

Some new estimates for Fejér type inequalities in quantum analysis

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Abstract. In this paper we derive some new quantum estimates of Fejér type inequalities which involve Riemann type of quantum integrals via some classes of convex functions. We also discuss some special cases which can be deduced from the main results of this paper.

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1. Introduction

A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, I is an interval, is said to be a convex function on I if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \quad (1.1)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in (1.1) holds, then f is said to be concave. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I and $a, b \in I$ with $a < b$. Then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \int_a^b p(x)dx \leq \int_a^b f(x)p(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b p(x)dx, \quad (1.2)$$

where $p : [a, b] \rightarrow \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$. This inequality is known as the Fejér inequality for convex functions (see [2, 3, 16, 17]).

Theory of convexity plays an important role in different fields of pure and applied sciences. Due to its importance in recent years several new generalizations of classical convexity have been proposed in the literature. Breckner [1] introduced the notion of s -convex function, as

Definition 1.1 ([1]). Let $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex in the second sense if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y) \quad (1.3)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$. This class of s -convex functions is usually denoted by K_s^2 .

For more information on s -convex functions, see [4].

In 2007, Varošanec [15] introduced the notion of h -convex functions, which not only generalizes the class of convex functions but also some other classes of convex functions, see [15]. Thus it was noticed that the class of h -convex functions is quite unifying one. This class is defined as:

Definition 1.2 ([15]). Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a positive function and $[0, 1] \subset J$. We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is h -convex function, or that f belong to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \quad (1.4)$$

If inequality (1.4) is reversed, then f is said to be h -concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

In [13], G. H. Toader defined the concept of m -convexity as the following:

Definition 1.3. The function $f : [0, b] \rightarrow \mathbb{R}$ is said to be m -convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \quad (1.5)$$

Denote by $K_m(b)$ the set of the m -convex functions on $[0, b]$.

In [6], V. G. Miheşan introduced the class of (s, m) -convex functions as the following:

Definition 1.4. The function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (s, m) -convex, where $(s, m) \in (0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y). \quad (1.6)$$

Denote by $K_m^s(b)$ the set of the (s, m) -convex functions on $[0, b]$.

In [17], Yang and Tseng established the following theorem

Theorem 1.5 (see [17], **Remark 6**). Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function and $p : [a, b] \rightarrow \mathbb{R}$ be a nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$. If H and F are defined on $[0, 1]$ by

$$H(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x) dx,$$

and

$$F(t) = \int_a^b \frac{1}{2} \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right) \right] dx,$$

then H, F are convex and increasing on $[0, 1]$ and for all $t \in [0, 1]$

$$f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx = H(0) \leq H(t) \leq H(1) = \int_a^b f(x)p(x) dx$$

and

$$\int_a^b f(x)p(x)dx = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2} \int_a^b p(x)dx.$$

In [12], M. Z. Sarikaya, E. Set and M. E. Özdemir established the following inequality:

Theorem 1.6. *Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$, $f \in L_1([a, b])$ and $p : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then, for $h(\frac{1}{2}) \neq 0$, we have*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b p(x)dx &\leq \int_a^b f(x)p(x)dx \\ &\leq \frac{f(a) + f(b)}{2} (h(t) + h(1-t)) \int_a^b p(x)dx. \end{aligned} \tag{1.7}$$

In [14], the following inequalities of Fejér type via s -convex function was derived:

Theorem 1.7. *Let $f \in K_s^2$, $a, b \in [0, \infty[$ with $a < b$ and $p : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\begin{aligned} 2^{s-1} f\left(\frac{a+b}{2}\right) \int_a^b p(x)dx &\leq \int_a^b f(x)p(x)dx \\ &\leq \frac{f(a) + f(b)}{2} \int_a^b \left(\left(\frac{b-x}{b-a}\right)^s + \left(\frac{x-a}{b-a}\right)^s \right) p(x)dx. \end{aligned} \tag{1.8}$$

Again, in [14], the authors proved the following theorems:

Theorem 1.8 ([14]). *Let $f : [0, \infty[\rightarrow \mathbb{R}$ be an m -convex function with $m \in]0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[0, b]$, then*

$$\int_a^b f(x)p(x)dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\} \int_a^b p(x)dx.$$

Theorem 1.9 ([14]). *Let $f : [0, \infty[\rightarrow \mathbb{R}$ be an m -convex function with $m \in]0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[0, b]$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b p(x)dx &\leq \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} p(x)dx \\ &\leq \frac{1}{8} \left(f(a) + f(b) + 2m \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) + m^2 \left(f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right) \right) \right) \int_a^b p(x)dx. \end{aligned}$$

The aim of this work is to establish the q -analogue of Fejér inequalities for some convex type functions. For this we recall some basic concepts of quantum calculus. Let $0 < q < 1$, the q -Jackson integral from 0 to b is defined by [5] as:

$$\int_0^b f(x)d_q x = (1-q)b \sum_{n=0}^{\infty} f(bq^n)q^n \tag{1.9}$$

provided the sum converge absolutely.

The q -Jackson integral in a generic interval $[a, b]$ is given by [5]

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \quad (1.10)$$

In [10], the authors presented a Riemann-type q -integral by:

$$R_q(f; a, b) = (b - a)(1 - q) \sum_{k=0}^{\infty} f(a + (b - a)q^k) q^k. \quad (1.11)$$

We can get another definition from the Riemman-type q -integral:

$$\begin{aligned} & \frac{2}{b-a} \int_a^b f(x) d_q^R x \\ &= (1-q) \sum_{k=0}^{\infty} \left(f\left(\frac{a+b}{2} + q^k \left(\frac{b-a}{2}\right)\right) + f\left(\frac{a+b}{2} - q^k \left(\frac{b-a}{2}\right)\right) \right) q^k \end{aligned}$$

From the q -Jackson integral we can write:

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x) d_q^R x &= \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t \\ &= \int_{-1}^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) d_q t. \end{aligned} \quad (1.12)$$

Contrary to the q -Jackson integral, if

$$f(x) \leq g(x), \quad x \in [a, b]$$

then

$$\int_a^b f(x) d_q^R x \leq \int_a^b g(x) d_q^R x. \quad (1.13)$$

In [11], the authors established the q -analogue of Hermite-Hadamard inequalities for convex function

Theorem 1.10. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) d_q^R t \leq \frac{f(a) + f(b)}{2}.$$

For some recent studies on quantum integral inequalities, see [7, 8, 9].

2. Main results

In this section, we discuss main results of the paper. For this we need the following Lemma:

Lemma 2.1. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function. Then the following inequality holds for all $s, t, u, v \in [0, 1]$ with $s \leq t \leq u \leq v$ and $t + u = s + v$*

$$f(t) + f(u) \leq f(s) + f(v) \quad (2.1)$$

Proof. Since f is convex function, for all $s, t, u, v \in [0, 1]$ with $s \leq t \leq u \leq v$ and $t + u = s + v$, we obtain

$$\frac{f(v) - f(u)}{v - u} \geq \frac{f(u) - f(t)}{u - t} \geq \frac{f(t) - f(s)}{t - s} = \frac{f(t) - f(s)}{v - u}$$

then, we have

$$f(v) - f(u) \geq f(t) - f(s).$$

The proof is completed. \square

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function and $p : [a, b] \rightarrow \mathbb{R}$ be a non-negative, integrable and symmetric about $x = \frac{a+b}{2}$. If H and F are defined on $[0, 1]$ by

$$H(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x) d_q^R x,$$

and

$$F(t) = \int_a^b \frac{1}{2} \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right) \right] d_q^R x,$$

then H, F are convex and increasing on $[0, 1]$ and for all $t \in [0, 1]$

$$f\left(\frac{a+b}{2}\right) \int_a^b p(x) d_q^R x = H(0) \leq H(t) \leq H(1) = \int_a^b f(x) p(x) d_q^R x \quad (2.2)$$

and

$$\int_a^b f(x) p(x) d_q^R x = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2} \int_a^b p(x) d_q^R x. \quad (2.3)$$

Proof. For all $t_1, t_2, \lambda \in [0, 1]$ and $x \in [a, b]$, by convexity of f , we have

$$\begin{aligned} & f\left((\lambda t_1 + (1-\lambda)t_2)x + (1 - (\lambda t_1 + (1-\lambda)t_2))\frac{a+b}{2}\right) \\ &= f\left(\lambda\left(t_1x + (1-t_1)\frac{a+b}{2}\right) + (1-\lambda)\left(t_2x + (1-t_2)\frac{a+b}{2}\right)\right) \\ &\leq \lambda f\left(t_1x + (1-t_1)\frac{a+b}{2}\right) + (1-\lambda)f\left(t_2x + (1-t_2)\frac{a+b}{2}\right). \end{aligned} \quad (2.4)$$

Utilizing the inequality (1.13) and by $p(\cdot)$ a nonnegative function, we get

$$\begin{aligned} & H(\lambda t_1 + (1-\lambda)t_2) \\ &= \int_a^b f\left((\lambda t_1 + (1-\lambda)t_2)x + (1 - (\lambda t_1 + (1-\lambda)t_2))\frac{a+b}{2}\right) p(x) d_q^R x \\ &\leq \lambda \int_a^b f\left(t_1x + (1-t_1)\frac{a+b}{2}\right) p(x) d_q^R x \\ &+ (1-\lambda) \int_a^b f\left(t_2x + (1-t_2)\frac{a+b}{2}\right) p(x) d_q^R x \\ &= \lambda H(t_1) + (1-\lambda)H(t_2), \end{aligned}$$

which shows that H is convex on $[0, 1]$.

Now, let $0 \leq s \leq t \leq 1$ and $w \in [0, 1]$, we have

$$\begin{aligned} & t \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-t) \frac{a+b}{2} \\ & \leq s \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-s) \frac{a+b}{2} \\ & \leq s \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-s) \frac{a+b}{2} \\ & \leq t \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-t) \frac{a+b}{2}, \end{aligned}$$

and for all $w \in [-1, 0]$, we have

$$\begin{aligned} & t \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-t) \frac{a+b}{2} \\ & \leq s \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-s) \frac{a+b}{2} \\ & \leq s \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-s) \frac{a+b}{2} \\ & \leq t \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-t) \frac{a+b}{2}, \end{aligned}$$

for all $w \in [-1, 1]$, we get

$$\begin{aligned} & \left(s \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-s) \frac{a+b}{2} \right) \\ & + \left(s \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-s) \frac{a+b}{2} \right) \\ & = \left(t \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-t) \frac{a+b}{2} \right) \\ & + \left(t \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-t) \frac{a+b}{2} \right). \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} & f \left(s \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-s) \frac{a+b}{2} \right) \\ & + f \left(s \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-s) \frac{a+b}{2} \right) \\ & \leq f \left(t \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-t) \frac{a+b}{2} \right) \\ & + f \left(t \left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-t) \frac{a+b}{2} \right). \end{aligned}$$

Then,

$$\begin{aligned}
H(s) &= \int_a^b f\left(sx + (1-s)\frac{a+b}{2}\right) p(x) d_q^R x \\
&= \frac{b-a}{2} \int_{-1}^1 f\left(s\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + (1-s)\frac{a+b}{2}\right) \\
&\quad \times p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\
&= \frac{b-a}{4} \int_{-1}^1 f\left(s\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + (1-s)\frac{a+b}{2}\right) \\
&\quad \times p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\
&\quad + \frac{b-a}{4} \int_{-1}^1 f\left(s\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + (1-s)\frac{a+b}{2}\right) \\
&\quad \times p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w.
\end{aligned}$$

Since $p(\cdot)$ is nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$, we get

$$\begin{aligned}
&\frac{b-a}{4} \int_{-1}^1 f\left(s\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + (1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\
&= \frac{b-a}{4} \int_{-1}^1 f\left(s\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + (1-s)\frac{a+b}{2}\right) \\
&\times p\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
&= \frac{b-a}{4} \int_{-1}^1 f\left(s\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + (1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right) d_q w \\
&= \frac{b-a}{4} \int_{-1}^1 f\left(s\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right) + (1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\
&= \frac{b-a}{4} \int_{-1}^1 f\left(s\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) + (1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w,
\end{aligned}$$

then, we obtain

$$\begin{aligned}
 H(s) &= \frac{b-a}{4} \int_{-1}^1 f \left(s \left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-s)\frac{a+b}{2} \right) \\
 &\quad \times p \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) d_q w \\
 &+ \frac{b-a}{4} \int_{-1}^1 f \left(s \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-s)\frac{a+b}{2} \right) \\
 &\quad \times p \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) d_q w \\
 &\leq \frac{b-a}{4} \int_{-1}^1 f \left(t \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) + (1-t)\frac{a+b}{2} \right) \\
 &\quad \times p \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) d_q w \\
 &+ \frac{b-a}{4} \int_{-1}^1 f \left(t \left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) \right) + (1-t)\frac{a+b}{2} \right) \\
 &\quad \times p \left(\frac{1-w}{2}a + \frac{1+w}{2}b \right) d_q w \\
 &= H(t).
 \end{aligned}$$

Thus, H is increasing on $[0, 1]$ and the inequality (2.2) holds for all $t \in [0, 1]$.

For all $t_1, t_2, \lambda \in [0, 1]$ and $x \in [a, b]$, by convexity of f , we get

$$\begin{aligned}
 &f \left(\frac{1+\lambda t_1 + (1-\lambda)t_2}{2}a + \frac{1-\lambda t_1 - (1-\lambda)t_2}{2}x \right) \\
 &= f \left(\frac{\lambda(1+t_1) + (1-\lambda)(1+t_2)}{2}a + \frac{\lambda(1-t_1) + (1-\lambda)(1-t_2)}{2}x \right) \\
 &\leq \lambda f \left(\frac{(1+t_1)}{2}a + \frac{(1-t_1)}{2}x \right) + (1-\lambda) f \left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x \right). \quad (2.5)
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 &f \left(\frac{1+\lambda t_1 + (1-\lambda)t_2}{2}b + \frac{1-\lambda t_1 - (1-\lambda)t_2}{2}x \right) \\
 &\leq \lambda f \left(\frac{(1+t_1)}{2}b + \frac{(1-t_1)}{2}x \right) + (1-\lambda) f \left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}x \right), \quad (2.6)
 \end{aligned}$$

then, using (2.5), (2.6), (1.13) and the fact that $p(\cdot)$ is nonnegative function, we obtain

$$\begin{aligned}
 & F(\lambda t_1 + (1 - \lambda)t_2) \\
 & \leq \lambda \int_a^b f\left(\frac{1+t_1}{2}a + \frac{1-t_1}{2}x\right) p\left(\frac{x+a}{2}\right) d_q^R x \\
 & + (1 - \lambda) \int_a^b f\left(\frac{1+t_2}{2}a + \frac{1-t_2}{2}x\right) p\left(\frac{x+a}{2}\right) d_q^R x \\
 & + \lambda \int_a^b f\left(\frac{1+t_1}{2}b + \frac{1-t_1}{2}x\right) p\left(\frac{x+b}{2}\right) d_q^R x \\
 & + (1 - \lambda) \int_a^b f\left(\frac{1+t_2}{2}b + \frac{1-t_2}{2}x\right) p\left(\frac{x+b}{2}\right) d_q^R x \\
 & = \lambda F(t_1) + (1 - \lambda)F(t_2).
 \end{aligned}$$

Thus, F is convex on $[0, 1]$.

For all $w \in [-1, 1]$, and $0 \leq s \leq t \leq 0$, we have

$$\begin{aligned}
 & \frac{1+t}{2}a + \frac{1-t}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\
 & \leq \frac{1+s}{2}a + \frac{1-s}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\
 & \leq \frac{1+s}{2}b + \frac{1-s}{2}\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
 & \leq \frac{1+t}{2}b + \frac{1-t}{2}\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right),
 \end{aligned}$$

where

$$\begin{aligned}
 & \frac{1+s}{2}a + \frac{1-s}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\
 & + \frac{1+s}{2}b + \frac{1-s}{2}\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
 & = \frac{1+t}{2}a + \frac{1-t}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\
 & + \frac{1+t}{2}b + \frac{1-t}{2}\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right).
 \end{aligned}$$

Using Lemma 2.1, and the fact that $p(\cdot)$ is nonnegative function, we have

$$\begin{aligned}
& f\left(\frac{1+s}{2}a + \frac{1-s}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
& + f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
& \leq f\left(\frac{1+t}{2}a + \frac{1-t}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
& + f\left(\frac{1+t}{2}b + \frac{1-t}{2}\left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right).
\end{aligned}$$

Integrating with respect to w on $[-1, 1]$, we have

$$\begin{aligned}
& \int_{-1}^1 f\left(\frac{1+s}{2}a + \frac{1-s}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
& + \int_{-1}^1 f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
& \leq \int_{-1}^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
& + \int_{-1}^1 f\left(\frac{1+t}{2}b + \frac{1-t}{2}\left(a + b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w,
\end{aligned}$$

where, using the fact that $p(\cdot)$ is nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$, we get

$$\begin{aligned}
& \int_{-1}^1 f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
& = \int_{-1}^1 f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(a+b - \frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
& = \int_{-1}^1 f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(b + \frac{1+w}{2}a + \frac{1-w}{2}b\right)\right) d_q w \\
& = \int_a^b f\left(\frac{1+s}{2}b + \frac{1-s}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_{-1}^1 f\left(\frac{1+t}{2}b + \frac{1-t}{2}\left(a+b - \left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
& = \int_a^b f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x,
\end{aligned}$$

then,

$$\begin{aligned}
F(s) & = \frac{1}{2} \int_a^b \frac{1}{2} f\left(\frac{1+s}{2}a + \frac{1-s}{2}x\right) \\
& \times p\left(\frac{a+x}{2}\right) d_q^R x + \int_a^b \frac{1}{2} f\left(\frac{1+s}{2}b + \frac{1-s}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x \\
& \leq \int_a^b \frac{1}{2} f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) \\
& \times p\left(\frac{a+x}{2}\right) d_q^R x + \int_a^b \frac{1}{2} f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x \\
& = F(t).
\end{aligned}$$

Thus, F is increasing on $[0, 1]$ and the inequality (2.3) holds for all $t \in [0, 1]$. \square

Theorem 2.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a h -convex function and $p : [a, b] \rightarrow \mathbb{R}$ be positive, integrable, and symmetric about $x = \frac{a+b}{2}$. Then the following inequalities hold:

$$\begin{aligned} & \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b p(x) d_q^R x \\ & \leq \int_a^b f(x) p(x) d_q^R x \\ & \leq f(a) \int_a^b h\left(\frac{b-x}{b-a}\right) p(x) d_q^R x + f(b) \int_a^b h\left(\frac{x-a}{b-a}\right) p(x) d_q^R x \end{aligned}$$

Proof. Since f is h -convex function, we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right) (f(x) + f(y)), \quad (2.7)$$

for all $x, y \in [a, b]$.

In (2.7), if we choose $x = \frac{1-w}{2}a + \frac{1+w}{2}b$ and $y = \frac{1+w}{2}a + \frac{1-w}{2}b$, $w \in [-1, 1]$ and by p is positive, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & = f\left(\frac{1}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + \frac{1}{2}\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right)\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & \leq h\left(\frac{1}{2}\right) f\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & + h\left(\frac{1}{2}\right) f\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \end{aligned}$$

Integrating with respect w over $[-1, 1]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \int_{-1}^1 p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\ & \leq h\left(\frac{1}{2}\right) \int_{-1}^1 f\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\ & + h\left(\frac{1}{2}\right) \int_{-1}^1 f\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w, \end{aligned}$$

Then, by p is symmetric function about $\frac{a+b}{2}$, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \frac{2}{b-a} \int_a^b p(x) d_q^R x \\ & \leq h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x) p(x) d_q^R x + h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x) p\left(\frac{a+b}{2} + \frac{a+b}{2} - x\right) d_q^R x \\ & = h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x) p(x) d_q^R x + h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x) p(x) d_q^R x, \end{aligned}$$

the first inequality is proved.

The proof of second inequality is given as

$$\begin{aligned} & f\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & \leq f(a)h\left(\frac{1-w}{2}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & + f(b)h\left(\frac{1+w}{2}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \end{aligned}$$

we integrate w on $[-1, 1]$, we obtain

$$\begin{aligned} & \int_{-1}^1 f\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\ & \leq \int_{-1}^1 f(a)h\left(\frac{1-w}{2}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\ & + \int_{-1}^1 f(b)h\left(\frac{1+w}{2}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w. \end{aligned}$$

Then

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x)p(x)d_q^R x & \leq f(a)\frac{2}{b-a} \int_a^b h\left(\frac{b-x}{b-a}\right) p(x)d_q^R x \\ & + \frac{2}{b-a} f(b) \int_a^b h\left(\frac{x-a}{b-a}\right) p(x)d_q^R x. \end{aligned}$$

The proof is complete. □

Remark 2.4. If we choose $p(x) = 1$ and $h(t) = t$, then Theorem(2.3) reduces to Theorem(1.10).

Corollary 2.5. Let $f \in K_s^2$, $a, b \in [0, \infty[$ with $a < b$ and $p : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then

$$\begin{aligned} & 2^{s-1} f\left(\frac{a+b}{2}\right) \int_a^b p(x)d_q^R x \leq \int_a^b f(x)p(x)d_q^R x \\ & \leq f(a) \int_a^b \left(\frac{b-x}{b-a}\right)^s p(x)d_q^R x + f(b) \int_a^b \left(\frac{x-a}{b-a}\right)^s p(x)d_q^R x \end{aligned}$$

Remark 2.6. If we choose $s = 1$, $h(t) = t$ and $p(x) = 1$, then Corollary (2.5) reduces to Theorem(1.10).

Theorem 2.7. Let $f : [0, \infty[\rightarrow \mathbb{R}$ be an m -convex function with $m \in]0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[0, b]$, then

$$\int_a^b f(x)p(x)d_q^R x \leq \min \left\{ \int_a^b L_x(a, b)p(x)d_q^R x, \int_a^b L_x(b, a)p(x)d_q^R x \right\}.$$

with $L_x(a, b) = \left(f(a) \left(\frac{b-x}{b-a}\right) + m f\left(\frac{b}{m}\right) \left(\frac{x-a}{b-a}\right) \right)$.

Proof. Since f is m -convex and for all $w \in [-1, 1]$, we have

$$\begin{aligned} & f\left(\frac{1-w}{2}a + m\frac{1+w}{2m}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & \leq \left(\frac{1-w}{2}\right) f(a) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & + m\left(\frac{1+w}{2}\right) f\left(\frac{b}{m}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right), \end{aligned}$$

and

$$\begin{aligned} & f\left(\frac{1-w}{2}b + m\frac{1+w}{2m}a\right) p\left(\frac{1-w}{2}b + \frac{1+w}{2}a\right) \\ & \leq \left(\frac{1-w}{2}\right) f(b) p\left(\frac{1-w}{2}b + \frac{1+w}{2}a\right) \\ & + m\left(\frac{1+w}{2}\right) f\left(\frac{a}{m}\right) p\left(\frac{1-w}{2}b + \frac{1+w}{2}a\right). \end{aligned}$$

Then, by integrating both sides with respect w on $[-1, 1]$, we have

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x) p(x) d_q^R x & \leq f(a) \frac{2}{b-a} \int_a^b \left(\frac{b-x}{b-a}\right) p(x) d_q^R x \\ & + m f\left(\frac{b}{m}\right) \frac{2}{b-a} \int_a^b \left(\frac{x-a}{b-a}\right) p(x) d_q^R x \end{aligned}$$

and

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x) p(x) d_q^R x & \leq f(b) \frac{2}{b-a} \int_a^b \left(\frac{a-x}{a-b}\right) p(x) d_q^R x \\ & + m f\left(\frac{a}{m}\right) \frac{2}{b-a} \int_a^b \left(\frac{x-b}{a-b}\right) p(x) d_q^R x. \end{aligned}$$

This completes the proof. □

Theorem 2.8. Let $f : [0, \infty[\rightarrow \mathbb{R}$ be a m -convex function with $m \in]0, 1]$ and $p : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) \int_a^b p(x) d_q^R x & \leq \int_a^b \left(f(x) + m f\left(\frac{x}{m}\right)\right) p(x) d_q^R x \\ & \leq \left(f(a) + m f\left(\frac{a}{m}\right)\right) \int_a^b \left(\frac{b-x}{b-a}\right) p(x) d_q^R x \\ & + \left(m f\left(\frac{b}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right) \int_a^b \left(\frac{x-a}{b-a}\right) p(x) d_q^R x. \end{aligned}$$

Proof. According to the definition of m -convex function, for all $t \in [-1, 1]$, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &= f\left(\frac{1}{2}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \frac{m}{2m}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &\leq \frac{1}{2}f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &\quad + \frac{m}{2}f\left(\frac{\frac{1+t}{2}a + \frac{1-t}{2}b}{m}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right). \end{aligned}$$

Integrating with respect to t on $[-1, 1]$, we have

$$\begin{aligned} & \int_{-1}^1 f\left(\frac{a+b}{2}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t \\ &\leq \frac{1}{2} \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t \\ &\quad + \int_{-1}^1 \frac{m}{2} f\left(\frac{\frac{1+t}{2}a + \frac{1-t}{2}b}{m}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t. \end{aligned}$$

Since $p(\cdot)$ is symmetric function, so, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \frac{2}{b-a} \int_a^b p(x) d_q^R x \\ &\leq \frac{2}{2(b-a)} \int_a^b f(x) p(x) d_q^R x + \frac{2m}{2(b-a)} \int_a^b f\left(\frac{x}{m}\right) p(a+b-x) d_q^R x \\ &= \frac{1}{(b-a)} \int_a^b f(x) p(x) d_q^R x + \frac{m}{(b-a)} \int_a^b f\left(\frac{x}{m}\right) p(x) d_q^R x. \end{aligned}$$

Then

$$f\left(\frac{a+b}{2}\right) 2 \int_a^b p(x) d_q^R x \leq \int_a^b f(x) p(x) d_q^R x + m \int_a^b f\left(\frac{x}{m}\right) p(x) d_q^R x.$$

the first inequality is proved.

Now the proof of second inequality is given by

$$\begin{aligned} & f\left(\frac{1-t}{2}a + m\frac{1+t}{2m}b\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &\leq f(a) \frac{1-t}{2} p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &\quad + m \frac{1+t}{2} f\left(\frac{b}{m}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right), \end{aligned}$$

and

$$\begin{aligned} & mf \left(\frac{1-t}{2m}a + \frac{1+t}{2m^2}mb \right) p \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \\ & \leq mf \left(\frac{a}{m} \right) \frac{1-t}{2} p \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \\ & + m^2 \frac{1+t}{2} f \left(\frac{b}{m^2} \right) p \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right). \end{aligned}$$

Integrating both sides with respect to t on $[-1, 1]$, we obtain

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x)p(x)d_q^R x & \leq f(a) \frac{2}{b-a} \int_a^b \left(\frac{b-x}{b-a} \right) p(x)d_q^R x \\ & + f \left(\frac{b}{m} \right) \frac{2m}{b-a} \int_a^b \left(\frac{x-a}{b-a} \right) p(x)d_q^R x, \end{aligned}$$

and

$$\begin{aligned} m \frac{2}{b-a} \int_a^b f \left(\frac{x}{m} \right) p(x)d_q^R x & \leq mf \left(\frac{a}{m} \right) \frac{2}{b-a} \int_a^b \frac{b-x}{b-a} p(x)d_q^R x \\ & + m^2 f \left(\frac{b}{m^2} \right) \frac{2}{b-a} \int_a^b \frac{x-a}{b-a} p(x)d_q^R x. \end{aligned}$$

Then, the proof of Theorem (2.8) is completed. \square

Theorem 2.9. Let $f : [0, \infty[\rightarrow \mathbb{R}$ be an (s, m) -convex function with $(s, m) \in]0, 1]^2$ and $p : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then

$$\begin{aligned} 2^s f \left(\frac{a+b}{2} \right) \int_a^b p(x)d_q^R x & \leq \int_a^b f(x)p(x)d_q^R x + m(2^s - 1) \int_a^b f \left(\frac{x}{m} \right) p(x)d_q^R x \\ & \leq \left(f(a) + m(2^s - 1)f \left(\frac{a}{m} \right) \right) \int_a^b \left(\frac{b-x}{b-a} \right)^s p(x)d_q^R x \\ & + m \left(f \left(\frac{b}{m} \right) + m(2^s - 1)f \left(\frac{b}{m^2} \right) \right) \int_a^b \left(\frac{x-a}{b-a} \right)^s p(x)d_q^R x \end{aligned}$$

Proof. Since f is (s, m) -convex function on $[a, b]$, we can write

$$\begin{aligned} & f \left(\frac{1}{2} \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) + \frac{m}{2m} \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right) p \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \\ & \leq \left(\frac{1}{2} \right)^s f \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) p \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \\ & + m \left(1 - \left(\frac{1}{2} \right)^s \right) f \left(\frac{1}{m} \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right) p \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right). \end{aligned}$$

Integrating with respect to t on $[-1, 1]$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \frac{2}{b-a} \int_a^b p(x) d_q^R x &\leq \left(\frac{1}{2}\right)^s \frac{2}{b-a} \int_a^b f(x) p(x) d_q^R x \\ &\leq m \left(1 - \left(\frac{1}{2}\right)^s\right) \frac{2}{b-a} \int_a^b f\left(\frac{x}{m}\right) p(a+b-x) d_q^R x. \end{aligned}$$

Since $p(\cdot)$ is symmetric function, so, we have

$$2^s f\left(\frac{a+b}{2}\right) \int_a^b p(x) d_q^R x \leq \int_a^b f(x) p(x) d_q^R x + m(2^s - 1) \int_a^b f\left(\frac{x}{m}\right) p(x) d_q^R x.$$

Also, we have

$$\begin{aligned} &f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &= f\left(\frac{1-t}{2}a + \frac{m}{m}\left(1 - \left(\frac{1-t}{2}\right)\right)b\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &\leq \left(\frac{1-t}{2}\right)^s f(a) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &+ m \left(1 - \left(\frac{1-t}{2}\right)^s\right) f\left(\frac{b}{m}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right), \end{aligned}$$

and

$$\begin{aligned} &f\left(\frac{\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)}{m}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &= f\left(\frac{1-t}{2} \frac{a}{m} + \frac{m}{m^2}\left(1 - \left(\frac{1-t}{2}\right)\right)b\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &\leq \left(\frac{1-t}{2}\right)^s f\left(\frac{a}{m}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &+ m \left(1 - \left(\frac{1-t}{2}\right)^s\right) f\left(\frac{b}{m^2}\right) p\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right). \end{aligned}$$

Integrating both sides with respect to t on $[-1, 1]$, we get

$$\begin{aligned} &\frac{2}{b-a} \int_a^b f(x) p(x) d_q^R x \\ &\leq \frac{2}{b-a} f(a) \int_a^b \left(\frac{b-x}{b-a}\right)^s p(x) + m f\left(\frac{b}{m}\right) \frac{2}{b-a} \int_a^b \left(1 - \left(\frac{b-x}{b-a}\right)^s\right) p(x) d_q^R x, \end{aligned}$$

and

$$\begin{aligned} &\frac{2}{b-a} \int_a^b f\left(\frac{x}{m}\right) p(x) d_q^R x \\ &\leq \frac{2}{b-a} f\left(\frac{a}{m}\right) \int_a^b \left(\frac{b-x}{b-a}\right)^s p(x) d_q^R x + m f\left(\frac{b}{m^2}\right) \frac{2}{b-a} \int_a^b \left(1 - \left(\frac{b-x}{b-a}\right)^s\right) p(x) d_q^R x. \end{aligned}$$

The proof of Theorem (2.9) is completed. \square

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