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Quintic B-spline method for numerical solution of fourth order singular perturbation boundary value problems

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Abstract. In this communication, we have studied an efficient numerical approach based on uniform mesh for the numerical solutions of fourth order singular perturbation boundary value problems. Such type of problems arises in various fields of science and engineering, like electrical network and vibration problems with large Peclet numbers, Navier-Stokes flows with large Reynolds numbers in the theory of hydrodynamics stability, reaction-diffusion process, quantum mechanics and optimal control theory etc. In the present study, a quintic B-spline method has been discussed for the approximate solution of the fourth order singular perturbation boundary value problems. The convergence analysis is also carried out and the method is shown to have convergence of second order. The performance of present method is shown through some numerical tests. The numerical results are compared with other existing method available in the literature.

Mathematics Subject Classification (2010): 65L10.

Keywords: Fourth order singular perturbation boundary value problem, quintic B-spline, quasilinearization, uniform mesh, convergence analysis.

1. Introduction

We consider the fourth order singular perturbation boundary value problem

$$-\varepsilon y^{iv}(t) - p(t)\, y'''(t) + q(t)\, y''(t) + r(t)\, y(t) = f(t), \quad t \in [a,\ b], \tag{1.1}$$

$$y(a) = \eta_1, \quad y(b) = \eta_2, \quad y''(a) = \eta_3, \quad y''(b) = \eta_4. \tag{1.2}$$

where $\eta_1,\ \eta_2,\ \eta_3$ and η_4 are finite real constants and ε is a small positive parameter, such that $0 < \varepsilon \ll 1$. Moreover, we assume that the functions $p(t)$,$q(t)$,$r(t)$ and $f(t)$ are sufficiently smooth. Further, the problem (1.1) is called non-turning point problem if $p(t) \geq \alpha > 0$ throughout the interval $[a,\ b]$, where α is some positive constant and boundary layer will be in the neighbourhood of $t = a$ [9]. In the same vein, if the $p(t)$ vanishes at $t = 0$, then it becomes a turning point problem. In that scenario, the boundary layer will be at both the end points $t = a$ and $t = b$ [2].

Singular perturbation problems are engendered by multiplication of a small positive parameter ε to highest derivative term of differential equation with boundary conditions. Many scholars have studied the analytical and numerical solutions of these problems, but sometimes they found that the classical numerical methods failed to get good approximate solutions of singular perturbation problems. That's why they have gone for the non classical methods. In the last few decades, many researchers have discussed the numerical solutions of singular perturbation problems. Most of the researchers have studied the numerical solutions of second order singular perturbation problems [5, 10, 11, 12, 13, 17, 19, 20, 21, 22, 29]. Only a few researchers have focused the numerical solutions of higher order singular perturbation problems [3, 24, 23, 28, 27]. Lodhi and Mishra [14, 15] have suggested the computational technique for numerical solutions of fourth order singular singularly perturbed and self adjoint boundary value problems. Raja and Tamilselvan [23] have designed a shooting method on a Shishkin mesh to solve reaction-diffusion type problems. Mishra and Saini [18] have used initial value technique for the numerical solution of fourth order singularly perturbed boundary value problems. Sarakhsi et al [25] have studied the existence of boundary layer problem. Parameter uniform numerical scheme to solve fourth order singularly perturbed turning points problems have been presented by Geetha and Tamilselvan [7]. Sharma et al. [26] have done the survey on singularly perturbed turning point and interior layers problem. Geetha et al. [8] have applied parameter uniform numerical method based on Shishkin mesh for third order singularly perturbed turning point problems exhibiting boundary layers.

This paper describes a quintic B-spline approach for the numerical solution of fourth order singular perturbation boundary value problems and it has been proved to be second order convergence. The paper is organized as follows: In section 2, we describe the quintic B-spline method. Convergence analysis is established in section 3. Quasilinearization method is discussed in section 4. Section 5 gives the numerical results which substantiate the theoretical aspects. Finally, we discuss the conclusions in section 6.

2. Quintic B-spline Method

We divide the interval $[a,\ b]$ into N equal subinterval and we choose piecewise uniform mesh points represented by $\pi = \{t_0, t_1, t_2, ..., t_N\}$,such that $t_0 = a$, $t_N = b$ and $h = \frac{b-a}{N}$ is the piecewise uniform spacing. We define $L_2[a,\ b]$ as a vector space of all the integrable functions on $[a,\ b]$, and X be the linear subspace of $L_2[a,\ b]$. Now we define

$$B_i(t)=\frac{1}{h^5}\begin{cases}(t-t_{i-3})^5\,, & \text{if } t\in[t_{i-3},\ t_{i-2}]\\ h^5+5h^4(t-t_{i-2})+10h^3(t-t_{i-2})^2+10h^2(t-t_{i-2})^3 & \\ +5h(t-t_{i-2})^4-5(t-t_{i-2})^5\,, & \text{if } t\in[t_{i-2},\ t_{i-1}]\\ 26h^5+50h^4(t-t_{i-1})+20h^3(t-t_{i-1})^2-20h^2(t-t_{i-1})^3 & \\ -20h(t-t_{i-1})^4+10(t-t_{i-1})^5\,, & \text{if } t\in[t_{i-1},\ t_i]\\ 26h^5+50h^4(t_{i+1}-t)+20h^3(t_{i+1}-t)^2-20h^2(t_{i+1}-t)^3 & \\ -20h(t_{i+1}-x)^4+10(t_{i+1}-t)^5\,, & \text{if } t\in[t_i,\ t_{i+1}]\\ h^5+5h^4(t_{i+2}-t)+10h^3(t_{i+2}-t)^2+10h^2(t_{i+2}-t)^3 & \\ +5h(t_{i+2}-t)^4-5(t_{i+2}-t)^5\,, & \text{if } t\in[t_{i+1},\ t_{i+2}]\\ (t_{i+3}-t)^5\,, & \text{if } t\in[t_{i+2},\ t_{i+3}]\\ 0 & \text{otherwise, for i}=0,1,2,...\text{N}.\end{cases} \tag{2.1}$$

We introduce six additional knots as $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_{N+3} > t_{N+2} > t_{N+1} > t_N$. From equation (2.1), we can easily check that each of the functions $B_i(t)$ is four times continuously differentiable on the entire real line. Also, the values of $B_i(t)$,$B_i'(t)$,$B_i''(t)$,$B_i'''(t)$ and $B_i^{iv}(t)$at the nodal points are given in Table 1.

Table 1. Quintic B-spline basis and its derivative function values at nodal points

$B(t)$	t_{i-3}	t_{i-2}	t_{i-1}	t	t_{i+1}	t_{i+2}	t_{i+3}
$B_i(t)$	0	1	26	66	26	1	0
$B_i'(t)$	0	$5/h$	$50/h$	0	$-50/h$	$-5/h$	0
$B_i''(t)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0
$B_i'''(t)$	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0
$B_i^{iv}(t)$	0	$120/h^4$	$-480/h^4$	$720/h^4$	$-480/h^4$	$120/h^4$	0

Let $\Omega = \{B_{-2}, B_{-1}, B_0, B_1, \ldots\ldots, B_{N-1}, B_N, B_{N+1}, B_{N+2}\}$ and let $\phi_5(\pi) = \operatorname{span}\Omega$. The function Ω is linearly independent on $[a,\ b]$, thus $\phi_5(\pi)$ is $(N+5)$-dimensional. Even one can show that $\phi_5(\pi) \subseteq_{\text{subspace}} X$. Let L be a linear operator whose domain is X and whose range is also in X. Now we define

$$S(t)=\sum_{i=-2}^{N+2} c_i B_i(t)\,, \tag{2.2}$$

be the approximate solution of the problem (1.1) with boundary conditions (1.2), where $c_i's$ is an unknown coefficient and $B_i(t)'s$ a fifth degree spline function. To solve fourth order singularly perturbed two point boundary value problems, the spline functions are evaluated at nodal points $t = t_i\ (i = 0, 1, 2, \ldots, N)$ which are needed for the solution.

From Table 1 and equation (2.2), we obtain the following relationships:

$$y(t_i)=S(t_i)=c_{i-2}+26c_{i-1}+66c_i+26c_{i+1}+c_{i+2} \tag{2.3}$$

$$m(t_i)=S'(t_i)=\frac{1}{h}(-5c_{i-2}-50c_{i-1}+50c_{i+1}+5c_{i+2}) \tag{2.4}$$

$$M_i = S''(t_i) = \frac{1}{h^2}\left(20c_{i-2} + 40c_{i-1} - 120c_i + 40c_{i+1} + 20c_{i+2}\right) \quad (2.5)$$

$$T_i = S'''(t_i) = \frac{1}{h^3}\left(-60c_{i-2} + 120c_{i-1} - 120c_{i+1} + 60c_{i+2}\right) \quad (2.6)$$

$$F_i = S^{iv}(t_i) = \frac{1}{h^4}\left(120c_{i-2} - 480c_{i-1} + 720c_i - 480c_{i+1} + 120c_{i+2}\right) \quad (2.7)$$

Moreover, m_i, M_i, T_i and F_i can be used to approximate values of $y'(t_i)$, $y''(t_i)$, $y'''(t_i)$ and $y^{iv}(t_i)$.

Since $S(t)$ is an approximate solution, it will satisfy equation (1.1) with boundary conditions (1.2). Hence we get

$$-\varepsilon S^{iv}(t) - p(t) S'''(t) + q(t) S''(t) + r(t) S(t) = f(t), \quad (2.8)$$

and

$$S(a) = \eta_1, \quad S(b) = \eta_2, \quad S''(a) = \eta_3, \quad S''(b) = \eta_4. \quad (2.9)$$

Discretizing equation (2.8) at the nodal points $t_i (i = 0, 1, \ldots, N)$, we have

$$-\varepsilon S^{iv}(t_i) - p(t_i) S'''(t_i) + q(t_i) S''(t_i) + r(t_i) S(t_i) = f(t_i),$$

Using equations (2.3)-(2.7) in above equation and simplifying, we obtain

$$\begin{array}{l} -\frac{\varepsilon}{h^4}\left\{120c_{i-2} - 480c_{i-1} + 720c_i - 480c_{i+1} + 120c_{i+2}\right\} \\ -\frac{p_i}{h^3}\left\{-60c_{i-2} + 120c_{i-1} - 120c_{i+1} + 60c_{i+2}\right\} \\ +\frac{q_i}{h^2}\left\{20c_{i-2} + 40c_{i-1} - 120c_i + 40c_{i+1} + 20c_{i+2}\right\} \\ +r_i\left\{c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}\right\} = f_i h^4, \end{array} \quad (2.10)$$

where $p_i = p(t_i)$, $q_i = q(t_i)$, $r_i = r(t_i)$ and $f_i = f(t_i)$. After simplifying above equation, we get

$$\gamma_1(t_i) c_{i-2} + \gamma_2(t_i) c_{i-1} + \gamma_3(t_i) c_i + \gamma_4(t_i) c_{i+1} + \gamma_5(t_i) c_{i+2} = f_i h^4, \quad (2.11)$$

where

$$\begin{array}{l} \gamma_1(t_i) = -120\varepsilon + 60p_i h + 20q_i h^2 + r_i h^4, \\ \gamma_2(t_i) = 480\varepsilon - 120p_i h + 40q_i h^2 + 26r_i h^4, \\ \gamma_3(t_i) = -720\varepsilon - 120q_i h^2 + 66r_i h^4, \\ \gamma_4(t_i) = 480\varepsilon + 120p_i h + 40q_i h^2 + 26r_i h^4, \\ \gamma_5(t_i) = -120\varepsilon - 60p_i h + 20q_i h^2 + r_i h^4, \quad \text{for} \quad i = 0, 1, \ldots, N. \end{array}$$

From the boundary conditions, we get the following equations

$$c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2 = \eta_1, \quad (2.12)$$

$$c_{N-2} + 26c_{N-1} + 66c_N + 26c_{N+1} + c_{N+2} = \eta_2, \quad (2.13)$$

$$20c_{-2} + 40c_{-1} - 120c_0 + 40c_1 + 20c_2 = \eta_3 h^2, \quad (2.14)$$

and

$$20c_{N-2} + 40c_{N-1} - 120c_N + 40c_{N+1} + 20c_{N+2} = \eta_4 h^2. \quad (2.15)$$

Coupling equations (2.11)-(2.15) lead to a system of $(N+5)$ linear equations $AY = D$ in the $(N+5)$ unknowns, where

$$Y = \left[c_{-2}, c_{-1}, c_0, c_1, \ldots, c_{N-1}, c_N, c_{N+1}, c_{N+2}\right]^T,$$

$$D = \left[\eta_1,\ \eta_3 h^2,\ f_0 h^4,\ f_1 h^4, \ldots,\ f_{N-1} h^4, f_N h^4,\ \eta_4 h^2,\ \eta_2\right]^T$$

and the coefficient matrix A is given by

$$A=\begin{bmatrix}
1 & 26 & 66 & 26 & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\
20 & 40 & -120 & 40 & 20 & 0 & 0 & 0 & \cdots & 0 & 0\\
\gamma_1(t_0) & \gamma_2(t_0) & \gamma_3(t_0) & \gamma_4(t_0) & \gamma_5(t_0) & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & \gamma_1(t_1) & \gamma_2(t_1) & \gamma_3(t_1) & \gamma_4(t_1) & \gamma_5(t_1) & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \gamma_1(t_i) & \gamma_2(t_i) & \gamma_3(t_i) & \gamma_4(t_i) & \gamma_5(t_i) & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & \gamma_1(t_{N-1}) & \gamma_2(t_{N-1}) & \gamma_3(t_{N-1}) & \gamma_4(t_{N-1}) & \gamma_5(t_{N-1}) & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \gamma_1(t_N) & \gamma_2(t_N) & \gamma_3(t_N) & \gamma_4(t_N) & \gamma_5(t_N)\\
0 & 0 & \cdots & 0 & 0 & 0 & 20 & 40 & -120 & 40 & 20\\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & 26 & 66 & 26 & 1
\end{bmatrix}. \tag{2.16}$$

Since A is a non-singular matrix, so we can solve the system $AY = D$ for c_{-2},c_{-1},c_0,c_1,c_2,…,c_{N-2},c_{N-1},c_N,c_{N+1},c_{N+2} substituting these values into equation (2.2), we get the required approximate solution.

3. Derivation for convergence

In this section, a technique is portrayed which will ascertain the truncation error for the quintic B-spline method over the whole range $a \le t \le b$. Here, we suppose that function $y(t)$ has continuous derivatives in the whole range.

We calculate the following relationships by comparing the coefficients of $c_i \quad (i = -2, -1, 0, 1, \dots, N, N+1, N+2)$. From equations (2.3)-(2.7), we have

$$\begin{aligned} &S'(t_{i-2}) + 26S'(t_{i-1}) + 66S'(t_i) + 26S'(t_{i+1}) + S'(t_{i+2}) \\ &\quad = \tfrac{1}{h}\left\{-5y(t_{i-2}) - 50y(t_{i-1}) + 50y(t_{i+1}) + 5y(i_{i+2})\right\} \end{aligned} \tag{3.1}$$

$$\begin{aligned} &S''(t_{i-2}) + 26S''(t_{i-1}) + 66S''(t_i) + 26S''(t_{i+1}) + S''(t_{i+2}) \\ &\quad = \tfrac{1}{h^2}\left\{20y(t_{i-2}) + 40y(t_{i-1}) - 120y(t_i) + 40y(t_{i+1}) + 20y(t_{i+2})\right\} \end{aligned} \tag{3.2}$$

$$\begin{aligned} &S'''(t_{i-2}) + 26S'''(t_{i-1}) + 66S'''(t_i) + 26S'''(t_{i+1}) + S'''(t_{i+2}) \\ &\quad = \tfrac{1}{h^3}\left\{-60y(t_{i-2}) + 120y(t_{i-1}) - 120y(t_{i+1}) + 60y(t_{i+2})\right\} \end{aligned} \tag{3.3}$$

$$\begin{aligned} & S^{iv}(t_{i-2})+26S^{iv}(t_{i-1})+66S^{iv}(t_i)+26S^{iv}(t_{i+1})+S^{iv}(t_{i+2}) \\ & =\tfrac{1}{h^4}\left\{120y(t_{i-2})-480y(t_{i-1})+720y(t_i)-480y(t_{i+1})+120y(t_{i+2})\right\} \end{aligned} \tag{3.4}$$

Using the operator notation [6, 16], the equations (3.1)-(3.4) can we written as

$$S'(t_i)=\frac{1}{h}\left(\frac{-5E^{-2}-50E^{-1}+50E+5E^2}{E^{-2}+26E^{-1}+66I+26E+E^2}\right)y(t_i) \tag{3.5}$$

$$S''(t_i)=\frac{1}{h^2}\left(\frac{20E^{-2}+40E^{-1}-120I+40E+20E^2}{E^{-2}+26E^{-1}+66I+26E+E^2}\right)y(t_i) \tag{3.6}$$

$$S'''(t_i)=\frac{1}{h^3}\left(\frac{-60E^{-2}+120E^{-1}-120E+60E^2}{E^{-2}+26E^{-1}+66I+26E+E^2}\right)y(t_i) \tag{3.7}$$

$$S^{iv}(t_i)=\frac{1}{h^4}\left(\frac{120E^{-2}-480E^{-1}+720I-480E+120E^2}{E^{-2}+26E^{-1}+66I+26E+E^2}\right)y(t_i) \tag{3.8}$$

where the operators are defined as $Ey(t_i)=y(t_i+h)$, $Dy(t_i)=y'(t_i)$ and $Iy(t_i)=y(t_i)$. Let $E=e^{hD}$ and expand them in powers of hD, we get

$$S'(t_i)=y'(t_i)+\frac{1}{5040}h^6y^7(t_i)-\frac{1}{21600}h^8y^9(t_i)+\frac{1}{1036800}h^{10}y^{11}(t_i)+0(h^{11}) \tag{3.9}$$

$$\begin{aligned} S''(t_i)=& y''(t_i)+\tfrac{1}{720}h^4y^6(t_i)-\tfrac{1}{3360}h^6y^8(t_i)+\tfrac{1}{86400}h^8y^{10}(t_i) \\ & +\tfrac{221}{239500800}h^{10}y^{12}(t_i)\ +0(h^{11}) \end{aligned} \tag{3.10}$$

$$\begin{aligned} S'''(t_i)=& y'''(t_i)-\tfrac{1}{240}h^4y^7(t_i)+\tfrac{11}{30240}h^6y^9(t_i)-\tfrac{1}{28800}h^8y^{11}(t_i) \\ & +\tfrac{37}{11404800}h^{10}y^{13}(t_i)+0(h^{11}) \end{aligned} \tag{3.11}$$

$$\begin{aligned} S^{iv}(t_i)=& y^{iv}(t_i)-\tfrac{1}{12}h^2y^6(t_i)+\tfrac{1}{240}h^4y^8(t_i)-\tfrac{1}{7560}h^6y^{10}(t_i) \\ & -\tfrac{13}{907200}h^8y^{12}(t_i)+\tfrac{643}{159667200}h^{10}y^{14}(t_i)+0(h^{11}) \end{aligned} \tag{3.12}$$

We now define $e\,(t)=y\,(t)-S\,(t)$ and substitute equations (3.9)-(3.12) in the Taylor series expansion of $e\,(t_i+\theta h)$ we obtain

$$\begin{aligned} e\,(t_i+\theta h)=& \left(\tfrac{\theta^2}{1440}-\tfrac{5\theta^4}{1440}\right)h^6y^6(t_i)+\left(\tfrac{\theta}{5040}-\tfrac{\theta^2}{1440}\right)h^7y^7(t_i) \\ & +\left(-\tfrac{\theta^2}{6720}+\tfrac{\theta^4}{5760}\right)h^8y^8(t_i)+0(h^9) \end{aligned} \tag{3.13}$$

where $a\le\theta\le b$. We abridge the above results in the following theorem:

Theorem 3.1. *Let $y\,(t)$ be the exact solution and $S\,(t)$ be the numerical solution of the singularly perturbed fourth order boundary value problem (1.1) with the boundary conditions (1.2) for sufficiently small h which further gives the truncation error of $O\left(h^6\right)$ and method of convergence of $O\left(h^2\right)$.*

4. Quasilinearization method

Let us consider the boundary value problem

$$-\varepsilon y^{iv}\,(t)=F\,(t,\,y,\,y'',\,y'''),\qquad t=[a,\;b], \tag{4.1}$$

$$y\,(a)=\eta_1,\quad y\,(b)=\eta_2,\quad y''\,(a)=\eta_3,\quad y''\,(b)=\eta_4, \tag{4.2}$$

where $F\left(t,\, y,\, y'',\, y'''\right)$ is a smooth function such that

$$\begin{cases} F_{y'''}\left(t,\, y,\, y'',\, y'''\right) \geq \alpha > 0, \\ F_{y''}\left(t,\, y,\, y'',\, y'''\right) \geq \beta > 0, \quad t \in \left[a,\ b\right], \\ 0 \geq F_{y}\left(t,\, y,\, y'',\, y'''\right) \geq -\lambda, \quad \lambda > 0. \end{cases} \tag{4.3}$$

In order to obtain the numerical solution of the boundary value problem (4.1) and (4.2), Newton's method of quasilinearization [1, 4] is applied to generate the sequence of $\{y_k\}_0^\infty$of successive approximations with a proper selection of initial guess y_0. We define y_{k+1}, for each fixed non-negative integer k,to be solution of the following linear problem:

$$-\varepsilon y_{k+1}^{iv}\left(t\right) - p_k\left(t\right) y_{k+1}'''\left(t\right) + q_k\left(t\right) y_{k+1}''\left(t\right) + r_k\left(t\right) y_{k+1}\left(t\right) = f_k\left(t\right), \quad t \in \left[a,\ b\right], \tag{4.4}$$

$$y_{k+1}\left(a\right) = \eta_1, \quad y_{k+1}\left(b\right) = \eta_2, \quad y_{k+1}''\left(a\right) = \eta_3, \quad y_{k+1}''\left(b\right) = \eta_4, \tag{4.5}$$

where

$$\begin{aligned} p_k\left(t\right) &= F_{y'''}\left(t,\, y_k,\, y_k'',\, y_k'''\right),\ \ q_k\left(t\right) = F_{y''}\left(t,\, y_k,\, y_k'',\, y_k'''\right), \\ r_k\left(t\right) &= F_{y}\left(t,\, y_k,\, y_k'',\, y_k'''\right),\ f_k\left(t\right) = F_{y}\left(t,\, y_k,\, y_k'',\, y_k'''\right) \\ &\quad - y_k F_{y}\left(t,\, y_k,\, y_k'',\, y_k'''\right) - y_k'' F_{y''}\left(t,\, y_k,\, y_k'',\, y_k'''\right) \\ &\quad - y_k''' F_{y'''}\left(t,\, y_k,\, y_k'',\, y_k'''\right). \end{aligned}$$

We make the following observations:

i) If the initial guess y_0 is sufficiently close to the solution $y\left(t\right)$ of (4.1) and (4.5), then the sequence $\{y_k\}_0^\infty$converges to $y(x)$. One can see the proof given in [4]. From (4.3), it follows that, for each fixed k,

$$\begin{aligned} &p_k\left(t\right) = F_{y'''}\left(t,\, y,\, y'',\, y'''\right) \geq \alpha > 0, \\ &q_k\left(t\right) = F_{y''}\left(t,\, y,\, y'',\, y'''\right) \geq \beta > 0, \\ &0 \geq r_k\left(t\right) = F_{y}\left(t,\, y,\, y'',\, y'''\right) \geq -\lambda, \quad \lambda > 0. \end{aligned} \tag{4.6}$$

ii) Problem (4.4) with the boundary conditions (4.5), for each fixed k,is a linear fourth order boundary value problem which is in the form of (1.1) and (1.2). Hence it can be solved by the method described in section 2.

iii) The following convergence criterion is used to terminate the iteration:

$$\left\|y_{k+1}\left(t_i\right) - y_k\left(t_i\right)\right\| \leq \varepsilon, \quad t_i \in \left[a,\ b\right], \quad k \geq 0. \tag{4.7}$$

5. Numerical results

In the present section, we have presented numerical results of the considered examples with the help of MATLAB software which verifies theoretical estimates. When the exact solutions of the considered examples are available then the maximum absolute errors E^N are evaluated using the following formula for the present method, which is given by

$$E^N = \max_{t_i \in [a,\, b]} \left|y_\varepsilon^N(t_i) - S_\varepsilon^N(t_i)\right|, \tag{5.1}$$

When the exact solutions of the considered examples are not available then the maximum absolute errors E_d^N are evaluated using the double mesh principle for the present method, which is given by

$$E_d^N = \max_{t_i \in [a,\, b]} \left| S_\varepsilon^N (t_i) - S_\varepsilon^{2N} (t_i) \right| , \tag{5.2}$$

The numerical order of convergence is computed using the following formula

$$\mathrm{Ord}^N = \frac{\ln E^N - \ln E^{2N}}{\ln 2}. \tag{5.3}$$

The exact and approximate solutions are denoted by y_ε^N and S_ε^N respectively.

Example 5.1 Consider the following singular perturbation boundary value problem [27]:

$$-\varepsilon y^{iv} (t) - 4y''' (t) = 1, \qquad t \in [0,\ 1] ,$$
$$y (0) = 1, \quad y (1) = 1, \qquad y'' (0) = -1, \qquad y'' (1) = -1.$$

The exact solution of Example 5.1 is given by

$$y (t) = \frac{1}{192\left(1-e^{-\frac{4}{\varepsilon}}\right)} \Big\{ -3\varepsilon^2 e^{-\frac{4t}{\varepsilon}} - 72t^2 - 8t^3 + 80t - 3t\varepsilon^2 + 192 + 3\varepsilon^2 + e^{-\frac{4}{\varepsilon}} \left(-192 + 96t^2 + 8t^3 - 104t + 3t\varepsilon^2\right) \Big\} .$$

Table 2. Maximum absolute errors and order of convergence of Example 5.1 for different values of ε and N.

N	$\varepsilon = 2^{-4}$	Ord	$\varepsilon = 2^{-5}$	Ord	$\varepsilon = 2^{-6}$	Ord	$\varepsilon = 2^{-7}$	Ord	$\varepsilon = 2^{-8}$	Ord
64	9.5153E-06	2.0442	1.1597E-05	2.2186	9.3841E-06	1.6594	7.6100E-06	1.6904	1.0217E-05	2.4174
128	2.3070E-06	2.0097	2.4916E-06	2.0418	2.9708E-06	2.2173	2.3579E-06	1.6494	1.9126E-06	1.6944
256	5.7289E-07	2.0010	6.0512E-07	2.0095	6.3885E-07	2.0421	7.5163E-07	2.2142	5.9098E-07	1.6435
512	1.4312E-07	2.1459	1.5029E-07	1.4799	1.5512E-07	2.0623	1.6198E-07	2.0885	1.8916E-07	2.2051
1024	3.2339E-08		5.3882E-08		3.7140E-08		3.8086E-08		4.1024E-08	

Example 5.2. Consider the following singular perturbation boundary value problem [7]:

$$-\varepsilon y^{iv} (t) + 5ty''' (t) + 4y'' (t) + 2y (t) = 0, \qquad t \in [-1,\ 1] ,$$
$$y (-1) = 1, \quad y (1) = 1, \qquad y'' (-1) = 1, \qquad y'' (1) = 1.$$

Table 3. Comparison of maximum absolute error and order of convergence of Example 5.2 for different values of N and $\varepsilon = 2^{-4}$.

N	Geetha and Tamilselvan [7]		Present Method	
	E_d^N	Ord	E_d^N	Ord
64	3.9249E-2	0.9770	4.4948E-04	2.5794
128	1.9940E-2	0.9886	7.5204E-05	2.0864
256	1.0049E-2	0.9944	1.7708E-05	2.0177
512	5.0440E-3	0.9972	4.3731E-06	2.3071
1024	2.5269E-3		8.8366E-07	

Example 5.3. Consider the following singular perturbation boundary value problem [7]:

$$-\varepsilon y^{iv}(t) + 5ty'''(t) + (4+t)\,y''(t) + \left(2+t^2\right) y(t) = -e^t + 5, \quad t \in [-1,\ 1],$$

$$y(-1) = 1, \quad y(1) = 1, \qquad y''(-1) = 2, \qquad y''(1) = 2.$$

Table 4. Comparison of maximum absolute error and order of convergence of Example 5.3 for different values of N and $\varepsilon = 2^{-4}$

N	Geetha and Tamilselvan [7]		Present Method	
	E_d^N	Ord	E_d^N	Ord
64	3.3778E-2	0.9823	4.1824E-04	2.5806
128	1.7097E-2	0.9913	6.9920E-05	2.0866
256	8.6002E-3	0.9957	1.6462E-05	2.0306
512	4.3130E-3	0.8693	4.0291E-06	2.1087
1024	2.3610E-3		9.3418E-07	

Example 5.4. Consider the following singular perturbation boundary value problem [7]:

$$-\varepsilon y^{iv}(t) + 5ty'''(t) + (4+t)\,y''(t) + 2y^2(t) = 0, \quad t \in [-1,\ 1],$$

$$y(-1) = 1, \quad y(1) = 1, \qquad y''(-1) = 2, \qquad y''(1) = 2.$$

Table 5. Comparison of maximum absolute error and order of convergence of Example 5.4 for different values of N and $\varepsilon = 2^{-4}$

N	Geetha and Tamilselvan [7]		Present Method	
	E_d^N	Ord	E_d^N	Ord
64	7.5762E-02	0.9731	1.1620e-03	2.5795
128	3.8593E-02	0.9867	1.9440e-04	2.0863
256	1.9475E-02	0.9934	4.5777e-05	2.0215
512	9.7821E-03	0.9967	1.1275e-05	1.7449
1024	4.9021E-03	-	3.3639e-06	-

6. Conclusions

In this article, we have used the quintic B-spline method for finding the approximate solution of fourth order linear and non-linear singular perturbation boundary value problems. We linearised the non-linear boundary value problem via quasilinearization method and solved the problem. It is a computationally proficient technique and the algorithm can easily be applied on a computer. The results obtained through this method are better than the existing method [7] with the same number of nodal points.

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