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Sufficient conditions of boundedness of L-index and analog of Hayman's Theorem for analytic functions in a ball

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Abstract. We generalize some criteria of boundedness of $\mathbf{L}$-index in joint variables for analytic in an unit ball functions. Our propositions give an estimate maximum modulus of the analytic function on a skeleton in polydisc with the larger radii by maximum modulus on a skeleton in the polydisc with the lesser radii. An analog of Hayman's Theorem for the functions is obtained. Also we established a connection between class of analytic in ball functions of bounded l_j-index in every direction $\mathbf{1}_j$, $j \in \{1,\ldots,n\}$ and class of analytic in ball of functions of bounded $\mathbf{L}$-index in joint variables, where $\mathbf{L}(z) = (l_1(z),\ldots,l_n(z))$, $l_j : \mathbb{B}^n \to \mathbb{R}_+$ is continuous function, $\mathbf{1}_j = (0,\ldots,0,\underbrace{1}_{j\text{-th place}},0,\ldots,0) \in \mathbb{R}^n_+$, $z \in \mathbb{C}^n$.

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1. Introduction

Recently, there was introduced a concept of analytic function in a ball in $\mathbb{C}^n$ of bounded $\mathbf{L}$-index in joint variables [8]. We also obtained criterion of boundedness of $\mathbf{L}$-index in joint variables which describes a local behavior of partial derivatives on a skeleton in the polydisc and established other important properties of analytic functions in a ball of bounded $\mathbf{L}$-index in joint variables. Those investigations used an idea of exhaustion of a ball in $\mathbb{C}^n$ by polydiscs.

The presented paper is a continuation of our investigations from [8]. We set the goal to prove new analogues of criteria of boundedness of $\mathbf{L}$-index in joint variables for analytic in a ball functions. Particular, we prove an estimate of maximum modulus on a greater polydisc by maximum modulus on a lesser polydisc (Theorems 3.1, 3.2) and obtain an analog of Hayman's Theorem for analytic functions in a ball of bounded

L-index in joint variables (Theorems 4.1 and 4.2). For entire functions similar propositions were obtained by A. I. Bandura, M. T. Bordulyak, O. B. Skaskiv [4, 5] in a case $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $z \in \mathbb{C}^n$. Also A. I. Bandura, N.V. Petrechko, O. B. Skaskiv [6, 7] deduced same results for analytic in a polydisc functions. Hayman's Theorem and its generalizations for different classes of analytic functions [1, 3, 5, 7, 12, 15, 20, 21] are very important in theory of functions of bounded index. The criterion is helpful [1, 9] to investigate boundedness of index of entire solutions of ordinary or partial differential equations.

Note that the corresponding theorems for entire functions of bounded l-index and of bounded L-index in direction were also applied to investigate infinite products (see bibliography in [21, 1]). Thus, those generalizations for analytic in a ball functions are necessary to study **L**-index in joint variables of analytic solutions of PDE's, its systems and multidimensional counterparts of Blaschke products. At the end of the paper, we present a scheme of application of Hayman's Theorem to study properties of analytic solutions in the unit ball.

2. Main definitions and notations

We need some standard notations. Denote
$\mathbb{R}_+ = (0, +\infty)$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$,
$\mathbf{1}_j = (0, \dots, 0, \underbrace{1}_{j\text{-th place}}, 0, \dots, 0) \in \mathbb{R}^n_+$,
$R = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $|z| = \sqrt{\sum_{j=1}^n |z_j|^2}$.

For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ we will use formal notations without violation of the existence of these expressions
$AB = (a_1 b_1, \cdots, a_n b_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$,
$A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\|A\| = a_1 + \cdots + a_n$,
and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \dots, n\}$; the relation $A \leq B$ is defined similarly. For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ denote $K! = k_1! \cdot \ldots \cdot k_n!$. The polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| < r_j, j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : |z_j - z_j^0| = r_j, j = 1, \dots, n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| \leq r_j, j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n[z^0, R]$. The open ball $\{z \in \mathbb{C}^n : |z - z^0| < r\}$ is denoted by $\mathbb{B}^n(z^0, r)$, its boundary is a sphere $\mathbb{S}^n(z^0, r) = \{z \in \mathbb{C}^n : |z - z^0| = r\}$, the closed ball $\{z \in \mathbb{C}^n : |z - z^0| \leq r\}$ is denoted by $\mathbb{B}^n[z^0, r]$, $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}, 1)$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : |z| < 1\}$.

For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ and the partial derivatives of an analytic in $\mathbb{B}^n$ function $F(z) = F(z_1, \dots, z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \cdots + k_n} F}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{B}^n) \colon \ l_j(z) > \beta/(1 - |z|), \ j \in \{1, \dots, n\}, \tag{2.1}$$

where $\beta > \sqrt{n}$ is a some constant. For a polydisc A.I. Bandura, N.V. Petrechko and O.B. Skaskiv [6, 7] imposed the restriction $(\forall z \in \mathbb{D}^n(\mathbf{0},\mathbf{1}))\colon\ l_j(z) > \beta/(1-|z_j|)$, $j \in \{1,\dots,n\}$. A similar condition is used in one-dimensional case by S.N. Strochyk, M.M. Sheremeta, V.O. Kushnir [22, 14, 21].

Note that if $R \in \mathbb{R}^n_+$, $|R| \le \beta$, $z^0 \in \mathbb{B}^n$ and $z \in \mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]$ then $z \in \mathbb{B}^n$ (see Remark 1 in [8]).

An analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ is said to be of *bounded* $\mathbf{L}$*-index (in joint variables),* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{B}^n$ and for all $J \in \mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}:\ K \in \mathbb{Z}^n_+,\ \|K\| \le n_0\right\}. \tag{2.2}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* F and is denoted by $N(F, \mathbf{L}, \mathbb{B}^n)$ (see [8]). Entire and analytic in polydisc functions of bounded $\mathbf{L}$-index in joint variables are considered in [4, 5, 6, 7, 10, 13, 19, 18, 16, 17].

By $Q(\mathbb{B}^n)$ we denote the class of functions $\mathbf{L}$, which satisfy (2.1) and the following condition

$$(\forall R \in \mathbb{R}^n_+, |R| \le \beta,\ j \in \{1,\dots,n\})\colon\ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty, \tag{2.3}$$

where $\lambda_{1,j}(R) = \inf\limits_{z^0\in\mathbb{B}^n} \inf\left\{l_j(z)/l_j(z^0) : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\},$

$\lambda_{2,j}(R) = \sup\limits_{z^0\in\mathbb{B}^n} \sup\left\{l_j(z)/l_j(z^0) : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}.$

$\Lambda_1(R) = (\lambda_{1,1}(R),\dots,\lambda_{1,n}(R)),\ \Lambda_2(R) = (\lambda_{2,1}(R),\dots,\lambda_{2,n}(R)).$

We need the following results.

Theorem 2.1 ([8])**.** *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$*. An analytic in* $\mathbb{B}^n$ *function* F *has bounded* $\mathbf{L}$*-index in joint variables if and only if for each* $R \in \mathbb{R}^n_+$*,* $|R| \le \beta$*, there exist* $n_0 \in \mathbb{Z}_+$*,* $p_0 > 0$ *such that for every* $z^0 \in \mathbb{B}^n$ *there exists* $K^0 \in \mathbb{Z}^n_+$*,* $\|K^0\| \le n_0$*, and*

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\| \le n_0,\ z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\} \le p_0\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{2.4}$$

Denote $\widetilde{\mathbf{L}}(z) = (\widetilde{l}_1(z),\dots,\widetilde{l}_n(z))$. The notation $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ means that there exist

$$\Theta_1 = (\theta_{1,j},\dots,\theta_{1,n}) \in \mathbb{R}^n_+,\ \Theta_2 = (\theta_{2,j},\dots,\theta_{2,n}) \in \mathbb{R}^n_+$$

such that $\forall z \in \mathbb{B}^n$ $\theta_{1,j}\widetilde{l}_j(z) \le l_j(z) \le \theta_{2,j}\widetilde{l}_j(z)$ for each $j \in \{1,\dots,n\}$.

Theorem 2.2 ([8])**.** *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$, $\mathbf{L} \asymp \widetilde{\mathbf{L}}$, $\beta|\Theta_1| > \sqrt{n}$*. An analytic in* $\mathbb{B}^n$ *function* F *has bounded* $\widetilde{\mathbf{L}}$*-index in joint variables if and only if* F *has bounded* $\mathbf{L}$*-index in joint variables.*

3. Local behaviour of maximum modulus of analytic in ball function

For an analytic in $\mathbb{B}^n$ function F we put

$$M(r, z^0, F) = \max\{|F(z)|\colon z \in \mathbb{T}^n(z^0, r)\},$$

where $z^0 \in \mathbb{B}^n$, $r \in \mathbb{R}^n_+$. Then $M(R, z^0, F) = \max\{|F(z)| \colon z \in \mathbb{D}^n[z^0, R]\}$, because the maximum modulus for an analytic function in a closed polydisc is attained on its skeleton.

The following proposition uses an idea about the possibility of replacing universal quantifier by existential quantifier in sufficient conditions of index boundedness [2]. To prove an analog of Hayman's Theorem we need this theorem which has an independent interest.

Theorem 3.1. *Let $\mathbf{L} \in Q^n$, $F : \mathbb{B}^n \to \mathbb{C}$ be analytic function. If there exist R', $R'' \in \mathbb{R}^n_+$, $R' < R''$, $|R''| < \beta$ and $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{C}^n$*

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \leq p_1 M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) \tag{3.1}$$

then F is of bounded $\mathbf{L}$-index in joint variables.

Proof. At first, we assume that $\mathbf{0} < R' < \mathbf{1} < R''$.
Let $z^0 \in \mathbb{B}^n$ be an arbitrary point. We expand a function F in power series

$$F(z) = \sum_{K \geq \mathbf{0}} b_K (z - z^0)^K = \sum_{k_1, \ldots, k_n \geq 0} b_{k_1, \ldots, k_n} (z_1 - z_1^0)^{k_1} \ldots (z_n - z_n^0)^{k_n}, \tag{3.2}$$

where $b_K = b_{k_1, \ldots, k_n} = \frac{F^{(K)}(z^0)}{K!}$.
Let $\mu(R, z^0, F) = \max\{|b_K| R^K \colon \ K \geq \mathbf{0}\}$ be a maximal term of power series (3.2) and

$$\nu(R) = \nu(R, z^0, F) = (\nu_1^0(R), \ldots, \nu_n^0(R))$$

be a set of indices such that

$$\mu(R, z^0, F) = |b_{\nu(R)}| R^{\nu(R)},$$

$$\|\nu(R)\| = \sum_{j=1}^{n} \nu_j(R) = \max\{\|K\| \colon K \geq \mathbf{0},\ |b_K| R^K = \mu(R, z^0, F)\}.$$

In view of inequality (3.8) we obtain for any $|R| < 1 - |z^0|$,

$$\mu(R, z^0, F) \leq M(R, z^0, F).$$

Then for given R' and R'' with $0 < |R'| < 1 < |R''| < \beta$ we conclude

$$M(R'R, z^0, F) \leq \sum_{k \geq \mathbf{0}} |b_k| (R'R)^k \leq \sum_{k \geq \mathbf{0}} \mu(R, z^0, F)(R')^k$$
$$= \mu(R, z^0, F) \sum_{k \geq \mathbf{0}} (R')^k = \prod_{j=1}^{n} \frac{1}{1 - r_j'} \mu(R, z^0, F).$$

Besides,

$$\begin{aligned}\ln\mu(R,z^0,F) &= \ln\{|b_{\nu(R)}|R^{\nu(R)}\} = \ln\left\{|b_{\nu(R)}|(RR'')^{\nu(R)}\frac{1}{(R'')^{\nu(R)}}\right\}\\ &= \ln\{|b_{\nu(R)}|(RR'')^{\nu(R)}\} + \ln\left\{\frac{1}{(R'')^{\nu(R)}}\right\}\\ &\leq \ln\mu(R''R,z^0,F) - \|\nu(R)\|\ln\min_{1\leq j\leq n} r''_j.\end{aligned}$$

This implies that

$$\begin{aligned}\|\nu(R)\| &\leq \frac{1}{\ln\min_{1\leq j\leq n} r''_j}(\ln\mu(R''R,z^0,F)-\ln\mu(R,z^0,F))\\ &\leq \frac{1}{\ln\min_{1\leq j\leq n} r''_j}\left(\ln M(R''R,z^0,F)-\ln(\prod_{j=1}^n(1-r'_j)M(R'R,z^0,F))\right)\\ &\leq \frac{1}{\ln\min_{1\leq j\leq n} r''_j}\left(\ln M(R''R,z^0,F)-\ln M(R'R,z^0,F)\right) - \frac{\sum_{j=1}^n\ln(1-r'_j)}{\min_{1\leq j\leq n} r''_j}\\ &= \frac{1}{\min_{1\leq j\leq n} r''_j}\ln\frac{M(R''R,z^0,F)}{M(R'R,z^0,F)} - \frac{\sum_{j=1}^n\ln(1-r_j)}{\min_{1\leq j\leq n} r''_j}. \end{aligned}\tag{3.3}$$

Put $R=\frac{\mathbf{1}}{\mathbf{L}(z^0)}$. Now let $N(F,z^0,\mathbf{L})$ be the **L**-index of the function F in joint variables at point z^0 i. e. it is the least integer for which inequality (2.2) holds at point z^0. Clearly that

$$N(F,z^0,\mathbf{L}) \leq \nu\left(\frac{1}{\mathbf{L}(z^0)},z^0,F\right) = \nu(R,z^0,F). \tag{3.4}$$

But

$$M\left(R''/\mathbf{L}(z^0),z^0,F\right) \leq p_1(R',R'')M\left(R'/\mathbf{L}(z^0),z^0,F\right). \tag{3.5}$$

Therefore, from (3.3), (3.4), (3.5) we obtain that $\forall z^0\in\mathbb{B}^n$

$$N(F,z^0,\mathbf{L}) \leq \frac{-\sum_{j=1}^2\ln(1-r'_j)}{\ln\min\{r''_1,r''_2\}} + \frac{\ln p_1(R',R'')}{\ln\min\{r''_1,r''_2\}}.$$

This means that F has bounded **L**-index in joint variables, if $\mathbf{0}<R'<\mathbf{1}<R''$, $|R''|<\beta$.

Now we will prove the theorem for any $\mathbf{0}<R'<R''$, $|R''|<\beta$. From (3.1) with $\mathbf{0}<R_1<R_2$ it follows that

$$\begin{aligned}&\max\left\{|F(z)|: z\in\mathbb{T}^n\left(z^0,\frac{2R''}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z^0)}\right)\right\}\\ &\leq P_1\max\left\{|F(z)|: z\in\mathbb{T}^n\left(z^0,\frac{2R'}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z^0)}\right)\right\}.\end{aligned}$$

Denoting $\widetilde{\mathbf{L}}(z) = \frac{2\mathbf{L}(z)}{R'+R''}$, we obtain

$$\max\left\{|F(z)| : z \in \mathbb{T}^n\left(z^0, \frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\right)\right\}$$
$$\leq P_1 \max\left\{|F(z)| : z \in \mathbb{T}^n\left(z^0, \frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\right)\right\},$$

where $\mathbf{0} < \frac{2R'}{R'+R''} < \mathbf{1} < \frac{2R''}{R'+R''}$. Taking into account the first part of the proof, we conclude that the function F has bounded $\widetilde{\mathbf{L}}$-index in joint variables. By Theorem 2.2, the function F is of bounded $\mathbf{L}$-index in joint variables. □

Also the corresponding necessary conditions are valid.

Theorem 3.2. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. If an analytic in $\mathbb{B}^n$ function F has bounded $\mathbf{L}$-index in joint variables then for any $R', R'' \in \mathbb{R}^n_+$, $R' < R''$, $|R''| \leq \beta$, there exists a number $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{B}^n$ inequality* (3.1) *holds.*

Proof. Let $N(F, \mathbf{L}) = N < +\infty$. Suppose that inequality (3.1) does not hold i.e. there exist R', R'', $0 < |R'| < |R''| < \beta$, such that for each $p_* \geq 1$ and for some $z^0 = z^0(p_*)$

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) > p_* M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right). \tag{3.6}$$

By Theorem 2.1, there exists a number $p_0 = p_0(R'') \geq 1$ such that for every $z^0 \in \mathbb{B}^n$ and some $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \leq N$, (i.e. $n_0 = N$, see proof of necessity of Theorem 2.1 in [8]) one has

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F^{(K^0)}\right) \leq p_0|F^{(K^0)}(z^0)|. \tag{3.7}$$

We put

$$b_1 = p_0\left(\prod_{j=2}^{n}\lambda^N_{2,j}(R'')\right)(N!)^{n-1}\left(\sum_{j=1}^{N}\frac{(N-j)!}{(r''_1)^j}\right)\left(\frac{r''_1 r''_2 \dots r''_n}{r'_1 r'_2 \dots r'_n}\right)^N,$$

$$b_2 = p_0\left(\prod_{j=3}^{n}\lambda^N_{2,j}(R'')\right)(N!)^{n-2}\left(\sum_{j=1}^{N}\frac{(N-j)!}{(r''_2)^j}\right)\left(\frac{r''_2 \dots r''_n}{r'_2 \dots r'_n}\right)^N\left\{1, \frac{1}{(r'_1)^N}\right\},$$

$$\dots$$

$$b_{n-1} = p_0\lambda^N_{2,n}(R')N!\left(\sum_{j=1}^{N}\frac{(N-j)!}{(r''_{n-1})^j}\right)\left(\frac{r''_{n-1}r''_n}{r'_{n-1}r'_n}\right)^N\max\left\{1, \frac{1}{(r'_1 \dots r'_{n-2})^N}\right\},$$

$$b_n = p_0\left(\sum_{j=1}^{N}\frac{(N-j)!}{(r''_n)^j}\right)\left(\frac{r''_n}{r'_n}\right)^N\max\left\{1, \frac{1}{(r'_1 \dots r'_{n-1})^N}\right\}$$

and

$$p_* = (N!)^n p_0\left(\frac{r''_1 r''_2 \dots r''_n}{r'_1 r'_2 \dots r'_n}\right)^N + \sum_{k=1}^{n} b_k + 1.$$

Let $z^0 = z^0(p_*)$ be a point for which inequality (3.6) holds and K^0 be such that (3.7) holds and

$$M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) = |F(z^*)|, \ M\left(\frac{r''}{\mathbf{L}(z^0)}, z^0, F^{(J)}\right) = |F^{(J)}(z_J^*)|$$

for every $J \in \mathbb{Z}_+^n$, $\|J\| \le N$. We apply Cauchy's inequality

$$|F^{(J)}(z^0)| \le J!\left(\frac{\mathbf{L}(z^0)}{R'}\right)^J |F(z^*)| \tag{3.8}$$

for estimate the difference

$$\begin{aligned}
&|F^{(J)}(z_{J,1}^*, z_{J,2}^*, \ldots, z_{J,n}^*) - F^{(J)}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*)| \\
&= \left|\int_{z_1^0}^{z_{J,1}^*} \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1}\partial z_2^{j_2}\ldots\partial z_n^{j_n}}(\xi, z_{J,2}^*, \ldots, z_{J,n}^*)d\xi\right| \\
&\le \left|\frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1}\partial z_2^{j_2}\ldots\partial z_n^{j_n}}(z^*_{(j_1+1, j_2, \ldots, j_n)})\right| \frac{r_1''}{l_1(z^0)}.
\end{aligned} \tag{3.9}$$

Taking into account $(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*) \in \mathbb{D}^n[z^0, \frac{R''}{\mathbf{L}(z^0)}]$, for all $k \in \{1, \ldots, n\}$,

$$|z_{J,k}^* - z_k^0| = \frac{r_k''}{l_k(z^0)}, \ l_k(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*) \le \lambda_{2,k}(R'')l_k(z^0)$$

and (3.8) with $J = K^0$, by Theorem 2.1 we have

$$\begin{aligned}
&|F^{(J)}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*)| \\
&\le \frac{J! l_1^{j_1}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*) \prod_{k=2}^n l_k^{j_k}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*)}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 |F^{(K^0)}(z^0)| \\
&\le \frac{J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 K^0! \left(\frac{\mathbf{L}(z^0)}{R'}\right)^{K^0} |F(z^*)| \\
&= \frac{p_0 J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{(R')^{K^0}} |F(z^*)|.
\end{aligned} \tag{3.10}$$

From inequalities (3.9) and (3.10) it follows that

$$\begin{aligned}
&\left|\frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1}\partial z_2^{j_2}\ldots\partial z_n^{j_n}}(z^*_{(j_1+1, j_2, \ldots, j_n)})\right| \\
&\ge \frac{l_1(z^0)}{r_1''}\left\{|F^{(J)}(z_j^*)| - |F^{(J)}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*)|\right\} \\
&\ge \frac{l_1(z_1^0)}{r_1''}|F^{(J)}(z_j^*)| - \frac{p_0 J! \mathbf{L}^{(j_1+1, j_2, \ldots, j_n)}(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{r_1''(R')^{K^0}} |F(z^*)|.
\end{aligned}$$

Then

$$|F^{(K^0)}(z_{K^0}^*)| \ge \frac{l_1(z^0)}{r_1''}\left|\frac{\partial^{\|K^0\|-1} f}{\partial z_1^{k_1^0-1}\partial z_2^{k_2^0}\ldots\partial z_n^{k_n^0}}(z^*_{(k_1^0-1, k_2^0, \ldots, k_n^0)})\right|$$

$$
\begin{aligned}
&-\frac{p_0(k_1^0-1)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')}{r_1''(R')^{K^0}}|F(z^*)|\\
&\geq \frac{l_1^2(z^0)}{(r_1'')^2}\left|\frac{\partial^{\|K^0\|-2}f}{\partial z_1^{k_1^0-2}\partial z_2^{k_2^0}\dots\partial z_n^{k_n^0}}(z^*_{(k_1^0-2,k_2^0,\dots,k_n^0)})\right|\\
&-\frac{p_0(k_1^0-2)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')}{(r_1'')^2(R')^{K^0}}|F(z^*)|\\
&-\frac{p_0(k_1^0-1)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(r_i'')}{r_1''(R')^{K^0}}|F(z^*)|\\
&\dots\\
&\geq \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}}\left|\frac{\partial^{\|K^0\|-k_1^0}f}{\partial z_2^{k_2^0}\dots\partial z_n^{k_n^0}}(z^*_{(0,k_2^0,\dots,k_n^0)})\right|\\
&-\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}|F(z^*)|\dots\\
&\geq \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}}\frac{l_2^{k_2^0}(z^0)}{(r_2'')^{k_2^0}}\left|\frac{\partial^{\|K^0\|-k_1^0-k_2^0}f}{\partial z_3^{k_3^0}\dots\partial z_n^{k_n^0}}(z^*_{(0,0,k_3^0,\dots,k_n^0)})\right|\\
&-\frac{l_1^{k_1^0}(z^0)p_0L^{(0,k_2^0,\dots,k_n^0)}(z^0)}{(r_1'')^{k_1^0}(R')^{K^0}}\left(\prod_{i=3}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_3^0!\dots k_n^0!\sum_{i_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r_2'')^{j_2}}|F(z^*)|\\
&-\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}|F(z^*)|\\
&\dots\\
&\geq \left(\frac{L(z^0)}{R''}\right)|F(z^*_{\mathbf{0}})|-|F(z^*)|\sum_{i=1}^b\tilde b_i, \qquad (3.11)
\end{aligned}
$$

where in view of the inequalities $\lambda_{2,i}(R'')\geq 1$ and $R''\geq R'$ we have

$$
\tilde b_1=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}
$$

$$
=\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left(\frac{R''}{R'}\right)^{K^0}p_0\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}\leq\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}b_1,
$$

$$
\tilde b_2=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=3}^n\lambda_{2,i}^{k_i^0}(R'')\right)\frac{k_3^0!\dots k_n^0!}{(r_1'')^{k_1^0}}\sum_{j_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r_2'')^{j_2}}\leq\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}b_2,
$$

$$\dots$$

$$\tilde{b}_{n-1} = \frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\lambda_{2,n}^{k_n^0}(R'')\frac{k_n^0!}{(r_1'')^{k_1^0}\dots(r_{n-2}'')^{k_{n-2}^0}}\times$$
$$\times\sum_{j_{n-1}=1}^{k_{n-1}^0}\frac{(k_{n-1}^0-j_{n-1})!}{(r_{n-1}'')^{j_{n-1}}} \le \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_{n-1},$$

$$\tilde{b}_n = \frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\frac{1}{(r_1'')^{k_1^0}\dots(r_{n-1}'')^{k_{n-1}^0}}\sum_{j_n=1}^{k_n^0}\frac{(k_n^0-j_n)!}{(r_n'')^{j_n}} \le \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_n.$$

Thus, (3.11) implies that

$$|F^{(K^0)}(z_{K^0}^*)| \ge \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{\frac{|F(z_{\mathbf{0}}^*)|}{|F(z^*)|}-\sum_{j=1}^n b_j\right\}.$$

But in view of (3.6) and a choice of p_* we have

$$\frac{|F(z_{\mathbf{0}}^*)|}{|F(z^*)|} \ge p_* > \sum_{j=1}^n b_j.$$

Thus, (3.7) and (3.8) imply

$$|F^{(K^0)}(z_{K^0}^*)| \ge \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{p_*-\sum_{j=1}^n b_j\right\}$$
$$\ge \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left\{p_*-\sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z^0)|(R')^{K^0}}{K^0!\mathbf{L}^{K^0}(z^0)}$$
$$\ge \left(\frac{r_1'\dots r_n'}{r_1''\dots r_n''}\right)^N\left\{p_*-\sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z_{K^0}^*)|}{p_0(n!)^n}.$$

Hence, we have $p_* \le p_0\left(\frac{r_1'\dots r_n'}{r_1''\dots r_n''}\right)^N (N!)^n + \sum_{j=1}^n b_j$, but this contradicts the choice of p_*. □

4. Analogue of Theorem of Hayman for analytic in a ball function of bounded L-index in joint variables

Theorem 4.1. *Let $\mathbf{L}\in Q(\mathbb{B}^n)$. An analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $p\in\mathbb{Z}_+$ and $c\in\mathbb{R}_+$ such that for each $z\in\mathbb{B}^n$*

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} :\ \|J\|=p+1\right\} \le c\cdot\max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} :\ \|K\|\le p\right\}. \tag{4.1}$$

Proof. Let $N=N(F,\mathbf{L},\mathbb{B}^n)<+\infty$. The definition of the boundedness of **L**-index in joint variables yields the necessity with $p=N$ and $c=((N+1)!)^n$.

We prove the sufficiency. For $F\equiv 0$ theorem is obvious. Thus, we suppose that $F\not\equiv 0$. Denote $\boldsymbol{\beta}=(\frac{\beta}{\sqrt{n}},\dots,\frac{\beta}{\sqrt{n}})$.

Assume that (4.1) holds, $z^0 \in \mathbb{B}^n$, $z \in \mathbb{D}^n[z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}]$. For all $J \in \mathbb{Z}^n_+$, $\|J\| \le p+1$, one has

$$\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z^0)} \le \Lambda_2^J(\boldsymbol{\beta}) \frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} \le c \cdot \Lambda_2^J(\boldsymbol{\beta}) \max\left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \ \|K\| \le p \right\}$$
$$\le c \cdot \Lambda_2^J(\boldsymbol{\beta}) \max\left\{ \Lambda_1^{-K}(2) \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)} : \|K\| \le p \right\} \le BG(z), \tag{4.2}$$

where $B = c \cdot \max\{\Lambda_2^K(\boldsymbol{\beta}) : \ \|K\| = p+1\} \max\{\Lambda_1^{-K}(\boldsymbol{\beta}) : \ \|K\| \le p\}$, and

$$G(z) = \max\left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)} : \ \|K\| \le p \right\}.$$

We choose

$$z^{(1)} = (z_1^{(1)}, \dots, z_n^{(1)}) \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$$

and

$$z^{(2)} = (z_1^{(2)}, \dots, z_n^{(2)}) \in \mathbb{T}^n(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)})$$

such that $F(z^{(1)}) \ne 0$ and

$$|F(z^{(2)})| = M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F \right) \ne 0. \tag{4.3}$$

These points exist, otherwise if $F(z) \equiv 0$ on skeleton

$$\mathbb{T}^n\left(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}\right) \quad \text{or} \quad \mathbb{T}^n\left(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right)$$

then by the uniqueness theorem $F \equiv 0$ in $\mathbb{B}^n$. We connect the points $z^{(1)}$ and $z^{(2)}$ with plane

$$\alpha = \begin{cases} z_2 = k_2 z_1 + c_2, \\ z_3 = k_3 z_1 + c_3, \\ \qquad \dots \\ z_n = k_n z_1 + c_n, \end{cases}$$

where

$$k_i = \frac{z_i^{(2)} - z_i^{(1)}}{z_1^{(2)} - z_1^{(1)}}, \ c_i = \frac{z_i^{(1)} z_1^{(2)} - z_i^{(2)} z_1^{(1)}}{z_1^{(2)} - z_1^{(1)}}, \ i = 2, \dots, n.$$

It is easy to check that $z^{(1)} \in \alpha$ and $z^{(2)} \in \alpha$. Let $\widetilde{G}(z_1) = G(z)|_\alpha$ be a restriction of the function G onto α.

For every $K \in \mathbb{Z}^n_+$ the function $F^{(K)}(z)\big|_\alpha$ is analytic function of variable z_1 and $\tilde{G}(z_1^{(1)}) = G(z^{(1)})\big|_\alpha \ne 0$ because $F(z^{(1)}) \ne 0$. Hence, all zeros of the function $F^{(K)}(z)\big|_\alpha$ are isolated as zeros of a function of one variable. Thus, zeros of the function $\tilde{G}(z_1)$ are isolated too. Therefore, we can choose piecewise analytic curve γ onto α as following

$$z = z(t) = (z_1(t), k_2 z_1(t) + c_2, \dots, k_n z_1(t) + c_n), \ t \in [0, 1],$$

which connect the points $z^{(1)}$, $z^{(2)}$ and such that $G(z(t)) \neq 0$ and

$$\int_0^1 |z_1'(t)|dt \le \frac{2\beta}{\sqrt{n} l_1(z_1^0)}.$$

For a construction of the curve we connect $z_1^{(1)}$ and $z_1^{(2)}$ by a line

$$z_1^*(t) = (z_1^{(2)} - z_1^{(1)})t + z_1^{(1)}, \ t \in [0,1].$$

The curve γ can cross points z_1 at which the function $\widetilde{G}(z_1) = 0$. The number of such points $m = m(z^{(1)}, z^{(2)})$ is finite. Let $(z_{1,k}^*)$ be a sequence of these points in ascending order of the value $|z_1^{(1)} - z_{1,k}^*|$, $k \in \{1, 2, ..., m\}$. We choose

$$r < \min_{1 \le k \le m-1} \{|z_{1,k}^* - z_{1,k+1}^*|, |z_{1,1}^* - z_1^{(1)}|, |z_{1,m}^* - z_1^{(2)}|, \frac{2\beta^2 - 1}{2\pi\sqrt{n}\beta l_1(z^0)}\}.$$

Now we construct circles with centers at the points $z_{1,k}^*$ and corresponding radii $r_k' < \frac{r}{2^k}$ such that $\widetilde{G}(z_1) \neq 0$ for all z_1 on the circles. It is possible, because $F \not\equiv 0$.

Every such circle is divided onto two semicircles by the line $z_1^*(t)$. The required piecewise-analytic curve consists with arcs of the constructed semicircles and segments of line $z_1^*(t)$, which connect the arcs in series between themselves or with the points $z_1^{(1)}, z_1^{(2)}$. The length of $z_1(t)$ in $\mathbb{C}$ (but not $z(t)$ in $\mathbb{C}^n$!) is lesser than

$$\frac{\beta/\sqrt{n}}{l_1(z^0)} + \frac{1}{2\sqrt{n}\beta l_1(z^0)} + \pi r \le \frac{2\beta}{\sqrt{n} l_1(z^0)}.$$

Then

$$\int_0^1 |z_s'(t)|dt = |k_s| \int_0^1 |z_1'(t)|dt \le \frac{|z_s^{(2)} - z_s^{(1)}|}{|z_1^{(2)} - z_1^{(1)}|} \frac{2\beta}{\sqrt{n} l_1(z^0)}$$

$$\le \frac{2\beta^2 + 1}{2\sqrt{n}\beta l_s(z^0)} \frac{2\sqrt{n}\beta l_1(z^0)}{2\beta^2 - 1} \frac{2\beta}{\sqrt{n} l_1(z^0)} \le \frac{2\beta(2\beta^2 + 1)}{(2\beta^2 - 1)\sqrt{n} l_s(z^0)}, \ s \in \{2, \dots, n\}.$$

Hence,

$$\int_0^1 \sum_{s=1}^n l_s(z^0)|z_s'(t)|dt \le \frac{2\beta(2\beta^2 + 1)\sqrt{n}}{2\beta^2 - 1} = S. \tag{4.4}$$

Since the function $z = z(t)$ is piece-wise analytic on $[0,1]$, then for arbitrary $K \in \mathbb{Z}_+^n$, $J \in \mathbb{Z}_+^n$, $\|K\| \le p$, either

$$\frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)} \equiv \frac{|F(J)(z(t))|}{\mathbf{L}^J(z^0)}, \tag{4.5}$$

or the equality

$$\frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)} = \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)} \tag{4.6}$$

holds only for a finite set of points $t_k \in [0;1]$.

Then for function $G(z(t))$ as maximum of such expressions $\frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ by all $\|J\| \le p$ two cases are possible:

1. In some interval of analyticity of the curve γ the function $G(z(t))$ identically equals simultaneously to some derivatives, that is (4.5) holds. It means that $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some J, $\|J\| \leq p$. Clearly, the function $F^{(J)}(z(t))$ is analytic. Then $|F^{(J)}(z(t))|$ is continuously differentiable function on the interval of analyticity except points where this partial derivative equals zero $|F^{(j_1,j_2)}(z_1(t), z_2(t))| = 0$. However, there are not the points, because in the opposite case $G(z(t)) = 0$. But it contradicts the construction of the curve γ.
2. In some interval of analyticity of the curve γ the function $G(z(t))$ equals simultaneously to some derivatives at a finite number of points t_k, that is (4.6) holds. Then the points t_k divide interval of analyticity onto a finite number of segments, in which of them $G(z(t))$ equals to one from the partial derivatives, i. e. $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some J, $\|J\| \leq p$. As above, in each from these segments the functions $|F^{(J)}(z(t))|$, and $G(z(t))$ are continuously differentiable except the points t_k.

The inequality

$$\frac{d}{dt}|f(t)| \leq \left|\frac{df(t)}{dt}\right|$$

holds for complex-valued functions of real argument outside a countable set of points. In view of this fact and (4.2) we have

$$\frac{d}{dt}G(z(t)) \leq \max\Big\{\frac{1}{\mathbf{L}^J(z^0)}\Big|\frac{d}{dt}F^{(J)}(z(t))\Big| : \ \|J\| \leq p\Big\}$$
$$\leq \max\Big\{\sum_{s=1}^{n}\Big|\frac{\partial^{\|J\|+1}F}{\partial z_1^{j_1}\dots\partial z_s^{j_s+1}\dots\partial z_n^{j_n}}(z(t))\Big|\frac{|z_s'(t)|}{\mathbf{L}^j(z^0)} : \ \|J\| \leq p\Big\}$$
$$\leq \max\Big\{\sum_{s=1}^{n}\Big|\frac{\partial^{\|J\|+1}F}{\partial z_1^{j_1}\dots\partial z_s^{j_s+1}\dots\partial z_n^{j_n}}(z(t))\Big|\frac{l_s(z^0)|z_s'(t)|}{l_1^{j_1}(z^0)\dots l_s^{j_1+1}(z^0)\dots l_n^{j_n}(z^0)} :$$
$$\|J\| \leq p\Big\} \leq \Big(\sum_{s=1}^{n} l_s(z^0)|z_s'(t)|\Big)\max\Big\{\frac{|F^{(j)}(z(t))|}{\mathbf{L}^J(z^0)} : \ \|J\| \leq p+1\Big\}$$
$$\leq \Big(\sum_{s=1}^{n} l_s(z^0)|z_s'(t)|\Big)BG(z(t)).$$

Therefore, (4.4) yields

$$\Big|\ln\frac{G(z^{(2)})}{G(z^{(1)})}\Big| = \Big|\int_0^1 \frac{1}{G(z(t))}\frac{d}{dt}G(z(t))dt\Big| \leq B\int_0^1 \sum_{s=1}^{n} l_s(z^0)|z_s'(t)|dt \leq S\cdot B.$$

Using (4.3), we deduce

$$M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F\right) \leq G(z^{(2)}) \leq G(z^{(1)})e^{SB}.$$

Since $z^{(1)} \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$, the Cauchy inequality holds

$$\frac{|F^{(J)}(z^{(1)})|}{\mathbf{L}^J(z^0)} \le J!(2\beta\sqrt{n})^{\|J\|} M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right).$$

for all $J \in \mathbb{Z}_+^n$. Therefore, for $\|J\| \le p$ we obtain

$$G(z^{(1)}) \le (p!)^n (2\beta\sqrt{n})^p M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right),$$

$$M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F\right) \le e^{SB}(p!)^n (2\beta\sqrt{n})^p M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right).$$

Hence, by Theorem 3.1 the function F has bounded **L**-index in joint variables. □

The following result was also obtained for other classes of holomorphic functions in [21, 11, 7].

Theorem 4.2. *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$*. An analytic function* F *in* $\mathbb{B}^n$ *has bounded* **L***-index in joint variables if and only if there exist* $c \in (0; +\infty)$ *and* $N \in \mathbb{N}$ *such that for each* $z \in \mathbb{B}^n$ *the inequality*

$$\sum_{\|K\|=0}^{N} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \ge c \sum_{\|K\|=N+1}^{\infty} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}. \tag{4.7}$$

Proof. Let $\frac{1}{\beta} < \theta_j < 1$, $j \in \{1, \dots, n\}$, $\Theta = (\theta_1, \dots, \theta_n)$. If the function F has bounded **L**-index in joint variables then by Theorem 2.2 F has bounded $\widetilde{\mathbf{L}}$-index in joint variables, where $\widetilde{\mathbf{L}} = (\widetilde{l}_1(z), \dots, \widetilde{l}_n(z))$, $\widetilde{l}_j(z) = \theta_j l_j(z)$, $j \in \{1, \dots, n\}$. Let $\widetilde{N} = N(F, \widetilde{L}, \mathbb{B}^n)$. Therefore,

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le \widetilde{N}\right\} = \max\left\{\frac{\Theta^K |F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} \colon \|K\| \le \widetilde{N}\right\}$$

$$\ge \prod_{s=1}^n \theta_s^{\widetilde{N}} \max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} \colon \|K\| \le \widetilde{N}\right\} \ge \prod_{s=1}^n \theta_s^{\widetilde{N}} \frac{|F^{(J)}(z)|}{J!\widetilde{\mathbf{L}}^J(z)} = \prod_{s=1}^n \theta_s^{\widetilde{N}-j_s} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}$$

for all $J \ge \mathbf{0}$ and

$$\sum_{\|J\|=\widetilde{N}+1}^{\infty} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^j(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le \widetilde{N}\right\} \sum_{\|J\|=\widetilde{N}+1}^{\infty} \theta_s^{j_s - \widetilde{N}}$$

$$= \prod_{i=1}^n \frac{\theta_s}{1-\theta_s} \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le \widetilde{N}\right\} \le \prod_{i=1}^n \frac{\theta_s}{1-\theta_s} \sum_{\|K\|=0}^{\widetilde{N}} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}.$$

Hence, we obtain (4.7) with $N = \widetilde{N}$ and

$$c = \prod_{i=1}^n \frac{\theta_s}{1-\theta_s}.$$

On the contrary, inequality (4.7) implies

$$\max\left\{\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}\colon \|J\|=N+1\right\}\leq \sum_{\|K\|=N+1}^{\infty}\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\leq\frac{1}{c}\sum_{\|K\|=0}^{N}\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}$$
$$\leq\frac{1}{c}\sum_{i=0}^{N}C_{n+i-1}^{i}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\leq N\right\}$$

and by Theorem 4.1 F is of bounded $\mathbf{L}$-index in joint variables. □

5. Some application for PDE: a scheme

Here we present a scheme of application of Hayman's Theorem to PDE. This is also appilcable in a more general situation.

Let us consider the following system of partial differential equations:

$$\begin{cases}F^{(2,0)}(z_1,z_2)=2\pi z_2\tan(\pi z_1z_2)F^{(1,0)}(z_1,z_2),\\ F^{(0,2)}(z_1,z_2)=2\pi z_1\tan(\pi z_1z_2)F^{(0,1)}(z_1,z_2).\end{cases}$$

Differentiate in variables z_1 and z_2 we deduce

$$\begin{cases}F^{(3,0)}(z_1,z_2)=\frac{2\pi^2z_2^2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+2\pi z_2\tan(\pi z_1z_2)F^{(2,0)}(z_1,z_2),\\ F^{(2,1)}(z_1,z_2)=2\pi\tan(\pi z_1z_2)F^{(1,0)}(z_1,z_2)+\frac{2\pi^2z_1z_2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+\\ \qquad +2\pi z_2\tan(\pi z_1z_2)F^{(1,1)}(z_1,z_2),\\ F^{(1,2)}(z_1,z_2)=2\pi\tan(\pi z_1z_2)F^{(0,1)}(z_1,z_2)+\frac{2\pi^2z_1z_2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+\\ \qquad +2\pi z_1\tan(\pi z_1z_2)F^{(1,1)}(z_1,z_2),\\ F^{(0,3)}(z_1,z_2)=\frac{2\pi^2z_1^2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+2\pi z_1\tan(\pi z_1z_2)F^{(2,0)}(z_1,z_2),\end{cases}\tag{5.1}$$

Let

$$\mathbf{L}(z_1,z_2)=(l_1(z_1,z_2),l_2(z_1,z_2))=\left(\frac{|z_2|+1}{(1-|z|)|\frac12-z_1z_2|},\frac{|z_1|+1}{(1-|z|)|\frac12-z_1z_2|}\right),$$

where $z=(z_1,z_2)$, $|z|=\sqrt{|z_1|^2+|z_2|^2}$. Now we will estimate all third order partial derivatives of the function $F(z_1,z_2)$ by its first and second order partial derivatives. From the first equation of system (5.1) we have for all $z\in\mathbb{B}^2$:

$$\frac{|F^{(3,0)}(z_1,z_2)|}{l_1^3(z_1,z_2)}\leq\frac{2\pi^2|z_2|^2|F^{(1,0)}(z_1,z_2)|}{|\cos^2(\pi z_1z_2)|l_1^3(z_1,z_2)}+2\pi|z_2\tan(\pi z_1z_2)|\frac{|F^{(2,0)}(z_1,z_2)|}{l_1^3(z_1,z_2)}$$
$$\leq\left(\frac{2\pi^2|z_2|^2}{|\cos^2(\pi z_1z_2)|l_1^2(z_1,z_2)}+\frac{2\pi|z_2\tan(\pi z_1z_2)|}{l_1(z_1,z_2)}\right)\max_{j\in\{1,2\}}\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)}\right\}$$
$$\leq\left(2\pi^2\frac{(1-|z|)^2|\frac12-z_1z_2|^2}{|\cos^2(\pi z_1z_2)|}+2\pi|\tan(\pi z_1z_2)|(1-|z|)\left|\frac12-z_1z_2\right|\right)$$

$$\times \max\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)}: j\in\{1,2\}\right\}$$

$$=\left(\frac{(1-|z|)^2|\frac{\pi}{2}-\pi z_1z_2|^2}{|\sin^2(\frac{\pi}{2}-\pi z_1z_2)|}+2|\sin(\pi z_1z_2)|(1-|z|)\frac{|\frac{\pi}{2}-\pi z_1z_2|}{|\sin(\frac{\pi}{2}-\pi z_1z_2)|}\right)$$

$$\times \max\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)}: j\in\{1,2\}\right\} \le C\max\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)}: j\in\{1,2\}\right\}.$$

Similarly, the second equation of system (5.1) yields

$$\frac{|F^{(2,1)}(z_1,z_2)|}{l_1^2(z_1,z_2)l_2(z_1,z_2)}\le\left(\frac{2\pi|\tan(\pi z_1z_2)|}{l_1(z_1,z_2)l_2(z_1,z_2)}+\frac{2\pi^2|z_1z_2|}{|\cos^2(\pi z_1z_2)|l_1(z_1,z_2)l_2(z_1,z_2)}\right)$$
$$\times\frac{|F^{(1,0)}(z_1,z_2)|}{l_1(z_1,z_2)}+\frac{2\pi|z_2\tan(\pi z_1z_2)|}{l_1(z_1,z_2)}\frac{|F^{(1,1)}(z_1,z_2)|}{l_1(z_1,z_2)l_2(z_1,z_2)}$$
$$\le\left(\frac{2\pi|\sin(\pi z_1z_2)|(1-|z|)^2|\frac{1}{2}-z_1z_2|^2}{|\cos(\pi z_1z_2)|}\right.$$
$$\left.+\frac{2\pi^2(1-|z|)^2|\frac{1}{2}-z_1z_2|^2}{|\cos^2(\pi z_1z_2)|}+\frac{2\pi|\sin(\pi z_1z_2)|(1-|z|)|\frac{1}{2}-z_1z_2|}{|\cos(\pi z_1z_2)|}\right)$$
$$\times\max\left\{\frac{|F^{(1,j)}(z_1,z_2)|}{l_1(z_1,z_2)l_2^j(z_1,z_2)}: j\in\{0,1\}\right\}$$
$$\le\left(\frac{2\pi|\sin(\pi z_1z_2)|(1-|z|)^2|\frac{1}{2}-z_1z_2|^2}{|\sin(\frac{\pi}{2}-\pi z_1z_2)|}+\frac{2(1-|z|)^2|\frac{\pi}{2}-\pi z_1z_2|^2}{|\sin^2(\frac{\pi}{2}-\pi z_1z_2)|}\right.$$
$$\left.+\frac{2|\sin(\pi z_1z_2)|(1-|z|)|\frac{\pi}{2}-\pi z_1z_2|}{|\sin(\frac{\pi}{2}-\pi z_1z_2)|}\right)\max\left\{\frac{|F^{(1,j)}(z_1,z_2)|}{l_1(z_1,z_2)l_2^j(z_1,z_2)}: j\in\{0,1\}\right\}$$
$$\le C\max\left\{\frac{|F^{(1,j)}(z_1,z_2)|}{l_1(z_1,z_2)l_2^j(z_1,z_2)}: j\in\{0,1\}\right\}.$$

By analogy, we can prove similar estimates for the third and the fourth equation of system (5.1). Combining all estimates, one has

$$\max\left\{\frac{|F^{(k,3-k)}(z_1,z_2)|}{l_1^k(z_1,z_2)l_2^{3-k}(z_1,z_2)}: k\in\{0,1,2,3\}\right\}$$
$$\le C\max\left\{\frac{|F^{(k,j)}(z_1,z_2)|}{l_1^k(z_1,z_2)l_2^j(z_1,z_2)}: 0\le k+j\le 2\right\}.$$

Hence, by Theorem 4.1 every analytic solution in $\mathbb{B}^2$ of system (5.1) has bounded **L**-index in joint variables with

$$\mathbf{L}(z_1,z_2)=\left(\frac{|z_2|+1}{(1-|z|)|\frac{1}{2}-z_1z_2|},\frac{|z_1|+1}{(1-|z|)|\frac{1}{2}-z_1z_2|}\right).$$

Particularly, the function $F(z_1, z_2) = \tan(\pi z_1 z_2)$ has the bounded $\mathbf{L}$-index in joint variables. Indeed, it is easy to see that the function F is analytic solution in $\mathbb{B}^2$ of system (5.1).

6. Boundedness of l_j-index in every direction $\mathbf{1}_j$

This section shows another application of Theorem 3.1. The boundedness of l_j-index of a function F in every variable z_j, generally speaking, does not imply the boundedness of $\mathbf{L}$-index in joint variables (see example in [4]). But, if F has bounded l_j-index in every direction $\mathbf{1}_j$, $j \in \{1, \dots, n\}$, then F is a function of bounded $\mathbf{L}$-index in joint variables.

Let $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{B}^n$ $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, $\beta > 1$.

For $\eta \in [0, \beta]$, $z \in \mathbb{B}^n$, we define
$\lambda_1^{\mathbf{b}}(z, \eta, L) = \inf\{L(z + t\mathbf{b})/L(z) : |t| \le \frac{\eta}{L(z)}\}$,
$\lambda_1^{\mathbf{b}}(\eta, L) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}^n\}$,
$\lambda_2^{\mathbf{b}}(z, \eta, L) = \sup\{L(z + t\mathbf{b})/L(z) : |t| \le \frac{\eta}{L(z)}\}$,
$\lambda_2^{\mathbf{b}}(\eta, L) = \sup\{\lambda_2^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}^n\}$.
By $Q_{\mathbf{b},\beta}(\mathbb{B}^n)$ we denote the class of all functions L satisfying $\forall \eta \in [0, \beta]$,

$$0 < \lambda_1^{\mathbf{b}}(\eta, L) \le \lambda_2^{\mathbf{b}}(\eta, L) < +\infty.$$

Analytic in $\mathbb{B}^n$ function $F(z)$ is called a function of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ that for every $m \in \mathbb{Z}_+$ and for every $z \in \mathbb{B}^n$ the following inequality is valid

$$\frac{1}{m!L^m(z)} \left| \frac{\partial^m F(z)}{\partial \mathbf{b}^m} \right| \le \max \left\{ \frac{1}{k!L^k(z)} \left| \frac{\partial^k F(z)}{\partial \mathbf{b}^k} \right| : 0 \le k \le m_0 \right\}, \tag{6.1}$$

where

$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z), \frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j, \ \overline{\mathbf{b}}\rangle, \frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}} \Big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}} \Big), \ k \ge 2.$$

The least such integer m_0 is called the *L-index in the direction* $\mathbf{b}$ *of the analytic function F and is denoted by $N_{\mathbf{b}}(F, L) = m_0$*. In the case $n = 1$, $\mathbf{b} = 1$ and $L = l$ we obtain a definition of analytic in an unit disc function of bounded l-index [22, 21].

We need the following theorem.

Theorem 6.1 ([3]). *Let $\beta > 1$, $L \in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. Analytic in $\mathbb{B}^n$ function $F(z)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for any r_1 and any r_2 with $0 < r_1 < r_2 \le \beta$, there exists number $P_1 = P_1(r_1, r_2) \ge 1$ such that for each $z^0 \in \mathbb{B}^n$*

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_1 \max\left\{|F(z^0 + t\mathbf{b})| : \ |t| = \frac{r_1}{L(z_0)}\right\}. \tag{6.2}$$

It is easy to see that if $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ and $\mathbf{L} \in Q(\mathbb{B}^n)$, then

$$l_j \in Q_{\mathbf{1}_j,\beta/\sqrt{n}}(\mathbb{B}^n), \ j \in \{1, \dots, n\}.$$

Theorem 6.2. *Let* $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $\mathbf{L} \in Q(\mathbb{B}^n)$. *If an analytic in* $\mathbb{B}^n$ *function* F *has bounded* l_j*-index in the direction* $\mathbf{1}_j$ *for every* $j \in \{1, \dots, n\}$, *then* F *is of bounded* **L***-index in joint variables.*

Proof. Let F be an analytic in $\mathbb{B}^n$ function of bounded l_j-index in every direction $\mathbf{1}_j$. Then by Theorem 6.1 for every $j \in \{1, \dots, n\}$ and arbitrary $0 < r'_j < 1 < r''_j \le \frac{\beta}{\sqrt{n}}$ there exists a number $p_j = p_j(r', r'')$ such that for every $(z_1, \dots, z_{j-1}, z_j^0, z_{j+1}, \dots, z_n) \in \mathbb{B}^n$,

$$\max\left\{|F(z)|\colon |z_j - z_j^0| = \frac{r''_j}{l_j(z_1, \dots, z_{j-1}, z_j^0, z_{j+1}, \dots, z_n)}\right\} \le p_j(r'_j, r''_j)$$
$$\times \max\left\{|F(z)|\colon |z_j - z_j^0| = \frac{r'_j}{l_j(z_1, \dots, z_{j-1}, z_j^0, z_{j+1}, \dots, z_n)}\right\}. \tag{6.3}$$

Obviously, if for every $j \in \{1, \dots, n\}$ $l_j \in Q_{\mathbf{1}_j, \beta/\sqrt{n}}(\mathbb{B}^n)$ then $\mathbf{L} \in Q(\mathbb{B}^n)$. Let z^0 be an arbitrary point in $\mathbb{B}^n$, and a point $z^* \in \mathbb{T}^n(z^0, \frac{R''}{\mathbf{L}(z^0)})$ is such that

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) = |F(z^*)|.$$

We choose R'' and R' such that $\mathbf{1} < R'' \le (\frac{\beta}{\sqrt{n}}, \dots, \frac{\beta}{\sqrt{n}})$ and $R' < \Lambda_1(R'')$. Then inequality (6.3) implies that

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \le \max\left\{|F(z_1, z_2^*, z_3^*, \dots, z_n^*)|\colon\ |z_1 - z_1^0| = \frac{r''_1}{l_1(z^0)}\right\}$$
$$= \max\left\{|F(z_1, z_2^*, \dots, z_n^*)|\colon |z_1 - z_1^0| = \frac{r''_1}{l_1(z_1^0, z_2^*, \dots, z_n^*)} \frac{l_1(z_1^0, z_2^*, \dots, z_n^*)}{l_1(z^0)}\right\}$$
$$\le \max\left\{|F(z_1, z_2^*, \dots, z_n^*)|\colon\ |z_1 - z_1^0| = \frac{r''_1 \lambda_{2,1}(R'')}{l_1(z_1^0, z_2^*, \dots, z_n^*)}\right\}$$
$$\le p_1(r'_1, r''_1\lambda_{2,1}(R''))\max\left\{|F(z_1, z_2^*, \dots, z_n^*)|\colon |z_1 - z_1^0| = \frac{r'_1}{l_1(z_1^0, z_2^*, \dots, z_n^*)}\right\}$$
$$= p_1(r'_1, r''_1\lambda_{2,1}(R''))$$
$$\times \max\left\{|F(z_1, z_2^*, \dots, z_n^*)|\colon |z_1 - z_1^0| = \frac{r'_1}{l_1(z^0)} \frac{l_1(z^0)}{l_1(z_1^0, z_2^*, \dots, z_n^*)}\right\}$$
$$\le p_1(r'_1, r''_1\lambda_{2,1}(R''))\max\left\{|F(z_1, z_2^*, \dots, z_n^*)|\colon\ |z_1 - z_1^0| = \frac{r'_1}{\lambda_{1,1}(R'')l_1(z^0)}\right\}$$
$$= p_1(r'_1, r''_1\lambda_{2,1}(R''))|F(z_1^{**}, z_2^*, \dots, z_n^*)| \le p_1(r'_1, r''_1\lambda_{2,1}(R''))$$
$$\times \max\left\{|F(z_1^{**}, z_2, z_3^*, \dots, z_n^*)|\colon\ |z_2 - z_2^0| = \frac{r''_2}{l_2(z^0)}\right\} = p_1(r'_1, r''_1\lambda_{2,1}(R''))$$
$$\times \max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r''_2}{l_2(z_1^{**}, z_2^0, \dots, z_n^*)} \frac{l_2(z_1^{**}, z_2^0, \dots, z_n^*)}{l_2(z^0)}\right\}$$
$$\le p_1(r'_1, r''_1\lambda_{2,1}(R''))\max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r''_2\lambda_{2,2}(R'')}{l_2(z_1^{**}, z_2^0, \dots, z_n^*)}\right\}$$

$$\leq \prod_{j=1}^{2} p_j(r'_j, r''_j\lambda_{2,j}(R''))$$

$$\times \max\left\{|F(z_1^{**}, z_2, \ldots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r'_2}{l_2(z_1^{**}, z_2^0, \ldots, z_n^*)}\right\}$$

$$\leq \prod_{j=1}^{2} p_j(r'_j, r''_j\lambda_{2,j}(R''))\max\left\{|F(z_1^{**}, z_2, \ldots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r'_2}{\lambda_{1,2}(R'')l_2(z^0)}\right\}$$

$$= \prod_{j=1}^{2} p_j(r'_j, r''_j\lambda_{2,j}(R''))|F(z_1^{**}, z_2^{**}, z_3^*, \ldots, z_n^*)| \leq \ldots \leq \prod_{j=1}^{n} p_j(r'_j, r''_j\lambda_{2,j}(R''))$$

$$\times \max\left\{|F(z_1, z_2, \ldots, z_n)|\colon\ |z_j - z_j^0| = \frac{r'_j}{\lambda_{1,j}(R'')l_j(z^0)}, j \in \{1, \ldots, n\}\right\}$$

$$= \prod_{j=1}^{n} p_j(r'_j, r''_j\lambda_{2,j}(R''))M\left(\frac{R'}{\Lambda_1(R'')\mathbf{L}(z^0)}, z^0, F\right).$$

Hence, by Theorem 3.1 the function F is of bounded **L**-index in joint variables. □

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