Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 3, 395–399
DOI: 10.24193/subbmath.2018.3.10

A note on the Wang-Zhang and Schwarz inequalities

Sever S. Dragomir

Abstract. In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

Mathematics Subject Classification (2010): 46C05, 26D15.

Keywords: Schwarz inequality, inner products, inequalities for sums.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex inner product space and $x, y \in H$ two nonzero vectors. One can define the *angle* between the vectors x, y either by

$$\Phi_{x,y} = \arccos \left(\frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \|y\|} \right) \text{ or by } \Psi_{x,y} = \arccos \left(\frac{|\langle x, y \rangle|}{\|x\| \|y\|} \right).$$

The function $\Psi_{x,y}$ is a natural metric on complex projective space [6].

In 1969 M. K. Kreĭn [5] obtained the following inequality for angles between two vectors

$$\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{z,y} \tag{1.1}$$

for any $x, y, z \in H \setminus \{0\}$.

By using the representation

$$\Psi_{x,y} = \inf_{\alpha,\beta \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x,\beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x,y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \Phi_{x,\beta y} \tag{1.2}$$

and Kreĭn's inequality (1.1), M. Lin [6] has shown recently that the following triangle inequality is also valid

$$\Psi_{x,y} \leq \Psi_{x,z} + \Psi_{z,y} \tag{1.3}$$

for any $x, y, z \in H \setminus \{0\}$.

The following inequality has been obtained by Wang and Zhang in [9] (see also [11, p. 195])

$$\sqrt{1-\frac{|\langle x,y\rangle|^2}{\|x\|^2\|y\|^2}} \le \sqrt{1-\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}} + \sqrt{1-\frac{|\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}} \tag{1.4}$$

for any $x,y,z \in H\setminus\{0\}$. Using the above notations it can be written as [6]

$$\sin\Psi_{x,y} \le \sin\Psi_{x,z} + \sin\Psi_{z,y} \tag{1.5}$$

for any $x,y,z \in H\setminus\{0\}$. It also provides another triangle type inequality complementing the Kreĭn and Lin inequalities above.

In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

2. Reverse of Schwarz inequality

In the sequel we assume that $(H,\langle\cdot,\cdot\rangle)$ is a complex inner product space. The inequality

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2 \text{ for } x,y\in H \tag{2.1}$$

is well know in the literature as the *Schwarz inequality*. The equality holds in (2.1) iff x and y are linearly dependent.

Theorem 2.1. *Let $x,y,z\in H$ with $\|z\|=1$ and $\alpha,\beta\in\mathbb{C}$, $r,s>0$ such that*

$$\|x-\alpha z\|\le r \text{ and } \|y-\beta z\|\le s. \tag{2.2}$$

Then

$$(0\le)\,\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \le (r\|y\|+s\|x\|)^2. \tag{2.3}$$

Proof. If we multiply (1.4) by $\|x\|\|y\|\|z\|>0$, then we get

$$\begin{aligned}&\|z\|\sqrt{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2}\\ &\le \|y\|\sqrt{\|x\|^2\|z\|^2-|\langle x,z\rangle|^2}+\|x\|\sqrt{\|y\|^2\|z\|^2-|\langle y,z\rangle|^2}\end{aligned} \tag{2.4}$$

for any $x,y,z\in H\setminus\{0\}$.

We observe that, if either $x=0$ or $y=0$, then the inequality (2.4) reduces to an equality.

Let $z\in H$ with $\|z\|=1$, and since (see for instance [2, Lemma 2.4])

$$\|x\|^2-|\langle x,z\rangle|^2=\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|^2 \text{ and } \|y\|^2-|\langle y,z\rangle|^2=\inf_{\mu\in\mathbb{C}}\|y-\mu z\|^2$$

then by (2.4) we have

$$\sqrt{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2}\le\|y\|\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|+\|x\|\inf_{\mu\in\mathbb{C}}\|y-\mu z\|, \tag{2.5}$$

for any $x,y,z\in H$ with $\|z\|=1$.

Since, by (2.2)

$$\inf_{\lambda\in\mathbb{C}} \|x-\lambda z\| \le \|x-\alpha z\| \le r \text{ and } \inf_{\mu\in\mathbb{C}} \|y-\mu z\| \le \|y-\beta z\| \le s,$$

then by (2.5) we obtain the desired result (2.3). □

Corollary 2.2. *Let $x, y, z \in H$ with $\|z\| = 1$ and $\lambda, \Lambda, \gamma, \Gamma \in \mathbb{C}$ with $\lambda \neq \Lambda$, $\gamma \neq \Gamma$ and such that either*

$$\operatorname{Re}\langle \Lambda z - x, x - \lambda z\rangle \ge 0 \textit{ and } \operatorname{Re}\langle \Gamma z - y, y - \gamma z\rangle \ge 0 \tag{2.6}$$

or, equivalently

$$\left\| x - \frac{\lambda+\Lambda}{2} z\right\| \le \frac{1}{2}|\Lambda-\lambda| \textit{ and } \left\| y - \frac{\gamma+\Gamma}{2} z\right\| \le \frac{1}{2}|\Gamma-\gamma|$$

are valid. Then

$$(0\le)\ \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \le \frac{1}{4}\left(|\Lambda-\lambda|\,\|y\| + |\Gamma-\gamma|\,\|x\|\right)^2. \tag{2.7}$$

Proof. Follows by Theorem 2.1 on observing that

$$\operatorname{Re}\langle \Delta e - u, u - \delta e\rangle = \frac{1}{4}|\Delta-\delta|^2 - \left\| u - \frac{\delta+\Delta}{2} e\right\|^2$$

for any $\delta, \Delta \in \mathbb{C}$ with $\delta \neq \Delta$ and $u, e \in H$ with $\|e\| = 1$. □

We give an example for n-tuples of complex numbers.

Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_n)$ be n-tuples of complex numbers, $p = (p_1, ..., p_n)$ a probability distribution, i.e. $p_i > 0$ $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$, with $\sum_{i=1}^n p_i |z_i|^2 = 1$ and $\lambda, \Lambda, \gamma, \Gamma \in \mathbb{C}$ with $\lambda \neq \Lambda$, $\gamma \neq \Gamma$ and such that

$$\operatorname{Re}\left[(\Lambda z_i - x_i)\left(\overline{x}_i - \overline{\lambda}\overline{z}_i\right)\right] \ge 0 \text{ and } \operatorname{Re}\left[(\Gamma z_i - \overline{y}_i)\left(\overline{y}_i - \overline{\gamma z_i}\right)\right] \ge 0$$

or, equivalently

$$\left| x_i - \frac{\lambda+\Lambda}{2} z_i\right| \le \frac{1}{2}|\Lambda-\lambda| \text{ and } \left| y_i - \frac{\gamma+\Gamma}{2} z_i\right| \le \frac{1}{2}|\Gamma-\gamma|$$

for any $i \in \{1, ..., n\}$. Then

$$\sum_{i=1}^n p_i \operatorname{Re}\left[(\Lambda z_i - x_i)\left(\overline{x}_i - \overline{\lambda}\overline{z}_i\right)\right] \ge 0 \text{ and } \sum_{i=1}^n p_i \operatorname{Re}\left[(\Gamma z_i - \overline{y}_i)\left(\overline{y}_i - \overline{\gamma z_i}\right)\right] \ge 0$$

and by applying Corollary 2.2 for the inner product $\langle\cdot,\cdot\rangle_p : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ with

$$\langle x, y\rangle_p = \sum_{i=1}^n p_i x_i \overline{y}_i,$$

we have

$$0 \leq \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \overline{y}_i \right|^2 \tag{2.8}$$
$$\leq \frac{1}{4} \left[|\Lambda - \lambda| \left(\sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} + |\Gamma - \gamma| \left(\sum_{i=1}^{n} p_i |x_i|^2 \right)^{1/2} \right]^2 .$$

If $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$ for any $i \in \{1, ..., n\}$ then by (2.8) we have for any $p = (p_1, ..., p_n)$ a probability distribution that

$$0 \leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2 \tag{2.9}$$
$$\leq \frac{1}{4} \left[(A - a) \left(\sum_{i=1}^{n} p_i b_i^2 \right)^{1/2} + (B - b) \left(\sum_{i=1}^{n} p_i a_i^2 \right)^{1/2} \right]^2 .$$

The interested reader may compare this new result with the classical reverses of Schwarz inequality obtained by Diaz and Metcalf [1], Ozeki [4], G. Pólya and G. Szegö [7], Shisha and Mond [8] and Cassels [10].

For other reverses of Schwarz inequality in complex inner product spaces see the monograph [3] and the references therein.

Acknowledgement. The author would like to thank the anonymous referee for some valuable suggestions that have been implemented in the final version of the manuscript.

References

[1] Diaz, J.B., Metcalf, F.T., *Stronger forms of a class of inequalities of G. Pólya-G. Szegö and L.V. Kantorovich*, Bull. Amer. Math. Soc., **69**(1963), 415-418.

[2] Dragomir, S.S., *Some Grüss type inequalities in inner product spaces*, J. Inequal. Pure Appl. Math., **4**(2003), No. 2, Article 42, 10 pp.

[3] Dragomir, S.S., *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp.

[4] Izumino, S., Pečarić, J., *A weighted version of Ozeki's inequality*, Sci. Math. Japonicae, **56**(2002), no. 3, 511-526.

[5] Kreĭn, M.K., *Angular localization of the spectrum of a multiplicative integral in a Hilbert space*, Funct. Anal. Appl., **3**(1969), 89–90.

[6] Lin, M., *Remarks on Kreĭn's inequality*, The Math. Intelligencer, **34**(2012), no. 1, 3-4.

[7] Pólya, G., Szegö, G., *Problems and Theorems in Analysis*, Volume 1: Series, Integral Calculus, Theory of Functions (in English), translated from german by D. Aeppli, corrected printing of the revised translation of the fourth German edition, Springer Verlag, New York, 1972.

[8] Shisha, O., Mond, B., *Bounds on Differences of Means, Inequalities*, Academic Press Inc., New York, 1967, pp. 293-308.

[9] Wang, B., Zhang, F., *A trace inequality for unitary matrices*, Amer. Math. Monthly, **101**(1994), 453–455.

[10] Watson, G.S., Alpargu, G., Styan, G.P.H., *Some comments on six inequalities associated with the inefficiency of ordinary least squares with one regressor*, Linear Algebra and its Appl., **264**(1997), 13-54.

[11] Zhang, F., *Matrix Theory: Basic Results and Techniques*, Springer-Verlag, New York, 2011

Sever S. Dragomir
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
sever.dragomir@vu.edu.au
http://rgmia.org/dragomir

School of Computer Science & Applied Mathematics
University of the Witwatersrand, Private Bag 3
Johannesburg 2050, South Africa
e-mail: `sever.dragomir@vu.edu.au`