x \in X$ and $0 \le r < T_0$ or equivalently,

$$(A+B)\int_0^r\int_0^t T(s)xdsdt = T(r)x - Cx$$

for all $x \in X$ and $0 \leq r < T_0$. Since $AB^n = B^nA$ and $B^nC(t) = C(t)B^n$ on $D(A)$, we have

$$\int_0^r \int_0^t T(s)(A+B)xdsdt = (A+B)\int_0^r \int_0^t T(s)xdsdt = T(r)x - Cx$$

for all $x \in D(A)$ and $0 \leq r < T_0$. It follows from [14, Theorem 2.5] that $T(\cdot)$ is a local C-cosine function on X with closed subgenerator $A+B$, and is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is. □

By slightly modifying the proof of Theorem 2.7 we also obtain the next perturbation theorem concerning local C-cosine functions which is still new even though $T_0 = \infty$.

Theorem 2.8. *Let A be a subgenerator of a local C-cosine function $C(\cdot)$ on X. Assume that B is a bounded linear operator on X such that $BC(\cdot) = C(\cdot)B$ on X. Then $A+B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on X satisfying*

$$T(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s)j_n(t-s)C(|t-2s|)B^nxds \tag{2.18}$$

for all $x \in X$ and $0 \leq t < T_0$. Moreover, $T(\cdot)$ is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is.

Proof. Suppose that B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Then

$$T(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s)j_n(t-s)C(|t-2s|)B^nxds$$

for all $x \in X$ and $0 \leq t < T_0$. Since the assumption of $BA \subset AB$ in the proof of Theorem 2.7 is only used to show that (2.10) and (2.17) hold, but both are automatically satisfied if $BA \subset AB$ is replaced by assuming that B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Therefore, the conclusion of this theorem is true. □

We end this paper with a simple illustrative example.

Example 2.9. Let $C(\cdot)$ $(= \{C(t)|0 \leq t < 1\})$ be a family of bounded linear operators on c_0 (family of all convergent sequences in $\mathbb{C}$ with limit 0), defined by

$$C(t)x = \{x_n e^{-n} \cosh nt\}_{n=1}^{\infty}$$

for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ and $0 \leq t < 1$, then $C(\cdot)$ is a local C-cosine function on c_0 with generator A defined by $Ax = \{n^2x_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ with $\{n^2x_n\}_{n=1}^{\infty} \in c_0$. Here $C = C(0)$. Let B be a bounded linear operator on c_0 defined by $Bx = \{x_n e^{-n} \cosh n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in D(A)$, then $C(\cdot)B = BC(\cdot)$ on c_0. Applying Theorem 2.8, we get that $A+B$ generates a local C-cosine function $T(\cdot)$ on c_0 satisfying (1.5).

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