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Inequalities for the area balance of absolutely continuous functions

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Abstract. We introduce the *area balance* function associated to a Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ by

$$AB_f(a,b,\cdot):[a,b]\to\mathbb{C},\ AB_f(a,b,x):=\frac{1}{2}\left[\int_x^b f(t)\,dt-\int_a^x f(t)\,dt\right].$$

We show amongst other that, if $f:I\to\mathbb{C}$ is an absolutely continuous function on the interval I and $[a,b]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is of bounded variation on $[a,b]$, then we have the inequality

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|$$
$$\leq\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')$$

for any $x\in[a,b]$.

If there exists the real numbers m,M such that

$$m\leq f'(t)\leq M \text{ for a.e. } t\in[a,b],$$

then also

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)-\frac{m+M}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|$$
$$\leq\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right](M-m)$$

for any $x\in[a,b]$.

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1. Introduction

For a *Lebesgue integrable* function $f : [a,b] \to \mathbb{C}$ and a number $x \in (a,b)$ we can naturally ask how far the integral $\int_x^b f(t)\,dt$ is from the integral $\int_a^x f(t)\,dt$. If f is nonnegative and continuous on $[a,b]$, then the above question has the geometrical interpretation of comparing the area under the curve generated by f at the right of the point x with the area at the left of x. The point x will be called a *median point*, if

$$\int_x^b f(t)\,dt = \int_a^x f(t)\,dt.$$

Due to the above geometrical interpretation, we can introduce the *area balance* function associated to a Lebesgue integrable function $f : [a,b] \to \mathbb{C}$ defined as

$$AB_f(a,b,\cdot) : [a,b] \to \mathbb{C},\ AB_f(a,b,x) := \frac{1}{2}\left[\int_x^b f(t)\,dt - \int_a^x f(t)\,dt\right].$$

Utilising the *cumulative function* notation $F : [a,b] \to \mathbb{C}$ given by

$$F(x) := \int_a^x f(t)\,dt$$

then we observe that

$$AB_f(a,b,x) = \frac{1}{2}F(b) - F(x),\ \ x \in [a,b].$$

If f is a *probability density*, i.e. f is nonnegative and $\displaystyle\int_a^b f(t)\,dt = 1$, then

$$AB_f(a,b,x) = \frac{1}{2} - F(x),\ \ x \in [a,b].$$

In this paper we obtain some inequalities concerning the area balance for absolutely continuous. Applications for differentiable functions whose derivatives are Lipschitzian functions are provided. Bounds involving the *Jensen difference*

$$\frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right)$$

are also established.

We notice that Jensen difference is closely related to the Hermite-Hadamard type inequalities where various bounds for the quantities

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)$$

are provided, see [1]-[6] and [8]-[18].

2. Preliminary results

The following representation result holds:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the representation*

$$AB_f(a,b,x) = \left(\frac{a+b}{2} - x\right) f(x) + \frac{1}{2}\left[\int_a^x (t-a) f'(t)\,dt + \int_x^b (b-t) f'(t)\,dt\right] \tag{2.1}$$

and

$$AB_f(a,b,x) = \frac{bf(b) + af(a)}{2} - \frac{f(b)+f(a)}{2} x - \frac{1}{2}\int_a^b |t-x| f'(t)\,dt \tag{2.2}$$

for any $x \in [a,b]$, where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Since f is absolutely continuous on $[a,b]$, then f is differentiable almost everywhere (a.e.) on $[a,b]$ and the Lebesgue integrals in the right hand side of the equations (2.1) and (2.2) exist.

Utilising the integration by parts formula for the Lebesgue integral, we have

$$\begin{aligned} &\int_a^x (t-a) f'(t)\,dt + \int_x^b (b-t) f'(t)\,dt \\ &= (t-a) f(t)\big|_a^x - \int_a^x f(t)\,dt + (b-t) f(t)\big|_x^b + \int_x^b f(t)\,dt \\ &= (x-a) f(x) - \int_a^x f(t)\,dt - (b-x) f(x) + \int_x^b f(t)\,dt \\ &= (2x-a-b) f(x) + 2AB_f(a,b,x) \end{aligned} \tag{2.3}$$

for any $x \in [a,b]$.

Dividing (2.3) by 2 and rearranging the equation, we deduce (2.1).

Integrating by parts, we also have

$$\begin{aligned} &\int_a^b |t-x| f'(t)\,dt \\ &= \int_a^x (x-t) f'(t)\,dt + \int_x^b (t-x) f'(t)\,dt \\ &= (x-t) f(t)\big|_a^x + \int_a^x f(t)\,dt + (t-x) f(t)\big|_x^b - \int_x^b f(t)\,dt \\ &= -(x-a) f(a) + (b-x) f(b) - 2AB_f(a,b,x) \\ &= bf(b) + af(a) - [f(b)+f(a)]\,x - 2AB_f(a,b,x) \end{aligned} \tag{2.4}$$

for any $x \in [a,b]$.

Dividing (2.4) by 2 and rearranging the equation, we deduce (2.2). $\square$

Corollary 2.2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'(t)\geq 0$ for a.e. $t\in[a,b]$, then*

$$\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x \geq AB_f(a,b,x) \geq \left(\frac{a+b}{2}-x\right)f(x) \tag{2.5}$$

for any $x\in[a,b]$.

In particular,

$$\frac{1}{4}(b-a)[f(b)-f(a)] \geq AB_f\left(a,b,\frac{a+b}{2}\right)\geq 0. \tag{2.6}$$

The constant $\frac{1}{4}$ is a best possible constant in the sense that it cannot be replaced by a smaller quantity.

Proof. The inequalities (2.5) follow from the representations (2.1) and (2.2) by taking into account that $f'(t)\geq 0$ for a.e. $t\in[a,b]$.

The inequality (2.6) follows by (2.5) for $x=\frac{a+b}{2}$.

Assume that the first inequality in (2.6) holds for a constant $C>0$, i.e.

$$C(b-a)[f(b)-f(a)]\geq AB_f\left(a,b,\frac{a+b}{2}\right) \tag{2.7}$$

Consider the function $f_n:[-1,1]\to\mathbb{R}$ given by

$$f_n(t)=\begin{cases} 0 & \text{if } t\in[-1,0] \\ nt & \text{if } t\in\left(0,\frac{1}{n}\right) \\ 1 & \text{if } t\in\left[\frac{1}{n},1\right] \end{cases}$$

where $n\geq 2$, a natural number. This functions is absolutely continuous and $f_n'(t)\geq 0$ for any $t\in(-1,1)$. We have for $a=-1, b=1$

$$C(b-a)[f_n(b)-f_n(a)]=2C$$

and

$$\begin{aligned} AB_{f_n}\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_0^1 f_n(t)\,dt-\int_{-1}^0 f_n(t)\,dt\right] \\ &= \frac{1}{2}\left(\int_0^{\frac{1}{n}} nt dt+\int_{\frac{1}{n}}^1 1dt\right) \\ &= \frac{1}{2}\left(\frac{1}{2n}+1-\frac{1}{n}\right)=\frac{1}{2}\left(1-\frac{1}{2n}\right). \end{aligned}$$

Replacing these values in (2.7) we get

$$2C \geq \frac{1}{2}\left(1-\frac{1}{2n}\right) \tag{2.8}$$

for any $n \geq 2$.

Taking the limit for $n \to \infty$ in (2.8) we get $C \geq \frac{1}{4}$, which proves that $\frac{1}{4}$ is best possible in the first inequality in (2.6) □

Remark 2.3. Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'(t) \geq 0$ for a.e. $t \in [a,b]$, then $AB_f(a,b,x) \geq 0$ for $x \in \left[a, \frac{a+b}{2}\right] \left(\left[\frac{a+b}{2}, b\right]\right)$.

Moreover, if $f(b) \neq -f(a)$ and

$$\frac{bf(b)+af(a)}{f(b)+f(a)} \in [a,b] \tag{2.9}$$

then

$$AB_f\left(a,b,\frac{bf(b)+af(a)}{f(b)+f(a)}\right) \leq 0. \tag{2.10}$$

Also, if $f(a), f(b) > 0$, then (2.9) holds and the inequality (2.10) is valid.

Corollary 2.4. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $\gamma \in \mathbb{C}$. Then we have the representation*

$$\begin{aligned} AB_f(a,b,x) &= \frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]+\left(\frac{a+b}{2}-x\right)f(x) \\ &+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} AB_f(a,b,x) &= \frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x \\ &-\frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right] \\ &-\frac{1}{2}\int_a^b |t-x|(f'(t)-\gamma)\,dt \end{aligned} \tag{2.12}$$

for any $x \in [a,b]$.

Proof. Let $e(t) = t, t \in [a,b]$. If we write the equality (2.1) for the function $f-\gamma e$ we have

$$\begin{aligned} AB_{f-\gamma e}(a,b,x) &= \left(\frac{a+b}{2}-x\right)(f(x)-\gamma x) \\ &+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \end{aligned} \tag{2.13}$$

for any $x \in [a,b]$.

Observe that

$$AB_{f-\gamma e}(a,b,x) = AB_f(a,b,x) - \gamma AB_e(a,b,x)$$

and

$$\begin{aligned} AB_e(a,b,x) &= \frac{1}{2}\left(\int_x^b tdt - \int_a^x tdt\right) \\ &= \frac{1}{2}\left(\frac{b^2-x^2}{2} - \frac{x^2-a^2}{2}\right) = \frac{1}{2}\left(\frac{a^2+b^2}{2} - x^2\right). \end{aligned}$$

From (2.13) we have

$$\begin{aligned} AB_f(a,b,x) &= \left(\frac{a+b}{2} - x\right)(f(x) - \gamma x) + \frac{1}{2}\gamma\left(\frac{a^2+b^2}{2} - x^2\right) \qquad (2.14) \\ &+ \frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt + \int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \\ &= \left(\frac{a+b}{2} - x\right) f(x) + \frac{1}{2}\gamma\left(\frac{a^2+b^2}{2} - x^2\right) - \gamma\left(\frac{a+b}{2} - x\right)x \\ &+ \frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt + \int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \qquad (2.15) \\ &= \frac{1}{2}\gamma\left[x^2 - (a+b)x + \frac{a^2+b^2}{2}\right] + \left(\frac{a+b}{2} - x\right) f(x) \\ &+ \frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt + \int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \end{aligned}$$

for any $x \in [a,b]$.

Since

$$x^2 - (a+b)x + \frac{a^2+b^2}{2} = \left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2$$

then from (2.14) we deduce the desired equality (2.11).

From (2.2) we have

$$\begin{aligned} AB_{f-\gamma e}(a,b,x) &= \frac{bf(b)+af(a)}{2} - \gamma\frac{b^2+a^2}{2} - \frac{f(b)+f(a)}{2}x + \gamma\frac{a+b}{2}x \\ &- \frac{1}{2}\int_a^b |t-x|(f'(t)-\gamma)\,dt \end{aligned}$$

and since

$$AB_{f-\gamma e}(a,b,x) = AB_f(a,b,x) - \gamma AB_e(a,b,x)$$

then

$$
\begin{aligned}
AB_f(a,b,x) &= \frac{1}{2}\gamma\left(\frac{a^2+b^2}{2}-x^2\right)+\frac{bf(b)+af(a)}{2}\\
&-\gamma\frac{b^2+a^2}{2}-\frac{f(b)+f(a)}{2}x+\gamma\frac{a+b}{2}x\\
&-\frac{1}{2}\int_a^b |t-x|\left(f'(t)-\gamma\right)dt\\
&=\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x\\
&-\frac{1}{2}\gamma\left[x^2-(a+b)x+\frac{a^2+b^2}{2}\right]-\frac{1}{2}\int_a^b |t-x|\left(f'(t)-\gamma\right)dt
\end{aligned}
$$

which proves the desired equality (2.12). □

Remark 2.5. We have the following equalities

$$
\begin{aligned}
AB_f\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{8}\gamma(b-a)^2 \\
&+\frac{1}{2}\left[\int_a^{\frac{a+b}{2}}(t-a)\left(f'(t)-\gamma\right)dt+\int_{\frac{a+b}{2}}^b (b-t)\left(f'(t)-\gamma\right)dt\right]
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
AB_f\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{4}(b-a)\left[f(b)-f(a)\right]-\frac{1}{8}\gamma(b-a)^2\\
&-\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right|\left(f'(t)-\gamma\right)dt
\end{aligned}
\tag{2.17}
$$

for any $\gamma \in \mathbb{C}$.

3. Bounds for absolutely continuous functions

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$
\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C}|\mathrm{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for each } t\in[a,b]\right\}
$$

and

$$
\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma| \text{ for each } t\in[a,b]\right\}.
$$

The following representation result may be stated.

Proposition 3.1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$
\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}
$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} \left| \Gamma - \gamma \right|$$

if and only if

$$\operatorname{Re}\left[\left(\Gamma - z \right) \left(\bar{z} - \bar{\gamma} \right) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} \left| \Gamma - \gamma \right|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}\left[\left(\Gamma - z \right) \left(\bar{z} - \bar{\gamma} \right) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3.2. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]} \left(\gamma, \Gamma \right) = \{ f : [a,b] \to \mathbb{C} \mid \ & \left(\operatorname{Re}\Gamma - \operatorname{Re} f \left(t \right) \right) \left(\operatorname{Re} f \left(t \right) - \operatorname{Re}\gamma \right) \\ & + \left(\operatorname{Im}\Gamma - \operatorname{Im} f \left(t \right) \right) \left(\operatorname{Im} f \left(t \right) - \operatorname{Im}\gamma \right) \geq 0 \text{ for each } t \in [a,b] \} . \end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}\left(\Gamma \right) \geq \operatorname{Re}\left(\gamma \right)$ and $\operatorname{Im}\left(\Gamma \right) \geq \operatorname{Im}\left(\gamma \right)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]} \left(\gamma, \Gamma \right) := \{ f : [a,b] \to \mathbb{C} \mid & \operatorname{Re}\left(\Gamma \right) \geq \operatorname{Re} f \left(t \right) \geq \operatorname{Re}\left(\gamma \right) \\ & \text{and } \operatorname{Im}\left(\Gamma \right) \geq \operatorname{Im} f \left(t \right) \geq \operatorname{Im}\left(\gamma \right) \text{ for each } t \in [a,b] \} . \end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]} \left(\gamma, \Gamma \right)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]} \left(\gamma, \Gamma \right) \subseteq \bar{U}_{[a,b]} \left(\gamma, \Gamma \right) . \tag{3.4}$$

Theorem 3.3. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. If there exists $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f' \in \bar{U}_{[a,b]} \left(\gamma, \Gamma \right)$ then*

$$\begin{aligned} & \left| AB_f \left(a,b,x \right) - \left(\frac{a+b}{2} - x \right) f \left(x \right) \right. \\ & \left. - \frac{\gamma + \Gamma}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4} \left(b-a \right)^2 \right] \right| \\ & \leq \frac{\left| \Gamma - \gamma \right|}{4} \left[\frac{1}{4} \left(b-a \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & \left| AB_f \left(a,b,x \right) - \frac{bf\left(b \right) + af\left(a \right)}{2} + \frac{f\left(b \right) + f\left(a \right)}{2} x \right. \\ & \left. + \frac{\gamma + \Gamma}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4} \left(b-a \right)^2 \right] \right| \\ & \leq \frac{\left| \Gamma - \gamma \right|}{4} \left[\frac{1}{4} \left(b-a \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{3.6}$$

for any $x \in [a, b]$.

Proof. From the equality (2.11) we have

$$AB_f(a,b,x) - \frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]-\left(\frac{a+b}{2}-x\right)f(x) = \frac{1}{2}\left[\int_a^x (t-a)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt+\int_x^b (b-t)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right] \tag{3.7}$$

for any $x \in [a, b]$.

If $f' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, then by taking the modulus in (3.7) we get

$$\begin{aligned}
&\left|AB_f(a,b,x) - \left(\frac{a+b}{2}-x\right)f(x) -\frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right| \\
&= \frac{1}{2}\left|\int_a^x (t-a)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt+\int_x^b (b-t)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right| \\
&\le \frac{1}{2}\left[\left|\int_a^x (t-a)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right|+\left|\int_x^b (b-t)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right|\right] \\
&\le \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t)-\frac{\gamma+\Gamma}{2}\right|dt+\int_x^b (b-t)\left|f'(t)-\frac{\gamma+\Gamma}{2}\right|dt\right] \\
&\le \frac{|\Gamma-\gamma|}{4}\left[\int_a^x (t-a)\,dt+\int_x^b (b-t)\,dt\right] \\
&= \frac{|\Gamma-\gamma|}{4}\left[\frac{(x-a)^2+(b-x)^2}{2}\right] = \frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right],
\end{aligned}$$

for any $x \in [a, b]$, which proves the inequality (3.5).

From the equality (2.12) we have

$$AB_f(a,b,x) - \frac{bf(b)+af(a)}{2}+\frac{f(b)+f(a)}{2}x + \frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right] = -\frac{1}{2}\int_a^b |t-x|\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt \tag{3.8}$$

for any $x \in [a, b]$.

Taking the modulus in (3.8) and using the fact that

$$f' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$$

we have

$$\begin{aligned}
&\Bigg|AB_f(a,b,x)-\frac{bf(b)+af(a)}{2}+\frac{f(b)+f(a)}{2}x \\
&+\frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\Bigg| \\
&\le\frac{1}{2}\int_a^b|t-x|\left|f'(t)-\frac{\gamma+\Gamma}{2}\right|dt \\
&\le\frac{|\Gamma-\gamma|}{4}\int_a^b|t-x|\,dt=\frac{|\Gamma-\gamma|}{4}\left[\int_a^x(x-t)\,dt+\int_x^b(t-x)\,dt\right] \\
&=\frac{|\Gamma-\gamma|}{4}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]=\frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]
\end{aligned}$$

for any $x\in[a,b]$, which proves the desired inequality (3.6). □

Remark 3.4. Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers m,M such that

$$m\le f'(t)\le M \text{ for a.e. } t\in[a,b],$$

then

$$\begin{aligned}
&\Bigg|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x) \\
&-\frac{m+M}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\Bigg| \\
&\le\frac{M-m}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\Bigg|AB_f(a,b,x)-\frac{bf(b)+af(a)}{2}+\frac{f(b)+f(a)}{2}x \\
&+\frac{m+M}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\Bigg| \\
&\le\frac{M-m}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]
\end{aligned}\tag{3.10}$$

for any $x\in[a,b]$.

Corollary 3.5. *With the assumptions of Theorem* 3.3 *we have*

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)-\frac{\gamma+\Gamma}{16}(b-a)^2\right|\le\frac{|\Gamma-\gamma|}{16}(b-a)^2\tag{3.11}$$

and

$$\left|\frac{1}{4}(b-a)[f(b)-f(a)]-\frac{\gamma+\Gamma}{16}(b-a)^2-AB_f\left(a,b,\frac{a+b}{2}\right)\right| \tag{3.12}$$
$$\leq\frac{|\Gamma-\gamma|}{16}(b-a)^2.$$

Theorem 3.6. *Let $f:I\to\mathbb{R}$ be an absolutely continuous function on the interval I and $[a,b]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is of bounded variation on $[a,b]$. Then we have the inequalities*

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right. \tag{3.13}$$
$$\left.-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|$$
$$\leq\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')$$

and

$$\left|AB_f(a,b,x)-\frac{bf(b)+af(a)}{2}+\frac{f(b)+f(a)}{2}x\right. \tag{3.14}$$
$$\left.+\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|$$
$$\leq\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')$$

for any $x\in[a,b]$.

Proof. From (2.11) for $\gamma=\frac{f'(a)+f'(b)}{2}$ we have the representation

$$AB_f(a,b,x) \tag{3.15}$$
$$-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]-\left(\frac{a+b}{2}-x\right)f(x)$$
$$=\frac{1}{2}\left[\int_a^x(t-a)\left(f'(t)-\frac{f'(a)+f'(b)}{2}\right)dt\right.$$
$$\left.+\int_x^b(b-t)\left(f'(t)-\frac{f'(a)+f'(b)}{2}\right)dt\right]$$

for any $x\in[a,b]$.

Taking the modulus in (3.15) we get

$$\begin{aligned}
&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right. \qquad (3.16)\\
&\left.-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|\\
&\le\frac{1}{2}\left[\int_a^x(t-a)\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right|dt\right.\\
&\left.+\int_x^b(b-t)\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right|dt\right]
\end{aligned}$$

for any $x\in[a,b]$.

For $t\in[a,x]$ we have

$$\begin{aligned}
\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right| &= \left|\frac{f'(t)-f'(a)+f'(t)-f'(b)}{2}\right|\\
&\le \frac{1}{2}\left[|f'(t)-f'(a)|+|f'(b)-f'(t)|\right]\\
&\le \frac{1}{2}\bigvee_a^b(f')
\end{aligned}$$

and similarly, for $t\in[x,b]$ we have

$$\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right|\le\frac{1}{2}\bigvee_a^b(f')$$

and then by (3.16) we get

$$\begin{aligned}
&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right.\\
&\left.-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|\\
&\le\frac{1}{4}\left[\int_a^x(t-a)\,dt+\int_x^b(b-t)\,dt\right]\bigvee_a^b(f')\\
&=\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')
\end{aligned}$$

for $t\in[a,b]$, and the inequality (3.13) is proved.

The second inequality goes along a similar way and we omit the details. □

Corollary 3.7. *With the assumptions of Theorem* 3.6 *we have*

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)-\frac{f'(a)+f'(b)}{16}(b-a)^2\right|\le\frac{1}{16}(b-a)^2\bigvee_a^b(f') \qquad (3.17)$$

and

$$\left|\frac{1}{4}(b-a)[f(b)-f(a)]-\frac{f'(a)+f'(b)}{16}(b-a)^2-AB_f\left(a,b,\frac{a+b}{2}\right)\right| \tag{3.18}$$
$$\leq \frac{1}{16}(b-a)^2\bigvee_a^b(f').$$

4. Bounds for Lipschitzian derivatives

We say that v is *Lipschitzian* with the constant $L>0$, if

$$|v(t)-v(s)|\leq L|t-s|$$

for any $t,s\in[a,b]$.

Theorem 4.1. *Let $f:I\to\mathbb{R}$ be an absolutely continuous function on the interval I and $[a,b]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is Lipschitzian with the constant $K>0$ on $[a,b]$. Then we have the inequalities*

$$\begin{aligned}&\Bigg|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\\&-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\Bigg|\\&\leq\frac{1}{12}(b-a)K\left[3\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\end{aligned} \tag{4.1}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)-\frac{1}{8}f'\left(\frac{a+b}{2}\right)(b-a)^2\right|\leq\frac{1}{48}K(b-a)^3. \tag{4.2}$$

The constant $\frac{1}{48}$ is best possible in (4.2).

Proof. We have from the equality (2.11) that

$$\begin{aligned}&AB_f(a,b,x)\\&-\left(\frac{a+b}{2}-x\right)f(x)-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\\&=\frac{1}{2}\left[\int_a^x(t-a)[f'(t)-f'(x)]\,dt+\int_x^b(b-t)[f'(t)-f'(x)]\,dt\right]\end{aligned} \tag{4.3}$$

for any $x\in[a,b]$.

Taking the modulus on (4.3) we have

$$\begin{aligned} &\Bigg| AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x) \qquad (4.4)\\ &-\frac{1}{2} f'(x) \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\Bigg| \\ &\leq \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t) - f'(x)\right| dt + \int_x^b (b-t)\left|f'(t) - f'(x)\right| dt\right] \\ &\leq \frac{1}{2} K \left[\int_a^x (t-a)(x-t)\, dt + \int_x^b (b-t)(t-x)\, dt\right] \end{aligned}$$

for any $x \in [a,b]$.

Since a simple calculation shows that

$$\int_c^d (t-c)(d-t)\, dt = \frac{1}{6}(d-c)^3,$$

then

$$\begin{aligned} &\int_a^x (t-a)(x-t)\, dt + \int_x^b (b-t)(t-x)\, dt \\ &= \frac{1}{6}\left[(x-a)^3 + (b-x)^3\right] \\ &= \frac{1}{6}(b-a)\left[3\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \end{aligned}$$

for any $x \in [a,b]$.

Utilising (4.4) we get the desired inequality (4.1).

Consider the function $f : [a,b] \to \mathbb{R}$,

$$f(t) := \begin{cases} -\left(t - \frac{a+b}{2}\right)^2 & \text{if } t \in \left[a, \frac{a+b}{2}\right) \\ \left(t - \frac{a+b}{2}\right)^2 & \text{if } t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Then f is differentiable and

$$\begin{aligned} f'(t) &= \begin{cases} -2\left(t - \frac{a+b}{2}\right) & \text{if } t \in \left[a, \frac{a+b}{2}\right) \\ 2\left(t - \frac{a+b}{2}\right) & \text{if } t \in \left[\frac{a+b}{2}, b\right]. \end{cases} \\ &= 2\left|t - \frac{a+b}{2}\right| \end{aligned}$$

for $t \in [a,b]$.

Since

$$\begin{aligned} \left|f'(t) - f'(s)\right| &= 2\left|\left|t - \frac{a+b}{2}\right| - \left|s - \frac{a+b}{2}\right|\right| \\ &\leq 2\left|t - s\right| \end{aligned}$$

for any $t, s \in [a, b]$, we conclude that f' is Lipschitzian with the constant $K = 2$.

We have

$$\begin{aligned} AB_f\left(a, b, \frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^{b} f(t)\,dt - \int_{a}^{\frac{a+b}{2}} f(t)\,dt\right] \\ &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right)^2 dt + \int_{a}^{\frac{a+b}{2}} \left(t - \frac{a+b}{2}\right)^2 dt\right] \\ &= \frac{1}{2}\int_{a}^{b} \left(t - \frac{a+b}{2}\right)^2 dt = \frac{1}{24}(b-a)^3. \end{aligned}$$

If we replace these values in (4.2) we get in both sides the same quantity $\frac{1}{24}(b-a)^3$. □

The following result also holds:

Theorem 4.2. *With the assumptions of Theorem* 4.1 *we have the inequalities*

$$\begin{aligned} &\left| AB_f(a, b, x) - \frac{bf(b) + af(a)}{2} + \frac{f(b) + f(a)}{2}x \right. \\ &\left. + \frac{1}{2}f'(x)\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \\ &\leq \frac{1}{12}(b-a)K\left[3\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \end{aligned} \tag{4.5}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} &\left|\frac{1}{4}(b-a)[f(b) - f(a)] - \frac{1}{8}f'\left(\frac{a+b}{2}\right)(b-a)^2 - AB_f\left(a, b, \frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{48}K(b-a)^3. \end{aligned} \tag{4.6}$$

The proof is similar to the above Theorem 4.1 and the details are omitted.

5. Inequalities for p-norms

For a Lebesgue measurable function $f : [c, d] \to \mathbb{C}$ we introduce the p-Lebesgue norms as

$$\|f\|_{[c,d],p} := \left(\int_{a}^{b} |f(t)|^p\,dt\right)^{1/p} \quad \text{if } p \geq 1$$

and

$$\|f\|_{[c,d],\infty} := ess \sup_{t \in [c,d]} |f(t)|$$

provided these quantities are finite. We denote $f \in L_p[c, d]$ and $f \in L_\infty[c, d]$.

Proposition 5.1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the inequalities*

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \le \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t)\right|dt+\int_x^b (b-t)\left|f'(t)\right|dt\right]:=B_1(x) \tag{5.1}$$

and

$$\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \le \frac{1}{2}\int_a^b |t-x|\left|f'(t)\right|dt:=B_2(x) \tag{5.2}$$

for any $x\in[a,b]$.

Moreover, we have

$$B_1(x)\le\frac{1}{2}\times\begin{cases}\frac{1}{2}(x-a)^2\left\|f'\right\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,x]\\ \frac{1}{(\alpha+1)^{1/\alpha}}(x-a)^{1+1/\alpha}\left\|f'\right\|_{[a,x],\beta} & \text{if } f'\in L_\beta[a,x],\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\alpha>1\\ (x-a)\left\|f'\right\|_{[a,x],1}\end{cases} \tag{5.3}$$
$$+\frac{1}{2}\times\begin{cases}\frac{1}{2}(b-x)^2\left\|f'\right\|_{[x,b],\infty} & \text{if } f'\in L_\infty[x,b]\\ \frac{1}{(\gamma+1)^{1/\gamma}}(b-x)^{1+1/\gamma}\left\|f'\right\|_{[x,b],\delta} & \text{if } f'\in L_\delta[x,b],\ \frac{1}{\gamma}+\frac{1}{\delta}=1,\gamma>1\\ (b-x)\left\|f'\right\|_{[x,b],1}\end{cases}$$

and

$$B_2(x)\le\frac{1}{2}\times\begin{cases}\frac{1}{2}(x-a)^2\left\|f'\right\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,x]\\ \frac{1}{(\alpha+1)^{1/\alpha}}(x-a)^{1+1/\alpha}\left\|f'\right\|_{[a,x],\beta} & \text{if } f'\in L_\beta[a,x],\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\alpha>1\\ (x-a)\left\|f'\right\|_{[a,x],1}\end{cases} \tag{5.4}$$
$$+\frac{1}{2}\times\begin{cases}\frac{1}{2}(b-x)^2\left\|f'\right\|_{[x,b],\infty} & \text{if } f'\in L_\infty[x,b]\\ \frac{1}{(\gamma+1)^{1/\gamma}}(b-x)^{1+1/\gamma}\left\|f'\right\|_{[x,b],\delta} & \text{if } f'\in L_\delta[x,b],\ \frac{1}{\gamma}+\frac{1}{\delta}=1,\gamma>1\\ (b-x)\left\|f'\right\|_{[x,b],1}\end{cases}$$

for any $x\in[a,b]$.

Proof. From (2.1) and (2.2) we have by taking the modulus

$$\begin{aligned}
&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \qquad (5.5)\\
&\le \frac{1}{2}\left[\left|\int_a^x (t-a)\,f'(t)\,dt\right|+\left|\int_x^b (b-t)\,f'(t)\,dt\right|\right]\\
&\le \frac{1}{2}\left[\int_a^x (t-a)\,|f'(t)|\,dt+\int_x^b (b-t)\,|f'(t)|\,dt\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \qquad (5.6)\\
&\le \frac{1}{2}\int_a^b |t-x|\,|f'(t)|\,dt\\
&= \frac{1}{2}\left[\int_a^x (x-t)\,|f'(t)|\,dt+\int_x^b (t-x)\,|f'(t)|\,dt\right]
\end{aligned}$$

for any $x\in[a,b]$.

Using the Hölder inequality we have

$$\begin{aligned}
&B_1(x)\\
\le\ & \frac{1}{2}\times\begin{cases}\frac{1}{2}(x-a)^2\,\|f'\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,x]\\[2ex] \frac{1}{(\alpha+1)^{1/\alpha}}(x-a)^{1+1/\alpha}\,\|f'\|_{[a,x],\beta} & \text{if } f'\in L_\beta[a,x],\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\alpha>1\\[2ex] (x-a)\,\|f'\|_{[a,x],1}\end{cases}\\
&+\frac{1}{2}\times\begin{cases}\frac{1}{2}(b-x)^2\,\|f'\|_{[x,b],\infty} & \text{if } f'\in L_\infty[x,b]\\[2ex] \frac{1}{(\gamma+1)^{1/\gamma}}(b-x)^{1+1/\gamma}\,\|f'\|_{[x,b],\delta} & \text{if } f'\in L_\delta[x,b],\ \frac{1}{\gamma}+\frac{1}{\delta}=1,\gamma>1\\[2ex] (b-x)\,\|f'\|_{[x,b],1}\end{cases}
\end{aligned}$$

and a similar inequality for B_2. □

Remark 5.2. We observe that

$$\begin{aligned}
B_1(x) &\le \frac{1}{4}(x-a)^2\,\|f'\|_{[a,x],\infty}+\frac{1}{4}(b-x)^2\,\|f'\|_{[x,b],\infty} \qquad (5.7)\\
&\le \left[\frac{1}{4}(x-a)^2+\frac{1}{4}(b-x)^2\right]\max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\\
&= \frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}
\end{aligned}$$

therefore

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \tag{5.8}$$
$$\le\frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}$$

for any $x\in[a,b]$.

Similarly,

$$\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \tag{5.9}$$
$$\le\frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)\right|\le\frac{1}{8}(b-a)^2\|f'\|_{[a,b],\infty} \tag{5.10}$$

and

$$\left|\frac{1}{4}(b-a)[f(b)-f(a)]-AB_f\left(a,b,\frac{a+b}{2}\right)\right|\le\frac{1}{8}(b-a)^2\|f'\|_{[a,b],\infty}. \tag{5.11}$$

6. Applications for twice differentiable functions

If we write the equalities (2.11) and (2.12) for the function $f=g'$, where $g: I\to\mathbb{R}$ is a differentiable function on the interior of the interval I with the derivative absolutely continuous on $[a,b]\subset\mathring{I}$, then we get

$$AB_{g'}(a,b,x) \tag{6.1}$$
$$=\frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]+\left(\frac{a+b}{2}-x\right)g'(x)$$
$$+\frac{1}{2}\left[\int_a^x(t-a)(g''(t)-\gamma)\,dt+\int_x^b(b-t)(g''(t)-\gamma)\,dt\right]$$

and

$$AB_{g'}(a,b,x)=\frac{bg'(b)+ag'(a)}{2}-\frac{g'(b)+g'(a)}{2}x \tag{6.2}$$
$$-\frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]$$
$$-\frac{1}{2}\int_a^b|t-x|(g''(t)-\gamma)\,dt$$

and since

$$AB_f(a,b,x) = \frac{1}{2}F(b) - F(x),$$

where $F(x) := \int_a^x f(t)\,dt$, then

$$\begin{aligned} AB_{g'}(a,b,x) &= \frac{1}{2}[g(b) - g(a)] - g(x) + g(a) \\ &= \frac{g(a)+g(b)}{2} - g(x) \end{aligned}$$

and by (6.1) and (6.2) we get the representations

$$\begin{aligned} g(x) &= \frac{g(a)+g(b)}{2} \\ &- \frac{1}{2}\gamma\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] - \left(\frac{a+b}{2} - x\right)g'(x) \\ &- \frac{1}{2}\left[\int_a^x (t-a)(g''(t)-\gamma)\,dt + \int_x^b (b-t)(g''(t)-\gamma)\,dt\right] \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} g(x) &= \frac{g(a)+g(b)}{2} - \frac{bg'(b)+ag'(a)}{2} + \frac{g'(b)+g'(a)}{2}x \\ &+ \frac{1}{2}\gamma\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \\ &+ \frac{1}{2}\int_a^b |t-x|(g''(t)-\gamma)\,dt \end{aligned} \tag{6.4}$$

for any $x \in [a,b]$.

If we assume that $g'' \in \bar{U}_{[a,b]}(\psi,\Psi)$ for some $\psi,\Psi \in \mathbb{C}$, $\psi \neq \Psi$, then, as above, we have the inequalities

$$\begin{aligned} &\left| g(x) - \frac{g(a)+g(b)}{2} \right. \\ &\left. + \frac{\psi+\Psi}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] + \left(\frac{a+b}{2} - x\right)g'(x)\right| \\ &\leq \frac{|\Psi-\psi|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \end{aligned} \tag{6.5}$$

and

$$\left| g(x) - \frac{g(a)+g(b)}{2} + \frac{bg'(b)+ag'(a)}{2} - \frac{g'(b)+g'(a)}{2}x \right. \tag{6.6}$$
$$\left. -\frac{\psi+\Psi}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right|$$
$$\leq \frac{|\Psi-\psi|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]$$

for any $x \in [a,b]$.

We have the particular inequalities

$$\left| g\left(\frac{a+b}{2}\right) - \frac{g(a)+g(b)}{2} + \frac{\psi+\Psi}{16}(b-a)^2 \right| \tag{6.7}$$
$$\leq \frac{|\Psi-\psi|}{16}(b-a)^2$$

and

$$\left| g\left(\frac{a+b}{2}\right) - \frac{g(a)+g(b)}{2} + \frac{1}{4}(b-a)[g'(b)-g'(a)] \right. \tag{6.8}$$
$$\left. -\frac{\psi+\Psi}{16}(b-a)^2 \right|$$
$$\leq \frac{|\Psi-\psi|}{16}(b-a)^2$$

Other similar results may be stated, however we do not present the details here.

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