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Integral estimates for a class $\mathcal{B}_n$ of operators

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Abstract. Let $\mathcal{P}_n$ be the class of polynomials of degree at most n. Rahman introduced a class $\mathcal{B}_n$ of operators $\mathcal{B}$ that map $\mathcal{P}_n$ into itself. In this paper, we establish certain integral estimates concerning $\mathcal{B}$-operator, and thereby obtain generalizations as well as improvements of some well known inequalities for polynomials.

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1. Introduction and Statement of Results

Let $\mathcal{P}_n$ be the class of polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n with complex coefficients. For $P \in \mathcal{P}_n$, define

$$\|P\|_0 := exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{i\theta})|d\theta\right\},$$

$$\|P\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \quad 0 < p < \infty,$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|.$$

It is known that if $P \in \mathcal{P}_n$, then

$$\|P'\|_\infty \leq n\|P\|_\infty \tag{1.1}$$

and for $R > 1$,

$$\|P(R\ .)\|_\infty \leq R^n\|P\|_\infty. \tag{1.2}$$

Inequality (1.1) is an immediate consequence of a famous result due to Bernstein on the derivative of a trigonometric polynomial (for reference see [6], [14]), where as

inequality (1.2) is a simple deduction from the maximum modulus principle.
Inequalities (1.1) and (1.2) can be obtained by letting $p \to \infty$ in

$$\|P'\|_p \leq n\|P\|_p, \quad p > 0 \tag{1.3}$$

and

$$\|P(R\ .)\|_p \leq R^n\|P\|_p, \quad R > 1, \quad \text{and} \quad p > 0. \tag{1.4}$$

Inequality (1.3) for $p \geq 1$ is due to Zygmund [18], where as inequality (1.4) is a simple consequence of a result due to Hardy [10]. Arestov [2] proved that (1.3) remains true for $0 < p < 1$ as well.
For the class of polynomials $P \in \mathcal{P}_n$ such that $P(z) \neq 0$ in $|z| < 1$, inequalities (1.1) and (1.2) can be replaced by

$$\|P'\|_\infty \leq \frac{n}{2}\|P\|_\infty \tag{1.5}$$

and

$$\|P(R\ .)\|_\infty \leq \frac{R^n + 1}{2}\|P\|_\infty, \quad R > 1. \tag{1.6}$$

Inequality (1.5) was conjectured by Erdös and later verified by Lax [11], where as Ankeny and Rivilin [1] used (1.5) to prove (1.6).

Inequalities (1.5) and (1.6) can be obtained by letting $p \to \infty$ in

$$\|P'\|_p \leq \frac{n}{\|1 + E_n\|_p}\|P\|_p, \quad p > 0, \tag{1.7}$$

and

$$\|P(R\ .)\|_p \leq \frac{\|E_n(R\ .) + 1\|_p}{\|1 + E_n\|_p}\|P\|_p, \quad p > 0, \tag{1.8}$$

where $E_n(z) := z^n$.

Inequality (1.7) was found out by de Brujin [8] for $p \geq 1$, whereas inequality (1.8) for $p \geq 1$ was proved by Boas and Rahman [7]. Rahman and Schmeisser [13] have shown that inequalities (1.7) and (1.8) remain true for $0 < p < 1$ as well.

As a compact generalisation of inequalities (1.7) and (1.8) , Aziz and Rather [4] proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $p > 0$,

$$\|P(Rz) - \phi(R, r, \alpha, \beta)P(rz)\|_p \leq \frac{C_p}{\|1 + z\|_p}\|P\|_p \tag{1.9}$$

where

$$C_p = \|(R^n + \phi(R, r, \alpha, \beta)r^n)z + (1 + \phi(R, r, \alpha, \beta))\|_p$$

and

$$\phi(R, r, \alpha, \beta) = \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} - \alpha. \tag{1.10}$$

Rahman [14] introduced a class $\mathcal{B}_n$ of operators B that map $P \in \mathcal{P}_n$ into itself. That is, the operator B carries $P \in \mathcal{P}_n$ into

$$B[P](z) := \lambda_o P(z) + \lambda_1\left(\frac{nz}{2}\right)\frac{P'(z)}{1!} + \lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!}, \tag{1.11}$$

where $\lambda_o,\ \lambda_1$and λ_2 are real or complex numbers such that all the zeros of

$$\mathcal{U}(z) := \lambda_o + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2,\ C(n,r) = \frac{n!}{r!(n-r)!}, \tag{1.12}$$

lie in the half plane

$$|z| \leq \left|z - \frac{n}{2}\right|.$$

He observed that if $P \in \mathcal{P}_n$, then for $R > 1$

$$|B[P](R\ .)| \leq R^n|\Lambda|\|P\|_\infty \quad \text{for} \quad |z| = 1 \tag{1.13}$$

where

$$\Lambda = \lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}. \tag{1.14}$$

On the other hand Shah and Liman [15] proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $R \geq 1$

$$|B[P](R\ .)| \leq \frac{1}{2}\{R^n|\Lambda| + |\lambda_o|\}\|P\|_\infty \quad \text{for} \quad |z| = 1. \tag{1.15}$$

While seeking the desired extension of inequality (1.13) to l_p-norm Shah and Liman [16] proved that, if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $R \geq 1$ and $p \geq 1$

$$\|B[P](R\ .)\|_p \leq \frac{\|R^n\Lambda z + \lambda_0\|_p}{\|1+z\|_p}\|P\|_p,$$

where $B \in \mathcal{B}_n$.
Later on Rather and Shah extended the result for $0 < p < 1$ as well.

In this paper, we investigate the dependence of

$$\|B[P](R\ .) + \phi(R,r,\alpha,\beta)B[P](r\ .)\|_p$$

on $\|P\|_p$, where $\phi(R,r,\alpha,\beta)$ is given by (1.10), $\alpha,\beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, $0 \leq p < \infty$ and establish certain generalised integral inequalities. The results obtained will not only generalise but also improve inequalities (1.5) to (1.8) as well. In fact we prove:

2. Main Results

Theorem 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$,then for $\alpha,\beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$,$R > r \geq 1$ and $0 \leq p < \infty$*

$$\|B[P](R\ .) + \phi(R,r,\alpha,\beta)B[P](r\ .)\|_p$$

$$\leq \frac{\|(R^n + \phi(R,r,\alpha,\beta)r^n)\Lambda z + (1+\phi(R,r,\alpha,\beta))\lambda_0\|_p}{\|1+z\|_p}\|P\|_p,$$

where $B \in \mathcal{B}_n,\ \phi(R,r,\alpha,\beta)$ and Λ are as defined by (1.10) and (1.14) respectively. The result is sharp and equality holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

If we assume that $\beta = 0$ so that $\phi(R,r,\alpha,\beta) = -\alpha$, then we get from Theorem 2.1 the following:

Corollary 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$, we have*

$$\|B[P](R\,.) - \alpha B[P](r\,.)\|_p \leq \frac{\|(R^n - \alpha r^n)\Lambda z + (1-\alpha)\lambda_o\|_p}{\|1+z\|_p}\|P\|_p,$$

where $B \in \mathcal{B}_n$ and Λ is defined by (1.14).
The result is best possible and equality holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.
Theorem 2.2. *Suppose $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$. If $\alpha, \beta \in \mathbb{C}$ are such that $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$, then for every $\gamma \in \mathbb{C}$ with $|\gamma| \leq 1$, and $0 \leq p < \infty$, we have*

$$\begin{aligned}&\Bigg\| B[P](R\,.) + \phi(R,r,\alpha,\beta)B[P](r\,.)\\ &+\frac{\gamma}{2}\Bigg\{|(R^n + \phi(R,r,\alpha,\beta)r^n)\Lambda| - |1+\phi(R,r,\alpha,\beta\|\lambda_0|\Bigg\}m\Bigg\|_p\\ &\leq \frac{\|(R^n + \phi(R,r,\alpha,\beta)r^n)\Lambda z + (1+\phi(R,r,\alpha,\beta))\lambda_0\|_p}{\|1+z\|_p}\|P\|_p,\end{aligned}$$

where $m = \min_{|z|=1}|P(z)|$, $B \in \mathcal{B}_n$, $\phi(R,r,\alpha,\beta)$ and Λ are defined in (1.10) *and* (1.14) *respectively.*
The result is sharp and equality holds for $P(z) = az^n + b, |a| = |b| \neq 0$.
If we assume that $\beta = 0$, so that $\phi(R,r,\alpha,\beta) = -\alpha$, then we get from Theorem 2.2 the following:
Corollary 2.2. *Suppose $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$. If $\alpha \in \mathbb{C}$ is such that $|\alpha| \leq 1$, then for every $\gamma \in \mathbb{C}$ with $|\gamma| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$, we have*

$$\begin{aligned}&\|B[P](R\,.) - \alpha B[P](r\,.) + \frac{\gamma}{2}\Big(|(R^n - \alpha r^n)\Lambda| - |1-\alpha\|\lambda_0|\Big)m\|_p\\ &\leq \frac{\|(R^n - \alpha r^n)\Lambda z + (1-\alpha)\lambda_o\|_p}{\|1+z\|_p}\|P\|_p.\end{aligned}$$

In particular if we let $p \to \infty$, we get

$$\begin{aligned}&\left|B[P](R\,.) - \alpha B[P](r\,.) + \frac{\gamma}{2}\Big(|(R^n - \alpha r^n)\Lambda| - |1-\alpha\|\lambda_0|\Big)m\right|\\ &\leq \frac{\|(R^n - \alpha r^n)\Lambda z + (1-\alpha)\lambda_o\|_\infty}{\|1+z\|_\infty}\|P\|_\infty.\end{aligned}$$

Choosing argument of γ suitably we get

$$\begin{aligned}&|B[P](R\,.) - \alpha B[P](r\,.)| + |\gamma|\frac{\|(R^n - \alpha r^n)\Lambda| - |1-\alpha\|\lambda_0\|m}{2}\\ &\leq \frac{\|(R^n - \alpha r^n)\Lambda z + (1-\alpha)\lambda_o\|_\infty}{\|1+z\|_\infty}\|P\|_\infty.\end{aligned}$$

That is for $|z| = 1$,

$$\begin{aligned}&|B[P](R\,.) - \alpha B[P](r\,.)|\\ &\leq \frac{|(R^n - \alpha r^n)\Lambda| + |(1-\alpha)\|\lambda_o|}{2}\|P\|_\infty - |\gamma|\frac{\|(R^n - \alpha r^n)\|\Lambda| - |1-\alpha\|\lambda_0\|m}{2}.\end{aligned}$$

Letting $|\gamma| \to 1$, we get

$$\begin{aligned} &|B[P](R\ .) - \alpha B[P](r\ .)| \\ &\le \frac{|(R^n - \alpha r^n)\Lambda| + |1-\alpha||\lambda_o|}{2} \max_{|z|=1} |P(z)| \\ &- \frac{(|(R^n - \alpha r^n)\Lambda| - |1-\alpha||\lambda_0|)}{2} \min_{|z|=1} |P(z)|. \end{aligned}$$

A result of Shah and Liman [15] and the result of Aziz and Dawood [3] are special cases of Corollary 2.2, when $\alpha = 0$

3. Lemmas

For the proofs of above theorems we need the following lemmas. The first lemma is due to Govil. et. al [9].

Lemma 3.1. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every $R > r \ge 1$ and $|z| = 1$,*

$$|P(Rz)| \ge \left(\frac{R+1}{r+1}\right)^n |P(rz)|.$$

The next Lemma follows from a result of Marden [12, Corollary 18.3, p.65]

Lemma 3.2. *Suppose all the zeros of a polynomial $P \in \mathcal{P}_n$ lie in $|z| \le 1$, then all the zeros of the polynomial $B[P](z)$ also lie in $|z| \le 1$.*

We also need the following Lemma due to Wali. et. al [17]

Lemma 3.3. *Suppose $F(z)$ and $P(z)$ are polynomials of degree n and m $(m \le n)$ respectively, such that on $|z| = 1$,*

$$|P(z)| \le |F(z)|.$$

If all the zeros of $F(z)$ are in $|z| \le 1$, then for arbitrary complex numbers α, β with $|\alpha| \le 1, |\beta| \le 1, R \ge r \ge 1$,

$$\begin{aligned} |B[P](Rz)+&\phi(R,r,\alpha,\beta)B[P](rz) + \lambda_1 \frac{n-m}{2} z\Big\{(P(Rz))' + \phi(R,r,\alpha,\beta)(P(rz))'\Big\} \\ &+ \lambda_2 \frac{n^2-m^2}{8} z^2\{(P(Rz))'' + \phi(R,r,\alpha,\beta)(P(rz))''\}| \\ &\le |B[F](Rz)| + \phi(R,r,\alpha,\beta)|B[F](rz)| \qquad (1 \le |z| < \infty). \end{aligned}$$

Here $\phi(R,r,\alpha,\beta)$ is defined by (1.10) and $B \in \mathcal{B}_n$.
$\lambda_0, \lambda_1, \lambda_2$ are such that $\lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2$ has all zeros in $\mathrm{Re(z)} \le \frac{\mathrm{n}}{4}$, and $v(z) = \lambda_0 + C(m,1)\lambda_1 z + C(m,2)\lambda_2 z^2$ has all zeros in $\mathrm{Re(z)} \le \frac{\mathrm{m}}{4}$.

Lemma 3.4. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for arbitrary real or complex numbers α, β with $|\alpha| \le 1, |\beta| \le 1, R > r \ge 1$ and $|z| \ge 1$*

$$|B[P](R\ .) + \phi(R,r,\alpha,\beta)B[P](r\ .)| \le |B[P^*](R\ .) + \phi(R,r,\alpha,\beta)B[P^*](r\ .)|$$

where $P^*(z) := z^n \overline{P(\frac{1}{\bar{z}})}, \phi(R,r,\alpha,\beta)$ is defined by (1.10) and $B \in \mathcal{B}_n$.

Proof of Lemma 3.4. By hypothesis the polynomial $P(z)$ of degree n does not vanish in $|z| < 1$, therefore all the zeros of polynomial $P^*(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ of degree n lie in

$|z| \leq 1$. Since $|P(z)| = |P^*(z)|$ for $|z| = 1$, therefore applying Lemma 3.3 with $F(z)$ replaced by $P^*(z)$, we get the desired result.
Next, we describe a result of Arestov [2].
For $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{C}^{n+1}$ and $P(z) := \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, we define

$$C_\gamma P(z) := \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator C_γ is said to be admissible if it preserves one of the following properties:

1. P(z) has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$.
2. P(z) has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov may now be stated as follows:
Lemma 3.5. [2, Theorem 4]. *Let $\phi(x) := \psi(\log x)$, where ψ is a convex non decreasing function on R. Then for all $P \in \mathcal{P}_n$ and each admissible operator C_γ*

$$\int_0^{2\pi} \phi(|C_\gamma P(e^{i\theta})|)d\theta \leq \int_0^{2\pi} \phi(C(\gamma, n)|P(e^{i\theta})|)d\theta$$

where $C(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.
In particular Lemma 3.5 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$ and $\phi : x \to \log x$ as well. Therefore, we have for $0 < p < \infty$,

$$\left\{ \int_0^{2\pi} |C_\gamma P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq C(\gamma, n) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}. \tag{3.1}$$

We also need the following lemma which is due to Aziz and Shah [5].
Lemma 3.6. *If A,B,C are non-negative real numbers such that $B + C \leq A$, then for every real number η*

$$|(A - C) + e^{i\eta}(B + C)| \leq |A + e^{i\eta}B|.$$

4. Proofs of Theorems

Proof of Theorem 2.1. Since $P(z)$ does not vanish in $|z| < 1$ and $P^*(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$, therefore by Lemma 3.4 we have

$$|B[P](Rz) + \phi(R, r, \alpha, \beta)B[P](rz)| \leq |B[P^*](Rz) + \phi(R, r, \alpha, \beta)[P^*](rz)|. \tag{4.1}$$

Also

$$P^*(Rz) + \phi(R, r, \alpha, \beta)P^*(rz) = R^n z^n \overline{P\left(\frac{1}{R\overline{z}}\right)} + \phi(R, r, \alpha, \beta) r^n z^n \overline{P\left(\frac{1}{r\overline{z}}\right)},$$

implies

$$B[P^*](Rz) + \phi(R, r, \alpha, \beta)B[P^*](rz)$$

$$= \lambda_o\Big[R^n z^n\overline{P\Big(\frac{1}{R\overline{z}}\Big)} + \phi(R,r,\alpha,\beta)r^n z^n\overline{P\Big(\frac{1}{r\overline{z}}\Big)}\Big] + \lambda_1\Big(\frac{nz}{2}\Big)\Big[nR^n z^{n-1}\overline{P\Big(\frac{1}{R\overline{z}}\Big)}$$
$$-R^{n-1}z^{n-2}\overline{P'\Big(\frac{1}{R\overline{z}}\Big)} + \phi(R,r,\alpha,\beta)\Big\{nr^n z^{n-1}\overline{P\Big(\frac{1}{r\overline{z}}\Big)} - r^{n-1}z^{n-2}\overline{P'\Big(\frac{1}{r\overline{z}}\Big)}\Big\}\Big]$$
$$+\lambda_2\frac{1}{2}\Big(\frac{nz}{2}\Big)^2\Big[n(n-1)R^n z^{n-2}\overline{P\Big(\frac{1}{R\overline{z}}\Big)} - 2(n-1)R^{n-1}z^{n-3}\overline{P'\Big(\frac{1}{R\overline{z}}\Big)}$$
$$+R^{n-2}z^{n-4}\overline{P''\Big(\frac{1}{R\overline{z}}\Big)} + \phi(R,r,\alpha,\beta)\Big\{(n-1)r^n z^{n-2}\overline{P\Big(\frac{1}{r\overline{z}}\Big)}$$
$$-2(n-1)r^{n-1}z^{n-3}\overline{P'\Big(\frac{1}{r\overline{z}}\Big)} + r^{n-2}z^{n-4}\overline{P''\Big(\frac{1}{r\overline{z}}\Big)}\Big\}\Big]$$
$$= \Big(\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}\Big)\Big[R^n\overline{P\Big(\frac{1}{R\overline{z}}\Big)} + \phi(R,r,\alpha,\beta)r^n\overline{P\Big(\frac{1}{r\overline{z}}\Big)}\Big]z^n$$
$$+\Big(\lambda_1\frac{n}{2} + \lambda_2\frac{n^2(n-1)}{4}\Big)\Big[-R^{n-1}\overline{P'\Big(\frac{1}{R\overline{z}}\Big)} - \phi(R,r,\alpha,\beta)r^{n-1}\overline{P'\Big(\frac{1}{r\overline{z}}\Big)}\Big]z^{n-1}$$
$$+\lambda_2\frac{n^2}{8}\Big[R^{n-2}\overline{P''\Big(\frac{1}{R\overline{z}}\Big)} + \phi(R,r,\alpha,\beta)r^{n-2}\overline{P''\Big(\frac{1}{r\overline{z}}\Big)}\Big]z^{n-2}.$$

This gives,

$$(B[P^*](Rz))^* + \phi(R,r,\overline{\alpha},\overline{\beta})(B[P^*](rz))^*$$
$$= \Big(\overline{\lambda_o} + \overline{\lambda_1}\frac{n^2}{2} + \overline{\lambda_2}\frac{n^3(n-1)}{8}\Big)\Big\{R^nP\Big(\frac{z}{R}\Big) + \phi(R,r,\overline{\alpha},\overline{\beta})r^nP\Big(\frac{z}{r}\Big)\Big\}$$
$$-\Big(\overline{\lambda_1}\frac{n}{2} + \overline{\lambda_2}\frac{n^2(n-1)}{4}\Big)\quad\Big\{R^{n-1}P'\Big(\frac{z}{R}\Big) + \phi(R,r,\overline{\alpha},\overline{\beta})r^{n-1}zP'\Big(\frac{z}{r}\Big)\Big\}z$$
$$+\overline{\lambda_2}\frac{n^2}{8}\Big\{R^{n-2}P''\Big(\frac{z}{R}\Big) + \phi(R,r,\overline{\alpha},\overline{\beta})r^{n-2}P''\Big(\frac{z}{r}\Big)\Big\}z^2.$$

This shows for $|z| = 1$

$$|B[P^*](Rz) + \phi(R,r,\alpha,\beta)B[P^*](rz)|$$
$$= |(B[P^*](Rz))^* + \phi(R,r,\overline{\alpha},\overline{\beta})(B[P^*](rz))^*|. \tag{4.2}$$

Using (4.2) in (4.1), we get on $|z| = 1$,

$$|B[P](Rz) + \phi(R,r,\alpha,\beta)B[P](rz)|$$
$$\leq |(B[P^*](Rz))^* + \phi(R,r,\overline{\alpha},\overline{\beta})(B[P^*](rz))^*|. \tag{4.3}$$

Since all the zeros of $P^*(z)$ lie in $|z| < 1$, therefore as shown earlier all the zeros of

$$P^*(Rz) + \phi(R,r,\alpha,\beta)P^*(rz)$$

lie in $|z| < 1$ for all real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$. Hence by Lemma 3.2, all the zeros of

$$B[P^*](Rz) + \phi(R,r,\alpha,\beta)B[P^*](rz)$$

lie in $|z| < 1$. This shows that all the zeros of

$$(B[P^*](Rz))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](rz))^*$$

lie in $|z| > 1$, and therefore

$$\frac{B[P](Rz)) + \phi(R, r, \alpha, \beta)(B[P](rz)}{(B[P^*](Rz))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](rz))^*}$$

is analytic in $|z| \leq 1$. Hence by maximum modulus principle

$$\begin{aligned} &|B[P](Rz) + \phi(R, r, \alpha, \beta)B[P](rz)| \\ &< |(B[P^*](Rz))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](rz))^*| \end{aligned} \tag{4.4}$$

for $|z| < 1$.
A direct application of Rouches theorem shows that

$$\begin{aligned} C_\gamma P(z) &= (B[P](Rz) + \phi(R, r, \alpha, \beta)B[P](rz))e^{i\eta} \\ &\quad +(B[P^*](Rz))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](rz))^* \\ &= \left\{(R^n + \phi(R, r, \alpha, \beta)r^n)\Lambda e^{i\eta} + (1 + \phi(R, r, \overline{\alpha}, \overline{\beta}))\overline{\lambda_0}\right\}a_n z^n \\ &\quad + \ldots + \left\{(R^n + \phi(R, r, \alpha, \beta)r^n)\overline{\Lambda} + e^{i\eta}(1 + \phi(R, r, \overline{\alpha}, \overline{\beta}))\lambda_0\right\}a_0 \end{aligned}$$

does not vanish in $|z| < 1$. Therefore C_γ is an admissible operator. Applying Lemma 3.5, we have, for each $p > 0$ and η real, $R > r \geq 1$,

$$\begin{aligned} &\int_0^{2\pi} \Bigg| \left\{ B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta}) \right\} e^{i\eta} \\ &\qquad +(B[P^*](Re^{i\theta}))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](re^{i\theta}))^* \Bigg|^p d\theta \\ &\leq |(R^n + \phi(R, r, \alpha, \beta)r^n)\Lambda e^{i\eta} + (1 + \phi(R, r, \overline{\alpha}, \overline{\beta}))\overline{\lambda_0}|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \\ &= |(R^n + \phi(R, r, \alpha, \beta)r^n)\Lambda e^{i\eta} + (1 + \phi(R, r, \alpha, \beta))\lambda_0|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \end{aligned} \tag{4.5}$$

Integrating both sides of (4.5) with respect to η from 0 to 2π, we have

$$\begin{aligned} &\int_0^{2\pi}\int_0^{2\pi} \Bigg| \left\{ B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta}) \right\} e^{i\eta} + (B[P^*](Re^{i\theta}))^* \\ &\qquad +\phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](re^{i\theta}))^* \Bigg|^p d\theta d\eta \\ &\leq \int_0^{2\pi} |(R^n + \phi(R, r, \alpha, \beta)r^n)\Lambda e^{i\eta} + (1 + \phi(R, r, \alpha, \beta))\lambda_0|^p d\eta \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \end{aligned} \tag{4.6}$$

Now it can be easily verified that for any real number t, and $s \geq 1$,

$$|s + e^{it}| \geq |1 + e^{it}|.$$

This implies for each $p > 0$

$$\int_0^{2\pi} |s + e^{it}|^p dt \geq \int_0^{2\pi} |1 + e^{it}|^p dt. \tag{4.7}$$

If $B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta}) \neq 0$, then we take

$$s = \frac{|(B[P^*](Re^{i\theta}))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](re^{i\theta}))^*|}{|B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})|}$$

so that by (4.3), $s \geq 1$. Using (4.7) we have

$$\begin{aligned}
&\int_0^{2\pi} \Big|\Big\{B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})\Big\}e^{i\eta} \\
&+(B[P^*](Re^{i\theta}))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](re^{i\theta}))^*)|^p d\eta \\
&= |B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})|^p \int_0^{2\pi} \Big|e^{i\eta} \\
&+ \frac{(B[P^*](Re^{i\theta}))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](re^{i\theta}))^*}{B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})}\Big|^p d\eta \\
&= |B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})|^p \int_0^{2\pi} \Big|e^{i\eta} \\
&+ \Big|\frac{(B[P^*](Re^{i\theta}))^* + \phi(R, r, \overline{\alpha}, \overline{\beta})(B[P^*](re^{i\theta}))^*}{B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})}\Big|\Big| d\eta \\
&\geq |B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})|^p \int_0^{2\pi} |1 + e^{i\eta}|^p d\eta.
\end{aligned} \tag{4.8}$$

For $B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta}) = 0$, this inequality is trivially true. Using (4.7) in (4.8), we conclude that for each $p > 0$

$$\begin{aligned}
&\int_0^{2\pi} |B[P](Re^{i\theta}) + \phi(R, r, \alpha, \beta)B[P](re^{i\theta})|^p d\theta \int_0^{2\pi} |1 + e^{i\eta}|^p d\eta \\
&\leq \int_0^{2\pi} |(R^n + \phi(R, r, \alpha, \beta)r^n)\Lambda e^{i\eta} + (1 + \phi(R, r, \alpha, \beta))\lambda_0|^p d\eta \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.
\end{aligned}$$

From this the conclusion of Theorem 2.1 follows immediately.

Proof of Theorem 2.2. By hypothesis $P \in \mathcal{P}_n$ does not vanish in $|z| < 1$. Let $m = \min_{|z|=1} |P(z)|$, then $|P(z)| \geq m$ for $|z| = 1$. If $m = 0$, then the result follows from Theorem 2.1. We assume that $m > 0$, that is, $P(z)$ has no zero on $|z| = 1$.

This gives for $|\delta| < 1$, $|\delta m z^n| < |P(z)|$ on $|z| = 1$. Since $P(z)$ has no zero in $|z| < 1$, therefore $F(z) = P(z) + \delta m z^n$ has no zero in $|z| < 1$. If $F^*(z) = z^n \overline{F\left(\frac{1}{\overline{z}}\right)}$, then $F^*(z) = P^*(z) + m\overline{\delta}$ and by Lemma 3.4 we have for arbitrary complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1, R \geq r \geq 1$,

$$|B[F](Rz) + \phi(R, r, \alpha, \beta)B[F](rz)| \leq |B[F^*](Rz) + \phi(R, r, \alpha, \beta)B[F^*](rz)|.$$

This gives

$$\begin{aligned} &|B[P(Rz) + \delta m R^n z^n] + \phi(R, r, \alpha, \beta)B[P(rz) + \delta m r^n z^n]| \\ &\leq |B[P^*(Rz) + \overline{\delta} m] + \phi(R, r, \alpha, \beta)B[P^*(rz) + \overline{\delta} m]|. \end{aligned} \tag{4.9}$$

Since $P(z)$ and z^n are of same degree and in this case B is linear, therefore

$$B[P(Rz) + \delta m R^n z^n] = B[P](Rz) + \delta m B[E_n](Rz),$$

where $E_n(z) = z^n$. Also it can be easily verified that

$$B[P^*(Rz) + \overline{\delta} m] = B[P^*](Rz) + \overline{\delta} m \lambda_o \qquad \text{for} \quad |z| = 1.$$

Therefore we have from inequality (4.9)

$$\begin{aligned} &|B[P](Rz) + \delta m B[E_n](Rz) + \phi(R, r, \alpha, \beta)\{B[P](rz) + \delta m B[E_n](rz)\}| \\ &\leq |B[P^*](Rz) + \overline{\delta} m \lambda_o + \phi(R, r, \alpha, \beta)\{B[P^*](rz) + \overline{\delta} m \lambda_o\}|, \quad |z| = 1. \end{aligned}$$

That is

$$\begin{aligned} &|B[P](Rz) + \phi(R, r, \alpha, \beta)B[P](rz) + \delta m\{B[E_n](Rz) + \phi(R, r, \alpha, \beta)B[E_n](rz)\}| \\ &\leq |B[P^*](Rz) + \phi(R, r, \alpha, \beta)B[P^*](rz) + m\overline{\delta}\lambda_o(1 + \phi(R, r, \alpha, \beta))|, \quad |z| = 1. \end{aligned} \tag{4.10}$$

Choosing the argument of δ suitably on the left hand side of (4.10) and using triangle inequality on the right hand side, we get for $|z| = 1$,

$$\begin{aligned} &|B[P](Rz) + \phi(R, r, \alpha, \beta)B[P](rz)| + |\delta| m |B[E_n](Rz) + \phi(R, r, \alpha, \beta)B[E_n](rz)| \\ &\leq |B[P^*](Rz) + \phi(R, r, \alpha, \beta)B[P^*](rz)| + |\delta| m |\lambda_o||1 + \phi(R, r, \alpha, \beta)|. \end{aligned}$$

Equivalently

$$\begin{aligned} &|B[P](Rz) + \phi(R, r, \alpha, \beta)B[P](rz)| \\ &\leq |B[P^*](Rz) + \phi(R, r, \alpha, \beta)B[P^*](rz)| \\ &-m|\delta|\{|B[E_n](Rz) + \phi(R, r, \alpha, \beta)B[E_n](rz)| - |\lambda_o||1 + \phi(R, r, \alpha, \beta)|\}. \end{aligned} \tag{4.11}$$

Since $|B[E_n](Rz)| = R^n|\Lambda|$ for $|z| = 1$, therefore we have from (4.11) after letting $|\delta| \to 1$

$$\begin{aligned} &|B[P](Rz) + \phi(R, r, \alpha, \beta)B[P](rz)| \\ &+\left\{\frac{|R^n\Lambda + \phi(R, r, \alpha, \beta)r^n\Lambda| - |1 + \phi(R, r, \alpha, \beta)||\lambda_o|}{2}\right\}m \\ &\leq |B[P^*](Rz) + \phi(R, r, \alpha, \beta)B[P^*](rz)| \\ &-\left\{\frac{|R^n\Lambda + \phi(R, r, \alpha, \beta)r^n\Lambda| - |1 + \phi(R, r, \alpha, \beta)||\lambda_o|}{2}\right\}m. \end{aligned} \tag{4.12}$$

If we take

$$L = |B[P^*](Rz) + \phi(R,r,\alpha,\beta)B[P^*](rz)|$$
$$M = |B[P](Rz) + \phi(R,r,\alpha,\beta)B[P](rz)|$$
$$N = \left\{\frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |1+\phi(R,r,\alpha,\beta)\|\lambda_o|}{2}\right\}m,$$

so that $M + N \leq L - N \leq L$, we get by using Lemma 3.5 for arbitrary complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1, R \geq r \geq 1$ and $|z| = 1$,

$$\Bigg| |B[P^*](Rz) + \phi(R,r,\alpha,\beta)B[P^*](rz)| - \left\{\frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |1+\phi(R,r,\alpha,\beta)\|\lambda_o|}{2}\right\}m + e^{i\eta}\Bigg[|B[P](Rz) + \phi(R,r,\alpha,\beta)B[P](rz)| + \left\{\frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |1+\phi(R,r,\alpha,\beta)\|\lambda_o|}{2}\right\}m\Bigg]\Bigg|$$
$$\leq \Big||B[P](Rz) + \phi(R,r,\alpha,\beta)B[P](rz)| + e^{i\eta}|B[P^*](Rz) + \phi(R,r,\alpha,\beta)B[P^*](rz)|\Big|$$
$$= \Big||B[P](Rz)+\phi(R,r,\alpha,\beta)B[P](rz)|+e^{i\eta}|(B[P^*](Rz))^*+\phi(R,r,\overline{\alpha},\overline{\beta})(B[P^*](rz))^*|\Big|$$

This gives for each $p > 0$ and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi} |G(\theta) + e^{i\eta}H(\theta)|^P d\theta \leq \int_0^{2\pi} \Big||B[P](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P](re^{i\theta})| + e^{i\eta}|(B[P^*](Re^{i\theta}))^* + \phi(R,r,\overline{\alpha},\overline{\beta})(B[P^*](re^{i\theta})|)^*\Big|^p d\theta, \tag{4.13}$$

where

$$H(\theta) := |B[P](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P](re^{i\theta})| + \left\{\frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |1+\phi(R,r,\alpha,\beta)\|\lambda_o|}{2}\right\}m$$

and

$$G(\theta) := |B[P^*](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P^*](re^{i\theta})| - \left\{\frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |1+\phi(R,r,\alpha,\beta)\|\lambda_o|}{2}\right\}m.$$

Integrating both sides of (4.13) with respect to η from 0 to 2π we get

$$\int_0^{2\pi}\int_0^{2\pi} |G(\theta) + e^{i\eta}H(\theta)|^P d\theta d\eta \leq \int_0^{2\pi}\int_0^{2\pi} \Big||B[P](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P](re^{i\theta})| + e^{i\eta}(|B[P^*](Re^{i\theta}))^* + \phi(R,r,\overline{\alpha},\overline{\beta})(B[P^*](re^{i\theta}))^*|\Big|^p d\theta d\eta.$$

This gives using inequality (4.5), for every real $\theta, 0 \leq \theta < 2\pi$ and η, $0 \leq \eta < 2\pi$

$$\int_0^{2\pi}\int_0^{2\pi} |G(\theta) + e^{i\eta}H(\theta)|^P d\theta d\eta$$

$$\leq \int_0^{2\pi} |(R^n + \phi(R,r,\alpha,\beta)r^n)\Lambda e^{i\eta} + (1+\phi(R,r,\alpha,\beta))\lambda_o|^p d\eta \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \quad (4.14)$$

Now, if $H(\theta) \neq 0$, then from (4.12),

$$t = \frac{|G(\theta)|}{|H(\theta)|} \geq 1$$

and we have by using (4.7),

$$\int_0^{2\pi} |G(\theta) + e^{i\eta}H(\theta)|^P d\eta = |H(\theta)|^p \int_0^{2\pi} \left|\frac{G(\theta)}{H(\theta)} + e^{i\eta}\right|^p d\eta$$

$$= |H(\theta)|^p \int_0^{2\pi} \left|\left|\frac{G(\theta)}{H(\theta)}\right| + e^{i\eta}\right|^p d\eta \geq |H(\theta)|^p \int_0^{2\pi} |1 + e^{i\eta}|^p d\eta. \quad (4.15)$$

Clearly inequality (4.15) is trivial in case $H(\theta) = 0$. Substituting for $H(\theta)$ and $G(\theta)$ and then integrating the two sides of (4.15) with respect to θ and using (4.14), we get for every $R > r \geq 1$ and $p > 0$,

$$\int_0^{2\pi} \Bigg||B[P](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P](re^{i\theta})|$$

$$- \frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |\lambda_o\|1+\phi(R,r,\alpha,\beta)|}{2} m\Bigg|^p d\theta$$

$$\leq \frac{\int_0^{2\pi} |(R^n + \phi(R,r,\alpha,\beta)r^n)e^{i\eta} + (1+\phi(R,r,\alpha,\beta))\lambda_o|^p d\eta}{\int_0^{2\pi} |1+e^{i\eta}|^p d\eta} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \quad (4.16)$$

Now, since we have for every γ, with $|\gamma| < 1$

$$\Bigg|B[P](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P](re^{i\theta})|$$

$$+ m\gamma \frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |\lambda_o\|1+\phi(R,r,\alpha,\beta)|}{2}\Bigg|$$

$$\leq |B[P](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P](re^{i\theta})|$$

$$+ m|\gamma| \left|\frac{|R^n + \phi(R,r,\alpha,\beta)r^n\|\Lambda| - |\lambda_o\|1+\phi(R,r,\alpha,\beta)|}{2}\right|$$

$$\leq |B[P](Re^{i\theta}) + \phi(R,r,\alpha,\beta)B[P](re^{i\theta})|$$

$$+m\left|\frac{|R^n+\phi(R,r,\alpha,\beta)r^n\|\Lambda|-|\lambda_o\|1+\phi(R,r,\alpha,\beta)|}{2}\right|. \tag{4.17}$$

Therefore using (4.17) in (4.16) the desired result follows.

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