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Coefficient estimates and subordination properties for certain classes of analytic functions of reciprocal order

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Abstract. In this work, we determine the coefficient bounds and subordination results for functions in certain subclasses of analytic functions of reciprocal order, which are introduced here by means of a Hadamard product of analytic functions. The results presented in this paper improve or generalize the recent works of other authors and also give rise to several new results.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of order α if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (\ 0 \leq \alpha < 1,\ z \in \mathbb{U}). \tag{1.2}$$

We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathbb{U}$. Also, a function $f \in \mathcal{A}$ is said to be convex of order α if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (\ 0 \leq \alpha < 1,\ z \in \mathbb{U}). \tag{1.3}$$

We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathbb{U}$. Clearly, we have $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{K}(\alpha) \subseteq \mathcal{K}(0) = \mathcal{K}$ and $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$ for $0 \leq \alpha < 1$.

For $|\beta| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is said to be β-spirallike of order α in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\beta \qquad (z \in \mathbb{U}). \tag{1.4}$$

The class of all such functions is denoted by $\mathcal{S}_\beta(\alpha)$ [8].

A function $f \in \mathcal{A}$ is said to be starlike of reciprocal order α if

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \qquad (\ 0 \leq \alpha < 1,\ z \in \mathbb{U}). \tag{1.5}$$

We denote the class of such functions by $\mathcal{S}^{-1}{}^*(\alpha)$. Furthermore, a function $f \in \mathcal{A}$ is said to be convex of reciprocal order α if

$$\operatorname{Re}\left\{\frac{1}{1+\frac{zf''(z)}{f'(z)}}\right\} > \alpha \qquad (\ 0 \leq \alpha < 1,\ z \in \mathbb{U}). \tag{1.6}$$

The class of all such convex functions of reciprocal order α is denoted by $\mathcal{K}^{-1}(\alpha)$.

We note that $f(z) \in \mathcal{K}^{-1}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^{-1}{}^*(\alpha)$.

In view of the fact that

$$\operatorname{Re}p(z) > 0 \Rightarrow \operatorname{Re}\frac{1}{p(z)} = \operatorname{Re}\frac{p(z)}{|p(z)|^2} > 0$$

it follows that $\mathcal{S}^{-1}{}^*(0) = \mathcal{S}^*$ and $\mathcal{K}^{-1}(0) = \mathcal{K}$. In particular, every starlike function of reciprocal order $\alpha \geq 0$ is starlike and hence univalent.

Example 1.1. The function $f(z) = ze^{(1-\alpha)z}$ is a starlike function of reciprocal order $1/(2-\alpha)$ [9, Example 2].

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.7}$$

Motivated and inspired by the work done by Owa et al. [11] and by making use of the Hadamard product (1.7), we now introduce the following subclass of $\mathcal{A}$.

Definition 1.2. Let $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \delta_n z^n$ and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ *be analytic in* $\mathbb{U}$, *such that* $\delta_n \geq 0, \mu_n \geq 0$ *and* $\delta_n \geq \mu_n$ *for* $n \geq 2$, *we say that* $f(z) \in \mathcal{A}$ *is in the class*

$\mathcal{S}^{-1}(\Phi,\Psi;\alpha,\beta)$ *if* $f(z)*\Phi(z)\neq 0,\ \ f(z)*\Psi(z)\neq 0$ *and*

$$\left|\frac{1}{e^{i\beta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}\right)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (\ \beta\in\mathbb{R},\ 0<\alpha<1,\ z\in\mathbb{U}). \tag{1.8}$$

Several known and new subclasses of analytic functions of reciprocal order α can be obtained from the class $\mathcal{S}^{-1}(\Phi,\Psi;\alpha,\beta)$, by suitably specializing the values of Φ, Ψ and β. We present below some of these subclasses of $\mathcal{S}^{-1}(\Phi,\Psi;\alpha,\beta)$ consisting of functions of the form (1.1).

Example 1.3. If $\Phi(z)=z(1-z)^{-2}$ and $\Psi(z)=z(1-z)^{-1}$, then

$$\mathcal{S}^{-1}\left(z(1-z)^{-2},z(1-z)^{-1};\alpha,\beta\right)$$
$$\equiv\mathcal{S}_\beta(\alpha):=\left\{f\in\mathcal{A}:\left|\frac{e^{-i\beta}f(z)}{zf'(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (\ \beta\in\mathbb{R},\ 0<\alpha<1,\ z\in\mathbb{U})\right\}.$$

Example 1.4. If $\Phi(z)=(z+z^2)(1-z)^{-3}$ and $\Psi(z)=z(1-z)^{-2}$, then

$$\mathcal{S}^{-1}\left((z+z^2)(1-z)^{-3},z(1-z)^{-2};\alpha,\beta\right)$$
$$\equiv\mathcal{K}_\beta(\alpha):=\left\{f\in\mathcal{A}:\left|\frac{e^{-i\beta}f'(z)}{f'(z)+zf''(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (\ \beta\in\mathbb{R},\ 0<\alpha<1,\ z\in\mathbb{U})\right\}.$$

The class $\mathcal{S}_\beta(\alpha)$, the β-spirallike functions of reciprocal order α and the class $\mathcal{K}_\beta(\alpha)$, the β-convexlike functions of reciprocal order α were studied by Owa et al. [11].

Example 1.5. If $\Phi(z)=z(1-z)^{-2}$, $\Psi(z)=z(1-z)^{-1}$ and $\beta=0$, then

$$\mathcal{S}^{-1}\left(z(1-z)^{-2},z(1-z)^{-1};\alpha,0\right)$$
$$\equiv\mathcal{M}(\alpha):=\left\{f\in\mathcal{A}:\left|\frac{f'(z)}{zf(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (0<\alpha<1,\ z\in\mathbb{U})\right\}.$$

Example 1.6. If $\Phi(z)=(z+z^2)(1-z)^{-3}$,$\Psi(z)=z(1-z)^{-2}$ and $\beta=0$, then

$$\mathcal{S}^{-1}\left((z+z^2)(1-z)^{-3},z(1-z)^{-2};\alpha,0\right)$$
$$\equiv\mathcal{N}(\alpha):=\left\{f\in\mathcal{A}:\left|\frac{f'(z)}{f'(z)+zf''(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (0<\alpha<1,z\in\mathbb{U})\right\}.$$

The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were studied by Owa et al. [10].

Furthermore, we have the following new classes:

Example 1.7. If $\Phi(z)=z(1-z)^{-2}$ and $\Psi(z)=z$, then

$$\mathcal{S}^{-1}\left(z(1-z)^{-2},z;\alpha,\beta\right)$$
$$\equiv\mathcal{P}_\beta(\alpha):=\left\{f\in\mathcal{A}:\left|\frac{1}{e^{i\beta}f'(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (\ \beta\in\mathbb{R},\ 0<\alpha<1,\ z\in\mathbb{U})\right\}.$$

Example 1.8. If $\Phi(z) = (z + z^2)(1 - z)^{-3}$ and $\Psi(z) = z$, then

$$\mathcal{S}^{-1}(z + z^2)(1 - z)^{-3}, z; \alpha, \beta)$$
$$\equiv \mathcal{R}_\beta(\alpha) := \left\{ f \in \mathcal{A} : \left| \frac{1}{e^{i\beta}\left((zf'(z))'\right)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (\ \beta \in \mathbb{R},\ 0 < \alpha < 1,\ z \in \mathbb{U}) \right\}.$$

In fact many new subclasses of functions of reciprocal order can be defined and studied by suitably choosing $\Phi(z), \Psi(z)$ and β.

The aim of the present paper is to investigate the coefficient estimates and subordination properties for the class $\mathcal{S}^{-1}(\Phi, \Psi; \alpha, \beta)$. Some interesting consequences of the results are also pointed out.

2. Coefficient Estimates

The sufficient condition for $f(z)$ to be in the class $\mathcal{S}^{-1}(\Phi, \Psi; \alpha, \beta)$ is given by using coefficient inequalities.

Theorem 2.1. *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \left[\delta_n + \left|\delta_n - 2\alpha e^{-i\beta}\mu_n\right|\right] |a_n| \le 1 - \left|1 - 2\alpha e^{-i\beta}\right| \tag{2.1}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in \mathcal{S}^{-1}(\Phi, \Psi; \alpha, \beta)$.

Proof. It suffices to show that

$$\left| \frac{2\alpha\left(f(z) * \Psi(z)\right) - e^{i\beta}\left(f(z) * \Phi(z)\right)}{e^{i\beta}\left(f(z) * \Phi(z)\right)} \right| < 1.$$

We observe that

$$\begin{aligned} \left| \frac{2\alpha\left(f(z) * \Psi(z)\right) - e^{i\beta}\left(f(z) * \Phi(z)\right)}{e^{i\beta}\left(f(z) * \Phi(z)\right)} \right| &= \left| \frac{\left(1 - 2\alpha e^{-i\beta}\right) + \sum_{n=2}^{\infty} \left(\delta_n - 2\alpha e^{-i\beta}\mu_n\right) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \delta_n a_n z^{n-1}} \right| \\ &\le \frac{\left|1 - 2\alpha e^{i\beta}\right| + \sum_{n=2}^{\infty} \left|\delta_n - 2\alpha e^{-i\beta}\mu_n\right| |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \delta_n |a_n| |z|^{n-1}} \\ &< \frac{\left|1 - 2\alpha e^{i\beta}\right| + \sum_{n=2}^{\infty} \left|\delta_n - 2\alpha e^{-i\beta}\mu_n\right| |a_n|}{1 - \sum_{n=2}^{\infty} \delta_n |a_n|}. \end{aligned}$$

It follows that the last term is bounded by 1 if

$$\sum_{n=2}^{\infty} \left[\delta_n + \left|\delta_n - 2\alpha e^{-i\beta}\mu_n\right|\right] |a_n| \le 1 - \left|1 - 2\alpha e^{-i\beta}\right|$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, which is equivalent to (2.1). Therefore, we have $f(z) \in \mathcal{S}^{-1}(\Phi, \Psi; \alpha, \beta)$ for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$. □

In the view of Examples 1.3 to 1.8, we state the following corollaries.

Corollary 2.2. ([11]) *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \left[n + \left|n - 2\alpha e^{-i\beta}\right|\right] |a_n| \leq 1 - \left|1 - 2\alpha e^{-i\beta}\right| \tag{2.2}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in \mathcal{S}_\beta(\alpha)$.

Corollary 2.3. ([11]) *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n \left[n + \left|n - 2\alpha e^{-i\beta}\right|\right] |a_n| \leq 1 - \left|1 - 2\alpha e^{-i\beta}\right| \tag{2.3}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in \mathcal{K}_\beta(\alpha)$.

Corollary 2.4. ([10]) *Let $0 < \alpha < 1$. If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \frac{1}{2} (1 - |1 - 2\alpha|) = \left\{ \begin{array}{cc} \alpha; & \text{if} \quad 0 < \alpha \leq \frac{1}{2} \\ 1 - \alpha; & \text{if} \quad \frac{1}{2} \leq \alpha < 1 \end{array} \right. , \tag{2.4}$$

then $f(z) \in \mathcal{M}(\alpha)$.

Corollary 2.5. ([10]) *Let $0 < \alpha < 1$. If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n (n - \alpha) |a_n| \leq \frac{1}{2} (1 - |1 - 2\alpha|) = \left\{ \begin{array}{cc} \alpha; & \text{if} \quad 0 < \alpha \leq \frac{1}{2} \\ 1 - \alpha; & \text{if} \quad \frac{1}{2} \leq \alpha < 1 \end{array} \right. , \tag{2.5}$$

then $f(z) \in \mathcal{N}(\alpha)$.

Corollary 2.6. *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} 2n |a_n| \leq 1 - \left|1 - 2\alpha e^{-i\beta}\right| \tag{2.6}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in \mathcal{P}_\beta(\alpha)$.

Corollary 2.7. *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} 2n^2 |a_n| \leq 1 - \left|1 - 2\alpha e^{-i\beta}\right| \tag{2.7}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in \mathcal{R}_\beta(\alpha)$.

3. Subordination Results

To proceed our main result in this section, let us first recall the following definitions and lemma.

Definition 3.1. (Subordination Principle). For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Definition 3.2. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1), $a_1 = 1$ is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in \mathbb{U}. \tag{3.1}$$

Lemma 3.3. ([14]) *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \quad (z \in \mathbb{U}). \tag{3.2}$$

Let $\mathcal{S}^{*^{-1}}(\Phi, \Psi; \alpha, \beta) \subseteq \mathcal{S}^{^{-1}}(\Phi, \Psi; \alpha, \beta)$ denote the subclass of functions $f \in \mathcal{A}$ whose coefficients a_n satisfy the inequalities (2.1) .

Employing the techniques used by Attiya [3], Singh [12] and Srivastava and Attiya [13] (see also, [1], [2], [4], [5], [6] and [7]), we state and prove the following theorem.

Theorem 3.4. *Let $f(z) \in \mathcal{S}^{*^{-1}}(\Phi, \Psi; \alpha, \beta)$ and $\delta_n + \left|\delta_n - 2\alpha e^{-i\beta}\mu_n\right|$ is increasing function for $n \geq 2$, $|\beta| < \frac{\pi}{2}$, $0 < \alpha < \cos\beta$. Then*

$$\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2\left(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|\right)}(f * g)(z) \prec g(z) \tag{3.3}$$

for every function $g(z)$ in the class $\mathcal{K}$ and

$$Re f(z) > -\frac{1 + \delta_2 - \left|1 - 2\alpha e^{-i\beta}\right| + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{\delta_2 + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} \tag{3.4}$$

for $z \in \mathbb{U}$.

The constant $\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|)}$ cannot be replace by any larger one.

Proof. Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in \mathcal{S}^{*^{-1}}(\Phi, \Psi; \alpha, \beta)$, and let $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\begin{aligned}
&\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2\left(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|\right)}(f * g)(z) \\
&= \frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2\left(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|\right)}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right).
\end{aligned}$$

Thus, by Definition 3.2, the assertion of our theorem will hold if the sequence

$$\left\{\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2\left(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|\right)} a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 3.3, this will be the case if and only if

$$\operatorname{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} a_n z^n\right\} > 0 \qquad (z \in \mathbb{U}). \tag{3.5}$$

Since $\delta_n + \left|\delta_n - 2\alpha e^{-i\beta}\mu_n\right|$ increasing for all $n \geq 2$, $|\beta| < \frac{\pi}{2}$, $0 < \alpha < \cos\beta$, we obtain

$$\begin{aligned}
&\operatorname{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} a_n z^n\right\} \\
&= \operatorname{Re}\left\{1 + \frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\lambda}\mu_2\right|}{1 + \delta_2 - |1 - 2\alpha e^{-i\lambda}| + |\delta_2 - 2\alpha e^{-i\lambda}\mu_2|} z + \right. \\
&\left. \frac{1}{1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} \sum_{n=2}^{\infty} \left(\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|\right) a_n z^n\right\} \\
&\geq 1 - \frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} r - \\
&\frac{1}{1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} \sum_{n=2}^{\infty} \left(\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|\right) |a_n| r^n \\
&> 1 - \frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} r \\
&- \frac{1 - \left|1 - 2\alpha e^{-i\beta}\right|}{1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} r > 0, \qquad |z| = r < 1.
\end{aligned}$$

This evidently proves the inequality (3.5) and hence also the subordination result (3.3). The inequality (3.4) follows from (3.3) by taking $g(z) = \frac{z}{1-z}$.

To prove the sharpness of the constant $\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|)}$, we consider the function

$$f_0(z) = z - \frac{1 - \left|1 - 2\alpha e^{-i\beta}\right|}{\delta_2 + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|} z^2 \qquad \left(|\beta| < \frac{\pi}{2},\ 0 < \alpha < \cos\beta\right),$$

which is a member of the class $\mathcal{S}^{*^{-1}}(\Phi, \Psi; \alpha, \beta)$. Thus from the relation (3.3), we obtain

$$\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2\left(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|\right)} f_0(z) \prec \frac{z}{1-z}.$$

It can be verified that

$$\min_{z \in \mathbb{U}} \left\{ \operatorname{Re}\left(\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2\left(1 + \delta_2 - |1 - 2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|\right)} f_0(z) \right) \right\} = \frac{-1}{2}.$$

This shows that the constant $\frac{\delta_2 + \left|\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2(1+\delta_2 - |1-2\alpha e^{-i\beta}| + |\delta_2 - 2\alpha e^{-i\beta}\mu_2|)}$ is best possible. $\square$

By taking different choices of Φ, Ψ and β in Theorem 3.4 and in view of the Examples 1.3 to 1.8 in Section 1, we state the following corollaries for the subclasses defined in those examples.

Corollary 3.5. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}_\beta^{-1}(\alpha)$; $|\beta| < \frac{\pi}{2}$, $0 < \alpha < \cos\beta$, where $\mathcal{S}_\beta^{-1}(\alpha)$ denote the subclasses of functions whose coefficients a_n satisfy the inequalities (2.2) and suppose that $g(z) \in \mathcal{K}$. Then*

$$\frac{1 + \left|1 - \alpha e^{-i\beta}\right|}{3 - |1 - 2\alpha e^{-i\beta}| + 2\left|1 - \alpha e^{-i\beta}\right|}(f * g)(z) \prec g(z) \tag{3.6}$$

and

$$Ref(z) > -\frac{3 - \left|1 - 2\alpha e^{-i\beta}\right| + 2\left|1 - \alpha e^{-i\beta}\right|}{2(1 + |1 - \alpha e^{-i\beta}|)}. \tag{3.7}$$

The constant factor $\frac{1+\left|1-\alpha e^{-i\beta}\right|}{3-|1-2\alpha e^{-i\beta}|+2|1-\alpha e^{-i\beta}|}$ in the subordination result (3.6) cannot be replaced by a larger one.

Corollary 3.6. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}_\beta^{-1}(\alpha)$; $|\beta| < \frac{\pi}{2}$, $0 < \alpha < \cos\beta$, where $\mathcal{K}_\beta^{-1}(\alpha)$ denote the subclasses of functions whose coefficients a_n satisfy the inequalities (2.3) and suppose that $g(z) \in \mathcal{K}$. Then*

$$\frac{2(1 + \left|1 - \alpha e^{-i\beta}\right|)}{5 - |1 - 2\alpha e^{-i\beta}| + 4\left|1 - \alpha e^{-i\beta}\right|}(f * g)(z) \prec g(z) \tag{3.8}$$

and

$$Ref(z) > -\frac{5 - \left|1 - 2\alpha e^{-i\beta}\right| + 4\left|1 - \alpha e^{-i\beta}\right|}{4(1 + |1 - \alpha e^{-i\beta}|)}. \tag{3.9}$$

The constant factor $\frac{2(1+\left|1-\alpha e^{-i\beta}\right|)}{5-|1-2\alpha e^{-i\beta}|+4|1-\alpha e^{-i\beta}|}$ in the subordination result (3.8) cannot be replaced by a larger one.

Corollary 3.7. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{M}^{-1}(\alpha); 0 < \alpha < 1$, where $\mathcal{M}^{-1}(\alpha)$ denote the subclasses of functions whose coefficients a_n satisfy the inequalities (2.4) and suppose that $g(z) \in \mathcal{K}$. Then*

$$\frac{2 - \alpha}{5 - 2\alpha - |1 - 2\alpha|}(f * g)(z) \prec g(z) \tag{3.10}$$

and

$$Ref(z) > -\frac{5-2\alpha-|1-2\alpha|}{2(1-\alpha)}. \tag{3.11}$$

The constant factor $\frac{2-\alpha}{5-2\alpha-|1-2\alpha|}$ *in the subordination result (*3.10*) cannot be replaced by a larger one.*

Corollary 3.8. *Let the function* $f(z)$ *defined by (*1.1*) be in the class* $\mathcal{N}^{-1}(\alpha)$; $0<\alpha<1$, *where* $\mathcal{N}^{-1}(\alpha)$ *denote the subclasses of functions whose coefficients* a_n *satisfy the inequalities (*2.5*) and suppose that* $g(z)\in\mathcal{K}$. *Then*

$$\frac{2(2-\alpha)}{9-4\alpha-|1-2\alpha|}(f*g)(z) \prec g(z) \tag{3.12}$$

and

$$Ref(z) > -\frac{9-4\alpha-|1-2\alpha|}{4(2-\alpha)}. \tag{3.13}$$

The constant factor $\frac{2(2-\alpha)}{9-4\alpha-|1-2\alpha|}$ *in the subordination result (*3.12*) cannot be replaced by a larger one.*

Corollary 3.9. *Let the function* $f(z)$ *defined by (*1.1*) be in the class* $\mathcal{P}_{\beta}^{-1}(\alpha)$; $|\beta|<\frac{\pi}{2}$, $0<\alpha<\cos\beta$, *where* $\mathcal{P}_{\beta}^{-1}(\alpha)$ *denote the subclasses of functions whose coefficients* a_n *satisfy the inequalities (*2.6*) and suppose that* $g(z)\in\mathcal{K}$. *Then*

$$\frac{2}{5-|1-2\alpha e^{-i\beta}|}(f*g)(z) \prec g(z) \tag{3.14}$$

and

$$Ref(z) > -\frac{5-\left|1-2\alpha e^{-i\beta}\right|}{4}. \tag{3.15}$$

The constant factor $\frac{2}{5-|1-2\alpha e^{-i\beta}|}$ *in the subordination result (*3.14*) cannot be replaced by a larger one.*

Corollary 3.10. *Let the function* $f(z)$ *defined by (*1.1*) be in the class* $\mathcal{R}_{\beta}^{-1}(\alpha)$; $|\beta|<\frac{\pi}{2}$, $0<\alpha<\cos\beta$, *where* $\mathcal{R}_{\beta}^{-1}(\alpha)$ *denote the subclasses of functions whose coefficients* a_n *satisfy the inequalities (*2.7*) and suppose that* $g(z)\in\mathcal{K}$. *Then*

$$\frac{4}{9-|1-2\alpha e^{-i\beta}|}(f*g)(z) \prec g(z) \tag{3.16}$$

and

$$Ref(z) > -\frac{9-\left|1-2\alpha e^{-i\beta}\right|}{8}. \tag{3.17}$$

The constant factor $\frac{4}{9-|1-2\alpha e^{-i\beta}|}$ *in the subordination result (*3.16*) cannot be replaced by a larger one.*

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