

Goldie absolute direct summand rings and modules

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Abstract. In the present paper, we introduce and study Goldie ADS modules and rings, which subsume two generalizations of Goldie extending modules due to Akalan et al. [3] and ADS-modules due to Alahmadi et al. [7]. A module M will be called a Goldie ADS module if for every decomposition $M = S \oplus T$ of M and every complement T' of S , there exists a submodule D of M such that $T'\beta D$ and $M = S \oplus D$. Various properties concerning direct sums of Goldie ADS modules are established.

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1. Introduction

The purpose of the present paper is to introduce and study Goldie ADS modules, which allow us to give a unified approach of Goldie extending modules and ADS-modules, introduced by E. Akalan et al. [3] and A. Alahmadi et al. [7], respectively. We define a Goldie ADS module by the property that for every decomposition $M = S \oplus T$ of M and every complement T' of S , there exists a submodule D of M such that $T'\beta D$ and $M = S \oplus D$. We study these modules, generalizing several results both on Goldie extending modules and ADS-modules. We show that a non-singular Goldie ADS module is an ADS module. We emphasize that our properties are of the same type as those for Goldie extending modules and ADS-modules, sharing similar limitations in studying certain properties, such as the closure of the respective class of modules under direct sums. We also analyze when a direct summand of Goldie ADS modules is a Goldie ADS and also when a direct sum of Goldie ADS module is Goldie ADS, by using the concepts of relative ejectivity. In the last section, we look at Goldie ADS property of some ring extensions.

2. Definitions and notions

In this paper, R will present an associative ring with identity and all modules over R are unitary right modules. We also write M_R to indicate that M is a right R -module. We shall denote the fact that a submodule N is essential in a module M by $N \leq_e M$. The following generalization of relative injectivity is introduced in [3, Definition 2.1]. Let N and M be modules. N is called *M-ejective* if, for each $K \leq M$ and each homomorphism $f : K \rightarrow N$, there exist a homomorphism $\bar{f} : M \rightarrow N$ and a $X \leq_e K$ such that $\bar{f}(x) = f(x)$, for all $x \in X$. M and N is called *mutually ejective* if M is N -ejective and N is M -ejective. A submodule K of M is called *fully invariant* if $f(K) \subseteq K$ for every $f \in \text{End}_R(M)$. Clearly 0 and M are fully invariant submodules of M . The right R -module M is called a *duo module* provided every submodule of M is fully invariant. The singular submodule of a module M will be denoted by $Z(M) = \{m \in M : mI = 0 \text{ for some } I \leq_e R_R\}$. A module M is called *singular* (respectively *non-singular*) if $Z(M) = M$ (respectively $Z(M) = 0$).

(CS): Every complement submodule of M is a direct summand of M .

(C2): Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M .

(C3): For any two direct summands A and B of M with $A \cap B = 0$, the sum $A + B$ is a direct summand of M .

A module M is called is called *continuous* (respectively, *quasi continuous*) if M satisfies (CS) and (C2) (respectively, (CS) and (C3)).

Let M be an R -module and $X, Y \leq M$. In [3], the notion of β relation on submodules X, Y of M , denoted by $X\beta Y$, is defined such as $X\beta Y$ if and only if $X \cap A = 0$ implies $Y \cap A = 0$ and $Y \cap B = 0$ implies $X \cap B = 0$ for all $A, B \leq M$. A right module M is *Goldie extending* if for each $X \leq M$, there exists a direct summand D of M such that $X\beta D$. M is Goldie extending if and only if for each closed submodule C of M there is a direct summand D of M such that $C\beta D$.

Another notion generalizing extending property, ADS (Absolute Direct Summand) modules, was recently considered in [7]. It was introduced by Fuchs [10] for abelian groups and for general modules by Alahmadi, Jain and Leroy [7]. As the authors pointed out in [7], if R is commutative then every cyclic R -module is ADS. Also every right quasi-continuous module is ADS, but the converse is not true. However, a right ADS module which is also CS is quasi-continuous. Also in [14], Quynh and Koşan proved that every ADS module satisfies (C3). Hence, this is a class of modules between quasi-continuous modules and modules satisfying the (C3) condition. Quynh and Koşan gave also different characterizations of ADS modules and showed how to characterize semisimple modules and semisimple artinian rings using the ADS. The SC and SI rings were also characterized by the ADS notion in [14].

3. Goldie absolute direct summand modules

A module M is called *Goldie absolute direct summand* (Goldie ADS) if, for every decomposition $M = S \oplus T$ of M and every complement T' of S , there exists a submodule D of M such that $T'\beta D$ and $M = S \oplus D$.

A ring R right Goldie ADS if the (right) R -module R is Goldie ADS. We know that extending modules are Goldie extending, but need not be ADS. Similarly, Goldie extending modules do not necessarily satisfy Goldie ADS. Hence, the notions extending, Goldie extending, ADS and the property Goldie ADS are not directly related.

Example 3.1. Let $R = \mathbb{Z}_2[x_1, x_2, \dots]$, where x_i are commuting indeterminates satisfying the relations: $x_i^3 = 0$ for all i , $x_i x_j = 0$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all i and j . Then R is a commutative, semiprimary ring with simple essential socle. But R is not a self-injective ring (see [13, Example 5.45]). On the other hand, R_R is soc- R_R -injective by [8, Example 5.7]. It follows that R_R is soc- $(R_R \oplus R_R)$ -injective by [8, Theorem 2.2(4)]. We have $\text{Soc}(R_R \oplus R_R) = \text{Soc}(R_R) \oplus \text{Soc}(R_R)$ is finitely generated. Therefore $R_R \oplus R_R$ is soc- $(R_R \oplus R_R)$ -injective by [8, Theorem 2.10]. Since $\text{Soc}(R_R \oplus R_R)$ is essential in $R_R \oplus R_R$, then $R_R \oplus R_R$ is self-ejective by [3, Corollary 2.5(iii)]. It follows that for each decomposition $R_R \oplus R_R = A \oplus B$, A and B are mutually ejective by Lemma 3.11(2). It shows that $R_R \oplus R_R$ is Goldie ADS by Lemma 3.6. On the other hand, $R_R \oplus R_R$ is not ADS. Indeed, if $R_R \oplus R_R$ is ADS, then R must be self-injective, a contradiction.

Example 3.2. Let R be a triangle matrix ring over a field K . Then R_R is CS. Note that R_R is non-singular. Since R_R is not a C3-module, R_R is not ADS. It follows that R_R is not Goldie ADS by Corollary 3.7.

Example 3.3 ([14, Example 2.10]). Let K be a field and let $R = K[x, y]/\langle x^2, xy, y^2 \rangle$. Assume that S is any simple injective R -module. Let $M = R \oplus S$. Then M is not a CS-module (since R is indecomposable and not uniform). On the other hand, R, S are relatively injective, and any two decompositions of M are isomorphic (since R and $\text{End}(S)$ are local rings). Hence M is an ADS module.

Let us mention the following equivalent conditions for Goldie ADS modules.

Lemma 3.4. *The following conditions are equivalent for a module M .*

1. M is Goldie ADS.
2. For every decomposition $M = S \oplus T$ of M and every complement T' of S , there exists a submodule D of M and X of M such that $X \leq_e T'$, $X \leq_e D$ and $M = S \oplus D$.

Proof. (1) \Rightarrow (2). Assume that M is Goldie ADS and M has a decomposition $M = S \oplus T$. Let T' be a complement of S . Then there exists a submodule D of M such that $T' \beta D$ and $M = S \oplus D$. Let $X = T' \cap D$. Hence $X \leq_e T'$, $X \leq_e D$.

(2) \Rightarrow (1) is obvious. □

It is well-known that in general the class of extending modules is not closed under direct sums, and this behavior is also carried on by Goldie extending modules and ADS-modules. Finding necessary and sufficient conditions for ensuring the closure of such classes under direct sums has been one of the most important open problems in the theory of extending modules and their generalizations. In the next parts of our work, we shall deal with such a problem for Goldie ADS modules.

In order to obtain when a direct sum of two Goldie ADS modules has the same property, the following concept generalizing relative injectivity will be useful.

Lemma 3.5 ([3, Theorem 2.7]). *Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. Then M_1 is M_2 -ejective if and only if for every $K \leq M$ such that $K \cap M_1 = 0$, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq_e K$.*

In [7, Lemma 3.1], it is shown that an R -module M is ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually injective.

Lemma 3.6. *An R -module M is Goldie ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually ejective.*

Proof. Suppose $M = A \oplus B$ is Goldie ADS. We will show that A is B -ejective. Let K be a submodule of M such that $K \cap A = 0$. So K is contained in a complement, say C , of A . Then, by hypothesis, there exists $D \leq M$ such that $C\beta D$ and $M = A \oplus D$. It is easy to see that $K \cap D \leq_e K$. Thus, we have A is B -ejective by Lemma 3.5.

Conversely, suppose for each decomposition $M = A \oplus B$, A and B are mutually ejective. Let C be a complement of A . By Lemma 3.5, there exists and $D \leq M$ such that $M = A \oplus D$ and $C \cap D \leq_e C$. So, $A \oplus (C \cap D) \leq_e M$. It follows that $C \cap D \leq_e D$. So we are done by Lemma 3.4. □

Let M_1 and M_2 be modules with $Z(M_1) = 0$ and $M = M_1 \oplus M_2$. In [3, Corollary 2.8], it is shown that M_1 is M_2 -injective if and only if M_1 is M_2 -ejective. As the authors pointed out in [3], if $Z(M) = 0$ and M is R -ejective, then M is injective (because of the Baer criterion).

Corollary 3.7. *A non-singular Goldie ADS module is ADS.*

Proof. By Lemma 3.6 and [3, Corollary 2.8]. □

We collect, in the following theorem, some fundamental properties of Goldie ADS modules.

Theorem 3.8. *Assume that M is Goldie ADS. Then the following statements hold.*

1. *Every direct summand of M is Goldie ADS.*
2. *M satisfies (C3) condition on fully invariant summands.*
3. *For any decomposition $M = A \oplus B$ and any $b \in B$, A is bR -ejective.*

Proof. (1) Assume that A is a direct summand of M , i.e., $M = A \oplus B$ for some $B \leq M$. Let $A = A_1 \oplus A_2$ and K be a complement of A_1 in A . Then we have $M = A_1 \oplus (A_2 \oplus B)$. First we show that $K \oplus B$ is a complement of A_1 in M . Let $C \leq M$ such that $K \oplus B \leq C$ and $C \cap A_1 = 0$. Then $K \leq C \cap A$ and $(C \cap A) \cap A_1 = 0$. Since K is a complement of A_1 in A , we can obtain that $K = C \cap A$. It follows that

$$K \oplus B = (C \cap A) \oplus B = C \cap (A \oplus B) = C.$$

Since M is Goldie ADS, there exists a submodule D of M such that $(K \oplus B)\beta D$ and $M = A_1 \oplus D$. Hence $A = A_1 \oplus (D \cap A)$. It is easy to see that $K\beta(D \cap A)$.

(2) Let A and B be fully invariant direct summands of M such that $A \cap B = 0$. We shall show that $A \oplus B$ is a direct summand of M . Write $M = A \oplus A'$ and $M = B \oplus B'$

for some submodules A', B' of M . By Lemma 3.6, A is A' -ejective. Hence there exists $M' \leq M$ such that $M = M' \oplus A$ and $B \cap M' \leq_e B$ by Lemma 3.5. Inasmuch as B is a fully invariant submodule of M , $B = (M' \cap B) \oplus (A \cap B) = M' \cap B$. It follows that $B \leq M'$. Then $M' = B \oplus (M' \cap B')$ and so $M = A \oplus B \oplus (M' \cap B')$.

(3) Suppose M has a decomposition $M = A \oplus B$. By Lemma 3.6, the module A is B -ejective. Let $K = A \oplus bR$ and X be a submodule of K such that $X \cap A = 0$. Since A is B -ejective, there exists $C \leq M$ such that $M = A \oplus C$ and $X \cap C \leq_e X$ by Lemma 3.5. Note that $K = A \oplus (C \cap K)$. It follows that $(C \cap K) \cap X = X \cap C \leq_e X$. Now A is bR -ejective for any $b \in B$ by Lemma 3.5. □

A module M_R is called *Goldie quasi continuous* if M is Goldie extending and satisfies (C3) (see [3]).

Proposition 3.9. *Every Goldie quasi continuous module is Goldie ADS.*

Proof. Assume that M has a decomposition $M = S \oplus T$ and T' is a complement of S in M . Since M is Goldie extending, there exists a direct summand D of M such that $T' \beta D$. Since $T' \cap S = 0$ we have $D \cap S = 0$ by the equivalence relation β . We have $S \oplus (T' \cap D) \leq_e M$ and obtain that $S \oplus D \leq_e M$. So, by (C3) property, we obtain that $M = S \oplus D$. □

In [11], Kuratomi defined the GQC (generalized quasi continuous) modules by using Goldie extending modules. M is said to be *GQC* if for every submodule X_1 and X_2 of M with $X_1 \cap X_2 = 0$ there exists an essential submodules $Y_i \leq_e X_i$ and a decomposition $M = A_1 \oplus A_2$ such that Y_i is a submodule of A_i for $i = 1, 2$. Let $\{M_i : i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M , there exists $\overline{M}_i \leq M_i$ ($i \in I$) such that $M = X \oplus (\oplus_I \overline{M}_i)$. A module M is said to have the *finite internal exchange property* (FIEP) if, any finite direct sum decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$ is exchangeable.

Corollary 3.10. *The following statements are equivalent for a duo module M :*

1. M is Goldie extending and Goldie ADS.
2. M is Goldie quasi continuous.
3. M is GQC with FIEP.

Proof. By Theorem 3.8, Proposition 3.9 and [11, Theorem 3.4]. □

We have the following direct sum decomposition theorem for Goldie extending submodules.

Lemma 3.11 ([11, Proposition 2.1]). *Let M, N, M_i and N_i be modules.*

1. *If N is M_1 -ejective and M_2 -ejective, then N is $M_1 \oplus M_2$ -ejective.*
2. *Let M_1 be a direct summand of M and N_1 a direct summand of N . If M is N -ejective, then M_1 is N_1 -ejective.*
3. *If M_1 and M_2 are N -ejective modules then so is $M_1 \oplus M_2$.*

The proof of the following proposition uses the similar argument as in [14].

Proposition 3.12. *Let $M = \bigoplus_{i=1}^n M_i$ be direct sum of fully invariant submodules M_i 's . Then M is Goldie ADS if and only if each M_i is Goldie ADS and M_i is M_j -ejective for all $i, j = 1, 2, \dots, n$ and $i \neq j$.*

Proof. \implies : By Theorem 3.8 (1), M_i 's are Goldie ADS. Again by Theorem 3.8 (2), $M_i \oplus M_j$, which is a direct summand of M for $i \neq j$, is Goldie ADS. But by Lemma 3.6, M_i is M_j -ejective for $i \neq j$.

\impliedby : Let $M = A \oplus B$. We claim that A is B -ejective. Since each M_i is fully invariant, we can obtain that $M_i = (A \cap M_i) \oplus (B \cap M_i)$ for all $i = 1, 2, \dots, n$ by [2]. It follows that $A = \bigoplus_{i=1}^n (A \cap M_i)$ and $B = \bigoplus_{i=1}^n (B \cap M_i)$. Since M_i is Goldie ADS, $(A \cap M_i)$ is $(B \cap M_i)$ -ejective. Since M_i is M_j -ejective then $(A \cap M_i)$ is $(B \cap M_j)$ -ejective for all $i, j = 1, 2, \dots, n$ by Lemma 3.11(2). It follows that $A \cap M_i$ is B -ejective for all $i = 1, 2, \dots, n$ by Lemma 3.11(1). Thus A is B -ejective by Lemma 3.11(3). \square

$E(-)$ denotes the injective hull for a module.

Theorem 3.13. *The following conditions are equivalent for a module M :*

1. M is Goldie ADS.
2. For every decomposition $M = A \oplus B$, for all $f \in Hom(E(B), E(A))$, there exists $D \leq M$ such that $M = A \oplus D$ and $D\beta X$, where $X = \{b + f(b) \mid b \in B, f(b) \in A\}$.

Proof. (1) \implies (2) We show that $X = \{b + f(b) \mid b \in B, f(b) \in A\}$ is a complement of A in M . First, we note that that $A \cap X = 0$. Let L be a submodule of M such that $L \cap A = 0$ and $X \leq L$. Consider the the natural projections π_A and π_B of M onto A and B , respectively.

Claim: $\pi_A(x) = f\pi_B(x)$ for all $x \in L$. Assume that there exists $x \in L$ such that $(\pi_A - f\pi_B)(x) \neq 0$. Since $A \leq_e E(A)$, there exists $r \in R$ such that $0 \neq (\pi_A - f\pi_B)(xr) \in A$. But $\pi_A(xr) - f\pi_B(xr) = xr - (\pi_B(xr) + f\pi_B(xr)) \in A \cap L = 0$, a contradiction. Thus $\pi_A(x) = f\pi_B(x)$ for all $x \in L$.

Now, let $x \in L$. Hence $x = a + b$, where $a \in A$ and $b \in B$. Then $\pi_A(x) = a$. By the claim, we can obtain $\pi_A(x) = a = f\pi_B(x)$ for all $x \in L$. Therefore, $x = a + b = f\pi_B(x) + b \in X$. It follows that $L = X$. The rest is clear from the definition of Goldie ADS.

(2) \implies (1) Let $M = A \oplus B$, and T be a complement of A in M . Then $T = \{k + f(k) \mid k \in K\}$ for some $K \leq B$ and $f \in Hom(E(B), E(A))$. In fact, let $\pi_B : A \oplus B \rightarrow B$ be the canonical projection. There exists $f : E(B) \rightarrow E(A)$ such that $f\pi_B(t) = t - \pi_B(t)$ for all $t \in T$. Thus $T = \{k + f(k) \mid k \in \pi_B(T)\}$. By (2), there exists $D \leq M$ such that $M = A \oplus D$ and $D\beta X$, where $X = \{b + f(b) \mid b \in B, f(b) \in A\}$. Note that $f(\pi_B(T)) \leq A$ and so $D \cap T \leq_e T$. Now we show that $T \cap D \leq_e D$. Let $d \in D, d \neq 0$. Assume $T \cap dR = 0$. If $(T \oplus dR) \cap A \neq 0$, write $a = c + dr \neq 0$ for some $a \in A, c \in T$ and $r \in R$. We have $c \neq 0$ and obtain that there exists $r' \in R$ such that $cr' \in T \cap D$ and $cr' \neq 0$. Therefore $ar' = cr' + drr' \in A \cap D = 0$, which implies $cr' + drr' = 0$. It follows that $cr' = -drr' \in T \cap dR = 0$ hence $cr' = 0$, a contradiction. Thus $(T \oplus dR) \cap A = 0$ and then $T = T \oplus dR$ by the maximality of

T . It follows that $d = 0$, again a contradiction. Thus we get $T \cap dR \neq 0$. Hence there exists $y \in R$ such that $dy \in T \cap D$ and $dy \neq 0$. So $D \cap T \leq_e D$. \square

4. Goldie ADS rings

A ring R is called a right Goldie ADS ring if R_R is a Goldie ADS module. We start this section with the following ring extension.

Theorem 4.1. *Let M be a $S - R$ -bimodule. Assume that*

$$T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$$

is right Goldie ADS. Then

1. R is right Goldie ADS
2. M_R is Goldie ADS.

Proof. (1) Let $R_R = A \oplus B$, $I \leq A$ and $f : I \rightarrow B$ an R -homomorphism. Let

$$\bar{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \bar{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \text{ and } \bar{I} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

It is easy to see that $\bar{A} \oplus \bar{B}$ is a direct summand of T_T . We define $\theta : \bar{I} \rightarrow \bar{B}$ via

$$\theta \left(\begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & f(r) \end{pmatrix}.$$

Then θ is a T -homomorphism. By the hypothesis, there exists a T -homomorphism $\phi : \bar{A} \rightarrow \bar{B}$ and $\bar{J} \leq_e \bar{I}$ such that $\phi(\bar{j}) = \theta(\bar{j})$ for every $\bar{j} \in \bar{J}$, where

$$\bar{J} = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}.$$

It is clear that ϕ is an R -homomorphism. Let $\iota : A \rightarrow \bar{A}$ via

$$\iota(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$$

and $\pi : \bar{B} \rightarrow B$ via

$$\pi \left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right) = b.$$

Then ι and π are R -homomorphisms. Since $\bar{J} \leq_e \bar{I}$ then $J \leq_e I$. Let say $\bar{f} := \pi\phi\iota$. So

$$\bar{f}(j) = \pi\phi\iota(j) = \pi\phi \left(\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix} \right) = \pi\theta \left(\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix} \right) = \pi \left(\begin{pmatrix} 0 & 0 \\ 0 & f(j) \end{pmatrix} \right) = f(j)$$

for every $j \in J$ so we are done by Lemma 3.6.

(2) Assume that that $M_R = M_1 \oplus M_2$, $N \leq M_1$ and $f : N \rightarrow M_2$ is an R -homomorphism. Let

$$\bar{M}_1 = \begin{pmatrix} S & M_1 \\ 0 & 0 \end{pmatrix}, \bar{M}_2 = \begin{pmatrix} 0 & M_2 \\ 0 & R \end{pmatrix} \text{ and } \bar{N} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $T_T = \bar{M}_1 \oplus \bar{M}_2$. We define $\theta : \bar{N} \rightarrow \bar{M}_2$ via

$$\theta \left(\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & f(n) \\ 0 & 0 \end{pmatrix}.$$

Then θ is a T -homomorphism. By the hypothesis, there exists a T -homomorphism $\phi : \bar{M}_1 \rightarrow \bar{M}_2$ and $\bar{J} \leq_e \bar{N}$ such that $\phi(\bar{j}) = \theta(\bar{j})$ for all $\bar{j} \in \bar{J}$. Then ϕ is an R -homomorphism. Let $\iota : M_1 \rightarrow \bar{M}_1$ via

$$\iota(m) = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$$

and $\pi : \bar{M}_2 \rightarrow M_2$ via

$$\pi \left(\begin{pmatrix} 0 & m \\ 0 & r \end{pmatrix} \right) = m.$$

Then ι and π are R -homomorphisms. Since $\bar{J} \leq_e \bar{N}$ then $J \leq_e N$. Say $\bar{f} := \pi\phi\iota$ so $\bar{f}(j) = f(j)$. □

We recall the following useful lemma proved in [5, Lemma 5].

Lemma 4.2. *Let M be a right R -module, and let L be a submodule of M , where $R = ReR$ for some $e^2 = e \in R$ and $S = eRe$. Then:*

1. L is essential in M if and only if Le is essential in $(Me)_S$;
2. L is a complement in M if and only if Le is a complement in $(Me)_S$;
3. L is a direct summand of M if and only if Le is a direct summand of $(Me)_S$.

Proposition 4.3. *Let M be a right R -module, where $R = ReR$ for some $e^2 = e \in R$ and $S = eRe$. Then:*

1. $(Me)_S$ is a Goldie ADS module if and only if M_R is a Goldie ADS module.
2. $(Re)_S$ is a Goldie ADS if and only if R_R is Goldie ADS.

Proof. (1) Assume that $(Me)_S$ is a Goldie ADS module. Let $M_R = X \oplus Y$ and Z be a submodule of M with $Z \cap X = 0$. Then $Me = Xe \oplus Ye$ and, by Lemma 4.2, $Ze \cap Xe = 0$. Since $(Me)_S$ is a Goldie ADS module, there exists a submodule D of Me such that $Ze \cap D$ is essential in Ze and $Me = Xe \oplus D$. Hence $M_R = X \oplus DR$ and $Z \cap DR$ is essential in Z . They imply that X is Y -ejective by Lemma 3.5, and hence M_R is a Goldie ADS module by Lemma 3.6.

Assume that M_R is a Goldie ADS module. Let $Me = D \oplus T$ and K be a submodule of $(Me)_S$ with $K \cap D = 0$. Then $M = DR \oplus TR$ and, by Lemma 4.2, $KR \cap DR = 0$. Since M_R is a Goldie ADS module, there exists a submodule X of M such that $KR \cap X$ is essential in KR and $M = DR \oplus X$. Hence $Me = (DR)e \oplus XS$ and $(KR)e \cap XeR \leq XS$. Now $DRe = DeRe = D$ and $KRe = KeRe = K$. This implies that $Me = D \oplus XS$ and $K \cap XS$ is essential in K . Thus D is T -ejective by Lemma 3.5, and hence $(Me)_S$ is a Goldie ADS module by Lemma 3.6.

(2) It is a direct consequence of (1). □

Theorem 4.4. *$M_n(R)$ is Goldie ADS if and only if $(\oplus_{i=1}^n R_i)_R$ is Goldie ADS, where each $R_i = R$.*

Proof. It is easy to obtain that $M_n(R) = M_n(R)eM_n(R)$, where e is the matrix unit with 1 in the $(1, 1)$ th position and zero elsewhere. The rest follows by Proposition 4.3. \square

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