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Differential subordinations obtained by using a fractional operator

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Abstract. We investigate several differential subordinations using the fractional operator $\mathbb{D}_{\lambda}^{\nu,n} : \mathcal{A} \to \mathcal{A}$, for $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, introduced in [7].

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1. Introduction

Let $\mathcal{H}(U)$ denote the class of functions which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $k \in \mathbb{N} = \{1, 2, \dots\}$, let

$$\mathcal{H}[a, k] = \{f \in \mathcal{H}(U) : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \dots\},$$

and

$$\mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + a_3 z^3 + \dots, z \in U\}.$$

In [3], the fractional integral operator $D_z^{-\mu}$ of order $\mu, \mu > 0$, for the function $f \in \mathcal{A}$, is defined by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt, z \in U,$$

where the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.

Also, the fractional derivative operator D_z^λ of order $\lambda, \lambda \geq 0$, for the function $f \in \mathcal{A}$, is defined by

$$D_z^\lambda f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int\limits_0^z \frac{f(t)}{(z-t)^\lambda} dt, & 0 \leq \lambda < 1 \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z), & n \leq \lambda < n+1 \end{cases}, n \in \mathbb{N}_0,$$

where the multiplicity of $(z-t)^{-\lambda}$ is understood in a similar way.

In [4] is defined the fractional differintegral operator $\Omega_z^\lambda : \mathcal{A} \to \mathcal{A}, -\infty < \lambda < 2$, by

$$\Omega_z^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), z \in U,$$

where $D_z^\lambda f(z)$ is the fractional integral of order$\lambda, -\infty < \lambda < 0$, and a fractional derivative of order $\lambda, 0 \leq \lambda < 2$.

In [6], the Sălăgean operator $\mathcal{D}^n$ of order n, $n \in \mathbb{N}_0$, for $f \in \mathcal{A}$, is defined by

$$\mathcal{D}^0 f(z) = f(z)$$

$$\mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z)$$

$$\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1} f(z)), n \in \mathbb{N}.$$

In [5], the Ruscheweyh operator $\mathcal{R}^\lambda : \mathcal{A} \to \mathcal{A}$ for $\lambda \geq -1$ is defined by

$$\mathcal{R}^\lambda f(z) = \frac{z}{(1-z)^{1+\lambda}} * f(z), z \in U,$$

where " $*$ " is the Hadamard product or convolution.

For $\lambda \in \mathbb{N}_0$ this operator is defined by

$$\mathcal{R}^\lambda f(z) = \frac{z(z^{\lambda-1} f(z))^\lambda}{\lambda!}, z \in U.$$

In [7], the fractional operator $\mathbb{D}_\lambda^{\nu,n} : \mathcal{A} \to \mathcal{A}$ for $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$, is defined as a composition of fractional differintegral operator, the Sălăgean operator and the Ruscheweyh operator:

$$\mathbb{D}_\lambda^{\nu,n} f(z) = \mathcal{R}^\nu \mathcal{D}^n \Omega_z^\lambda f(z).$$

The series expression of $\mathbb{D}_\lambda^{\nu,n} f(z)$ for $f \in \mathcal{A}$ of the form $f(z) = z + \sum\limits_{k=1}^\infty a_{k+1} z^{k+1}$ is given by

$$\mathbb{D}_\lambda^{\nu,n} f(z) = z + \sum_{k=1}^\infty \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1)^{n+1} a_{k+1} z^{k+1},$$

$-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, z \in U$, where the symbol $(\gamma)_k$ denotes the usual Pochhammer symbol, for $\gamma \in \mathbb{C}$, defined by

$$(\gamma)_k = \begin{cases} 1, k = 0 \\ \gamma(\gamma+1)\ldots(\gamma+k-1), k \in \mathbb{N} \end{cases} = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

Remark 1.1. [7] The fractional operator $\mathbb{D}_0^{\nu,0}$ is precisely the Ruscheweyh derivative operator $\mathcal{R}^\nu$ of order $\nu, \nu > -1$, and $\mathbb{D}_\lambda^{0,0}$ is the fractional differintegral operator Ω_z^λ of order $\lambda, -\infty < \lambda < 2$, while $\mathbb{D}_0^{0,n} = \mathcal{D}^n$ and $\mathbb{D}_\lambda^{1-\lambda,n} = \mathcal{D}^{n+1}$ are the Sălăgean operators, respectively, of order n and n+1, $n \in \mathbb{N}_0$.

Remark 1.2. [7] The operator $\mathbb{D}_\lambda^{\nu,n}$ satisfies the following identity:

$$\mathbb{D}_\lambda^{\nu+1,n} f(z) = \frac{\nu}{\nu+1}\mathbb{D}_\lambda^{\nu,n} f(z) + \frac{1}{\nu+1} z(\mathbb{D}_\lambda^{\nu,n} f(z))', \tag{1.1}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$.

Remark 1.3. [8] The operator $\mathbb{D}_\lambda^{\nu,n}$ satisfies the following identities:

$$\mathbb{D}_\lambda^{\nu,n+1} f(z) = z(\mathbb{D}_\lambda^{\nu,n} f(z))', \tag{1.2}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$, and

$$\mathbb{D}_{\lambda+1}^{\nu,n} f(z) = -\frac{\lambda}{1-\lambda}\mathbb{D}_\lambda^{\nu,n} f(z) + \frac{1}{1-\lambda} z(\mathbb{D}_\lambda^{\nu,n} f(z))', \tag{1.3}$$

where $-\infty < \lambda < 1, \nu > -1, n \in \mathbb{N}_0$.

Definition 1.4. [1, p. 4] *Let $f, F \in \mathcal{H}(U)$. The function f is said to be subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there exists a function $w \in \mathcal{H}(U)$, with $w(0) = 0$ and $\big|w(z)\big| < 1, z \in U$, such that $f(z) = F\big[w(z)\big], z \in U$.*

In order to prove our results we shall need the following lemma.

Lemma 1.5. [2] *Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha z q'(z),$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = q(0) + p_n z^n + \ldots \in \mathcal{H}[q(0), n]$$

and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec q(z),$$

and this result is sharp.

2. Main results

Theorem 2.1. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + \frac{1}{\nu+1} z g'(z), \nu > -1.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu+1,n} f(z)\right)' \prec h(z) \tag{2.1}$$

then

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z).$$

The result is sharp.

Proof. If we denote by

$$p(z) = \left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)',$$

where $p(z) \in \mathcal{H}[1,1]$, then, by (1.1), we get

$$\left(\mathbb{D}_{\lambda}^{\nu+1,n} f(z)\right)' = p(z) + \frac{1}{\nu+1} zp'(z), z \in U. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$p(z) + \frac{1}{\nu+1} zp'(z) \prec g(z) + \frac{1}{\nu+1} zg'(z) \equiv h(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)' \prec g(z).$$

This result is sharp. □

Theorem 2.2. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + \frac{1}{1-\lambda} zg'(z), -\infty < \lambda < 1.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)' \prec h(z) \tag{2.3}$$

then

$$\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)' \prec g(z).$$

The result is sharp.

Proof. If we denote by

$$p(z) = \left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)',$$

where $p(z) \in \mathcal{H}[1,1]$, then, by (1.3), we get

$$\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)' = p(z) + \frac{1}{1-\lambda} zp'(z), z \in U. \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$p(z) + \frac{1}{1-\lambda} zp'(z) \prec g(z) + \frac{1}{1-\lambda} zg'(z) \equiv h(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)' \prec g(z).$$

This result is sharp. □

Theorem 2.3. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), n \in \mathbb{N}_0.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu,n+1} f(z)\right)' \prec h(z) \tag{2.5}$$

then

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z).$$

The result is sharp.

Proof. If we denote by

$$p(z) = \left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)',$$

where $p(z) \in \mathcal{H}[1,1]$, then, by (1.2), we get

$$\left(\mathbb{D}_\lambda^{\nu,n+1} f(z)\right)' = p(z) + zp'(z), z \in U. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$p(z) + zp'(z) \prec g(z) + zg'(z) \equiv h(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z).$$

This result is sharp. □

Theorem 2.4. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec h(z), z \in U \tag{2.7}$$

then

$$\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z} \prec g(z).$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z}, z \in U.$$

Differentiating we obtain

$$p'(z) = \frac{\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)'}{z} - \frac{p(z)}{z}.$$

We get

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' = p(z) + zp'(z).$$

The subordination (2.7) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z} \prec g(z).$$

This result is sharp. □

Theorem 2.5. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right)' \prec h(z), z \in U, \tag{2.8}$$

then

$$\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}.$$

We obtain

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right)' = p(z) + zp'(z).$$

The subordination (2.8) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

This result is sharp. □

Theorem 2.6. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right)' \prec h(z), z \in U, -\infty < \lambda < 1, \tag{2.9}$$

then

$$\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}.$$

We obtain

$$\left(\frac{z\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \right)' = p(z) + zp'(z).$$

The subordination (2.9) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

This result is sharp. □

Theorem 2.7. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \right)' \prec h(z), z \in U, \tag{2.10}$$

then

$$\frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}.$$

We obtain

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \right)' = p(z) + zp'(z).$$

The subordination (2.10) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

This result is sharp. □

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