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Integral characterizations for the (h, k)-splitting of skew-evolution semiflows

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Abstract. The main aim of this paper is to give integral characterizations for a general concept of (h, k)-splitting for skew-evolution semiflows in Banach spaces. As consequences, criteria for the properties of (h, k)-dichotomy, nonuniform exponential splitting and exponential splitting are obtained.

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1. Introduction

The study of the asymptotic behaviours for dynamical systems represents a research area of large interest, with an impressive development in the last years.

An important starting point for the stability theory is due to E. A Barbashin and R. Datko, who establish integral characterizations for the property of uniform exponential stability in [2], respectively [8].

Recently, P.V. Hai ([10]) obtains discrete and continuous characterizations for the concept of (uniform) exponential stability in terms of Banach sequence (function) spaces. Also, in [20] and [25] are proved generalizations of the results obtained by E. A. Barbashin and R. Datko.

Significant results in the field of exponential dichotomy of skew-product flows are obtained in [7], [11], [13], [14], [22] and for the case of nonlinear differential equations, we emphasize the contributions of S. Elaydi and O. Hajek ([9]).

In [18], respectively [24], the authors give necessary and sufficient conditions for exponential dichotomy with input-output techniques, using spaces of continuous and bounded functions, respectively Lebesgue spaces. Also, the property of (uniform) exponential dichotomy is studied in [23] through the Banach function spaces.

Different concepts of dichotomy of exponential type or more general, with different growth rates, are treated in [4], [5], [6], [12], [16], [19] and the references therein.

As application, we mention the robustness property studied by L. Barreira, J. Chu, C. Valls in [3] and by M. Lizana in [15].

The notion of exponential splitting is a extension of the exponential dichotomy and it is studied for difference equations in [1] and [17]. Important characterizations for various concepts of splitting with growth rates are given in [21].

In this paper we approach the concept of (h,k)-splitting as generalization of (h,k)-dichotomy for skew-evolution semiflows in Banach spaces. Integral conditions of Datko and Barbashin type are given, considering invariant and strongly invariant families of projectors.
Also, we emphasize the results for (h,k)-dichotomy, nonuniform exponential splitting and exponential splitting.

2. Preliminaries

We denote by X a metric space, V a Banach space and $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators on V. The norms on V, respectively $\mathcal{B}(V)$ will be denoted $||\cdot||$.
Also, we consider the sets

$$\Delta = \{(t,t_0) \in \mathbb{R}_+^2 : t \geq t_0\},$$

$$T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

and $Y = X \times V$.

Definition 2.1. A continuous map $\varphi : \Delta \times X \to X$ is said to be *evolution semiflow* on X if it satisfies the following relations:

(es_1) $\varphi(s,s,x) = x,$ for all $(s,x) \in \mathbb{R}_+ \times X$;

(es_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x),$ for all $(t,s,t_0,x) \in T \times X$.

Definition 2.2. We say that $\Phi : \Delta \times X \to \mathcal{B}(V)$ is an *evolution cocycle* over the evolution semiflow φ if

(ec_1) $\Phi(s,s,x) = I$ (the identity operator on V), for all $(s,x) \in \mathbb{R}_+ \times X$;

(ec_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x),$ for all $(t,s,t_0,x) \in T \times X$;

(ec_3) $(t,s,x) \mapsto \Phi(t,s,x)v$ is continuous for every $v \in V$.

Definition 2.3. If φ is an evolution semiflow on X and Φ is an evolution cocycle over φ, then the pair $C = (\Phi,\varphi)$ is called *skew-evolution semiflow*.

Example 2.4. Let X be a compact metric space, V a Banach space, φ an evolution semiflow on X and $A : X \to \mathcal{B}(V)$ a continuous map. If $\Phi(t,s,x)v$ is the solution of the equation

$$\dot{v}(t) = A(\varphi(t,s,x))v(t), \quad t \geq s \geq 0,$$

then $C = (\Phi,\varphi)$ is a skew-evolution semiflow.

Definition 2.5. We say that a continuous map $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is a *family of projectors* on V if

$$P(s,x)^2 = P(s,x), \quad \text{for all } (s,x) \in \mathbb{R}_+ \times X.$$

Remark 2.6. If $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is a family of projectors for $C = (\Phi, \varphi)$, then $Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, $Q(t,x) = I - P(t,x)$ is also a family of projectors for C, called the *complementary family of projectors of* P.

Definition 2.7. A family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is called

(i) *invariant* for the skew-evolution semiflow $C = (\Phi, \varphi)$ if

$$P(t, \varphi(t,s,x))\Phi(t,s,x) = \Phi(t,s,x)P(s,x), \quad \text{for all } (t,s,x) \in \Delta \times X;$$

(ii) *strongly invariant* for the skew-evolution semiflow $C = (\Phi, \varphi)$ if it is invariant for C and for all $(t,s,x) \in \Delta \times X$, the map $\Phi(t,s,x)$ is an isomorphism from *Range* $Q(s,x)$ to *Range* $Q(t, \varphi(t,s,x))$.

Remark 2.8. An example of an invariant family of projectors for a skew-evolution semiflow which is not strongly invariant is given in [21].

Proposition 2.9. *If* $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ *is a strongly invariant family of projectors for* $C = (\Phi, \varphi)$, *then there exists an isomorphism* $\Psi : \Delta \times X \to \mathcal{B}(V)$ *from Range* $Q(t, \varphi(t,s,x))$ *to Range* $Q(s,x)$, *such that:*

(Ψ_1) $\Phi(t,s,x)\Psi(t,s,x)Q(t,\varphi(t,s,x)) = Q(t,\varphi(t,s,x))$;
(Ψ_2) $\Psi(t,s,x)\Phi(t,s,x)Q(s,x) = Q(s,x)$;
(Ψ_3) $\Psi(t,s,x)Q(t,\varphi(t,s,x)) = Q(s,x)\Psi(t,s,x)Q(t,\varphi(t,s,x))$;
(Ψ_4) $\Psi(t,t_0,x)Q(t,\varphi(t,t_0,x)) = \Psi(s,t_0,x)\Psi(t,s,\varphi(s,t_0,x))Q(t,\varphi(t,t_0,x))$,

for all $(t,s,t_0,x) \in T \times X$.

Proof. See [21], Proposition 2. □

Throughout this paper, we will consider two nondecreasing functions $h,k : \mathbb{R}_+ \to [1,+\infty)$ with $\lim_{t\to+\infty} h(t) = \lim_{t\to+\infty} k(t) = +\infty$ (*growth rates*).

Let $C = (\Phi, \varphi)$ be a skew-evolution semiflow and $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ an invariant family of projectors for C.

Definition 2.10. The pair (C,P) admits a (h,k)-*splitting* if there exist two constants $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and a nondecreasing map $N : \mathbb{R}_+ \to [1,+\infty)$ such that

(hs_1) $h(s)^\alpha ||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \le N(s)h(t)^\alpha ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$;
(ks_1) $k(t)^\beta ||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \le N(t)k(s)^\beta ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||$,

for all $(t,s,t_0,x_0,v_0) \in T \times Y$, where Q is the complementary family of projectors of P.

The constants α and β are called *splitting constants.*

As particular cases, we have:

(i) if the map N is constant, then we have the property of *uniform* (h,k)-*splitting*;
(ii) if $\alpha < 0 < \beta$, then we obtain the notion of (h,k)-*dichotomy*;
(iii) if $h(t) = k(t) = e^t$, $t \ge 0$, then we recover the concept of *nonuniform exponential splitting*;

(iv) if $h(t) = k(t) = e^t$ and $N(t) = Se^{\varepsilon t}$, with $t \geq 0$, $S \geq 1$ and $\varepsilon \geq 0$, then we obtain the concept of *exponential splitting.*

Remark 2.11. The pair (C, P) is (h, k)-dichotomic if and only if there are $a, b > 0$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1, +\infty)$ with

(hd_1) $h(t)^a ||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(s)^a ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$;

(kd_1) $k(t)^b ||\Phi(s, t_0, x_0)Q(t_0, x_0)v_0|| \leq N(t)k(s)^b ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Example 2.12. Let $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a constant family of projectors on V and $Q = I - P$.

Let $h, k : \mathbb{R}_+ \to [1, +\infty)$ be two growth rates and let $\alpha < \beta$ be two real constants.

For every two nondecreasing functions $u, v : \mathbb{R}_+ \to [1, +\infty)$ with

$$\sup_{t\geq 0} u(t) = \alpha \quad \text{and} \quad \sup_{t\geq 0} v(t) = \beta$$

we define $\Phi : \Delta \times X \to \mathcal{B}(V)$ by

$$\Phi(t, s, x) = \frac{u(s)}{u(t)} \left(\frac{h(t)}{h(s)}\right)^{\alpha} P(s, x) + \frac{v(t)}{v(s)} \left(\frac{k(t)}{k(s)}\right)^{\beta} Q(s, x),$$

which is an evolution cocycle over every evolution semiflow on X with

$$\Phi(t, s, x_1) = \Phi(t, s, x_2), \quad \text{for all} \quad (t, s, x_1), (t, s, x_2) \in \Delta \times X,$$

$$\Phi(t, t_0, x_0)P(t_0, x_0) = \frac{u(t_0)}{u(t)} \left(\frac{h(t)}{h(t_0)}\right)^{\alpha} P(t_0, x_0), \quad \text{for all} \quad (t, t_0, x_0) \in \Delta \times X,$$

$$\Phi(t, t_0, x_0)Q(t_0, x_0) = \frac{v(t)}{v(t_0)} \left(\frac{k(t)}{k(t_0)}\right)^{\beta} Q(t_0, x_0), \quad \text{for all} \quad (t, t_0, x_0) \in \Delta \times X.$$

Moreover,

$$h(s)^{\alpha} ||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| = \frac{u(t_0)}{u(t)} \left(\frac{h(s)}{h(t_0)}\right)^{\alpha} h(t)^{\alpha} ||P(t_0, x_0)v_0|| \leq$$

$$\leq u(s)h(t)^{\alpha} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(t)^{\alpha} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$$

and

$$k(t)^{\beta} ||\Phi(s, t_0, x_0)Q(t_0, x_0)v_0|| = \frac{v(s)}{v(t)} k(s)^{\beta} ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0|| \leq$$

$$\leq v(t)k(s)^{\beta} ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0|| \leq N(t)k(s)^{\beta} ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$, where $N(t) = u(t) + v(t)$, for every $t \geq 0$.

Finally, we obtain that (C, P) has a (h, k)-splitting, with the splitting constants α and β.

If we suppose that (C, P) is (h, k)-dichotomic, then it results that there exist $\gamma > 0$ and a nondecreasing function $N : \mathbb{R}_+ \to [1, +\infty)$ such that

$$h(t)^{\gamma} ||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(s)^{\gamma} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(t, s, t_0) \in T$ and all $(x_0, v_0) \in Y$.

From here, for $s = t_0 = 0$ we deduce

$$u(0)h(t)^{\alpha+\gamma} \leq N(0)h(0)^{\alpha+\gamma}u(t) \leq \alpha N(0)h(0)^{\alpha+\gamma}$$

and for $t \to +\infty$ we obtain a contradiction.

Remark 2.13. The previous example shows that for every two growth rates h, k and all two real constants $\alpha < \beta$ there is a skew-evolution semiflow which admits a (h,k)-splitting with the splitting constants α, β and which is not (h,k)-dichotomic.

Remark 2.14. The pair (C,P) has a (h,k)-splitting if and only if there exist $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and nondecreasing map $N : \mathbb{R}_+ \to [1,+\infty)$ such that

(hs_1') $h(t_0)^{\alpha}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq N(t_0)h(t)^{\alpha}||P(t_0,x_0)v_0||$;
(ks_1') $k(t)^{\beta}||Q(t_0,x_0)v_0|| \leq N(t)k(t_0)^{\beta}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||$,

for all $(t,t_0,x_0,v_0) \in \Delta \times Y$.

Definition 2.15. We say that (C,P) has a (h,k)-*growth* if there exist two constants $\omega_1, \omega_2 > 0$ and nondecreasing map $M : \mathbb{R}_+ \to [1,+\infty)$ such that

(hg_1) $h(s)^{\omega_1}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq M(t_0)h(t)^{\omega_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$;
(kg_1) $k(s)^{\omega_2}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq M(t)k(t)^{\omega_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||$,

for all $(t,s,t_0,x_0,v_0) \in T \times Y$.

In particular,

(neg) for $h(t) = k(t) = e^t$, $t \geq 0$, we have the property of *nonuniform exponential growth*;
(eg) for $h(t) = k(t) = e^t$ and $M(t) = Ge^{\gamma t}$, $t \geq 0$, $G \geq 1$ and $\gamma \geq 0$, we obtain the notion of *exponential growth.*

Proposition 2.16. *Let $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a strongly invariant family of projectors for $C = (\Phi,\varphi)$. Then (C,P) admits a (h,k)-splitting if and only if there exist two real constants $\alpha < \beta$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1,+\infty)$ such that*

(hs_1) $h(s)^{\alpha}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq N(s)h(t)^{\alpha}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$;
(ks_1'') $k(s)^{\beta}||\Psi(t,t_0,x_0)Q(t,\varphi(t,t_0,x_0))v_0|| \leq$
$\leq N(s)k(t_0)^{\beta}||\Psi(t,s,\varphi(s,t_0,x_0))Q(t,\varphi(t,t_0,x_0))v_0||$,

for all $(t,s,t_0,x_0,v_0) \in T \times Y$.

Proof. See [21], Proposition 3. □

Similarly, we obtain

Remark 2.17. Let $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a strongly invariant family of projectors for $C = (\Phi,\varphi)$. Then (C,P) has a (h,k)-growth if and only if there exist $\omega_1, \omega_2 > 0$ and nondecreasing function $M : \mathbb{R}_+ \to [1,+\infty)$ with

(hg_1) $h(s)^{\omega_1}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq M(t_0)h(t)^{\omega_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$;
(kg_1') $k(t_0)^{\omega_2}||\Psi(t,t_0,x_0)Q(t,\varphi(t,t_0,x_0))v_0|| \leq$
$\leq M(s)k(s)^{\omega_2}||\Psi(t,s,\varphi(s,t_0,x_0))Q(t,\varphi(t,t_0,x_0))v_0||$,

for all $(t,s,t_0,x_0,v_0) \in T \times Y$.

3. The main results

In this section we will denote with $\mathcal{H}_1$ the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with

$$\int_0^{+\infty} h(s)^c ds < +\infty, \quad \text{for all } c < 0.$$

Also, $\mathcal{K}_1$ represents the set of the growth rates $k : \mathbb{R}_+ \to [1, +\infty)$, with the property that there exists a constant $K \geq 1$ such that

$$\int_0^t k(s)^c ds \leq K k(t)^c, \quad \text{for all } c > 0,\ t \geq 0.$$

By $\mathcal{H}$ we denote the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there exists $H \geq 1$ such that

$$h(t)^c \leq H h(s)^c, \quad \text{for all} \quad (t, s) \in \Delta,\ t \leq s + 1,\ c \in \mathbb{R}.$$

Remark 3.1. If we denote by $e(t) = e^t$, $t \geq 0$, then $e \in \mathcal{H}_1 \cap \mathcal{K}_1 \cap \mathcal{H}$.

We consider $C = (\Phi, \varphi)$ a skew-evolution semiflow, $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ an invariant family of projectors for C.

A first characterization for the (h, k)-splitting property is given by

Theorem 3.2. *Let (C, P) be a pair with (h, k)-growth, where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) admits a (h, k)-splitting if and only if there exist $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$ and a nondecreasing mapping $D : \mathbb{R}_+ \to [1, +\infty)$ such that the following assertions hold:*

(Dhs_1)
$$\int_s^{+\infty} \frac{||\Phi(\tau, t_0, x_0) P(t_0, x_0) v_0||}{h(\tau)^{d_1}} d\tau \leq \frac{D(s)}{h(s)^{d_1}} ||\Phi(s, t_0, x_0) P(t_0, x_0) v_0||,$$
for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

(Dks_1)
$$\int_{t_0}^{t} \frac{||\Phi(\tau, t_0, x_0) Q(t_0, x_0) v_0||}{k(\tau)^{d_2}} d\tau \leq \frac{D(t)}{k(t)^{d_2}} ||\Phi(t, t_0, x_0) Q(t_0, x_0) v_0||,$$
for all $(t, t_0, x_0, v_0) \in \Delta \times Y$.

Proof. Necessity. It is a simple verification for $\alpha < d_1 < d_2 < \beta$ and

$$D(s) = N(s)[K + H h(s)^{d_1 - \alpha}],$$

where $H = \int_0^{+\infty} h(\tau)^{\alpha - d_1} d\tau$.

Sufficiency. We show that the relations from Definition 2.10 are verified.
(hs_1) *Case 1*: Let $t \geq s + 1$, $(s, t_0) \in \Delta$ and $(x_0, v_0) \in Y$. Then

$$h(s)^{d_1} ||\Phi(t, t_0, x_0) P(t_0, x_0) v_0|| \leq$$

$$\leq h(s)^{d_1} M(t_0) \int_{t-1}^{t} \left(\frac{h(t)}{h(\tau)} \right)^{\omega_1} ||\Phi(\tau, t_0, x_0) P(t_0, x_0) v_0|| d\tau =$$

$$= M(t_0)h(s)^{d_1}h(t)^{d_1}\int\limits_{t-1}^{t}\left(\frac{h(t)}{h(\tau)}\right)^{\omega_1-d_1}\frac{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||}{h(\tau)^{d_1}}d\tau \leq$$

$$\leq HM(s)h(s)^{d_1}h(t)^{d_1}\int\limits_{s}^{+\infty}\frac{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||}{h(\tau)^{d_1}}d\tau \leq$$

$$\leq N(s)h(t)^{d_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \quad \text{for all } t\geq s+1,\ s\geq t_0, (x_0,v_0)\in Y,$$

where $N(s)=HM(s)D(s)$, $s\geq 0$.
Case 2: Let $t\in[s,s+1]$, $s\geq t_0$ and $(x_0,v_0)\in Y$. We obtain

$$h(s)^{d_1}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq$$

$$\leq M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega_1-d_1}h(t)^{d_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| \leq$$

$$\leq N(s)h(t)^{d_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for all $t\in[s,s+1]$, $s\geq t_0,\ (x_0,v_0)\in Y$.
Then, we obtain that (hs_1) is verified for all $(t,s,t_0,x_0,v_0)\in T\times Y$.
(ks_1) *Case 1*: We consider $(t,s,t_0)\in T,\ t\geq s+1$, $(x_0,v_0)\in Y$. Then,

$$\int\limits_{s}^{s+1}k(t)^{d_2}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq$$

$$\leq k(t)^{d_2}\int\limits_{s}^{s+1}M(\tau)\left(\frac{k(\tau)}{k(s)}\right)^{\omega_2}||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq$$

$$\leq M(t)k(t)^{d_2}k(s)^{d_2}\int\limits_{s}^{s+1}\left(\frac{k(\tau)}{k(s)}\right)^{\omega_2+d_2}\frac{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}{k(\tau)^{d_2}}d\tau \leq$$

$$\leq HM(t)k(s)^{d_2}k(t)^{d_2}\int\limits_{t_0}^{t}\frac{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}{k(\tau)^{d_2}}d\tau \leq$$

$$\leq N(t)k(s)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||.$$

We obtain

$$k(t)^{d_2}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq N(t)k(s)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$

for all $t\geq s+1,\ s\geq t_0,\ (x_0,v_0)\in Y$.
Case 2: Let $t\in[s,s+1]$, $s\geq t_0$ and $(x_0,v_0)\in Y$. We deduce the following:

$$k(t)^{d_2}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq$$

$$\leq M(t)\left(\frac{k(t)}{k(s)}\right)^{\omega_2}k(t)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| =$$

$$= M(t)\left(\frac{k(t)}{k(s)}\right)^{\omega_2+d_2}k(s)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| \leq$$

$$\leq N(t)k(s)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||.$$

Thus, the condition (ks_1) holds for all $(t,s,t_0,x_0,v_0) \in T \times Y$.
In conclusion, the pair (C,P) has a (h,k)-splitting. □

As consequences, we obtain

Corollary 3.3. *Let (C,P) be a pair with (h,k)-growth, where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C,P) is (h,k)-dichotomic if and only if then there exist $d_1 < 0 < d_2$ and a nondecreasing function $D : \mathbb{R}_+ \to [1,+\infty)$ such that:*

$$(Dhd_1) \quad \int_s^{+\infty} \frac{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||}{h(\tau)^{d_1}}d\tau \leq \frac{D(s)}{h(s)^{d_1}}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for all $(s,t_0,x_0,v_0) \in \Delta \times Y$;

$$(Dkd_1) \quad \int_{t_0}^{t} \frac{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}{k(\tau)^{d_2}}d\tau \leq \frac{D(t)}{k(t)^{d_2}}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$

for all $(t,t_0,x_0,v_0) \in \Delta \times Y$.

Corollary 3.4. *We consider (C,P) a pair with nonuniform exponential growth. Then (C,P) has a nonuniform exponential splitting if and only if there are two constants $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$ and a nondecreasing map $D : \mathbb{R}_+ \to [1,+\infty)$ with:*

$$(Dnes_1) \quad \int_s^{+\infty} e^{-\tau d_1}||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq$$
$$\leq D(s)e^{-sd_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for all $(s,t_0,x_0,v_0) \in \Delta \times Y$;

$$(Dnes_2) \quad \int_{t_0}^{t} e^{-\tau d_2}||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq$$
$$\leq D(t)e^{-td_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$

for all $(t,t_0,x_0,v_0) \in \Delta \times Y$.

Corollary 3.5. *If (C,P) is a pair with exponential growth, then it admits an exponential splitting if and only if there exists some real constants $d_1 < d_2$, $D \geq 1$ and $\delta \geq 0$ such* *that:*

$$(Des_1) \qquad \int_s^{+\infty} e^{-\tau d_1} ||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0|| d\tau \leq$$
$$\leq De^{(\delta - d_1)s} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$
for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Des_2) \qquad \int_{t_0}^{t} e^{-\tau d_2} ||\Phi(\tau, t_0, x_0)Q(t_0, x_0)v_0|| d\tau \leq$$
$$\leq De^{(\delta - d_2)t} ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||,$$
for all $(t, t_0, x_0, v_0) \in \Delta \times Y$.

Remark 3.6. The results given by Theorem 3.2, Corollary 3.3, Corollary 3.4 and Corollary 3.5 are characterizations of Datko-type for the splitting concepts studied in this paper.

Further, $C = (\Phi, \varphi)$ represents a skew-evolution semiflow and $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ a strongly invariant family of projectors for C.
In this context, we obtain the following characterization for (h, k)-splitting:

Theorem 3.7. *Let (C, P) be a pair with (h, k)-growth,where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) admits a (h, k)-splitting if and only if there exist $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$ and a nondecreasing map $D : \mathbb{R}_+ \to [1, +\infty)$ such that the following inequalities are verified:*

$$(Dhs_1) \qquad \int_s^{+\infty} \frac{||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0||}{h(\tau)^{d_1}} d\tau \leq \frac{D(s)}{h(s)^{d_1}} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$
for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Dks_1') \qquad \int_{t_0}^{s} \frac{||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||}{k(\tau)^{d_2}} d\tau \leq$$
$$\leq \frac{D(s)}{k(s)^{d_2}} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||,$$
for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Proof. Necessity. It results from Proposition 2.16, for $\alpha < d_1 < d_2 < \beta$ and

$$D(s) = N(s)[K + Hh(s)^{d_1 - \alpha}],$$

where $H = \int_0^{+\infty} h(\tau)^{\alpha - d_1} d\tau$.

Sufficiency. We prove that the inequalities (hs_1) and (ks_1'') from Proposition 2.16 hold.

In a similar manner with the proof of Theorem 3.2 we obtain

$$h(s)^{d_1} ||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(t)^{d_1} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$, where $N(s) = HM(s)D(s)$, $s \geq 0$.

Thus, we consider $(t, s, t_0) \in T$, $s \geq t_0 + 1$, $(x_0, v_0) \in Y$ and it results that

$$k(s)^{d_2}||\Psi(t, t_0, x_0)Q(t, \varphi(t, t_0, x_0)v_0|| =$$

$$= k(s)^{d_2} \int_{t_0}^{t_0+1} ||\Psi(\tau, t_0, x_0)\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||d\tau \leq$$

$$\leq k(s)^{d_2} \int_{t_0}^{t_0+1} M(\tau) \left(\frac{k(\tau)}{k(t_0)}\right)^{\omega_2} ||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||d\tau \leq$$

$$\leq M(s)k(s)^{d_2}k(t_0)^{d_2} \int_{t_0}^{t_0+1} \left(\frac{k(\tau)}{k(t_0)}\right)^{\omega_2+d_2} \frac{||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||}{k(\tau)^{d_2}} d\tau \leq$$

$$\leq HM(s)k(s)^{d_2}k(t_0)^{d_2} \int_{t_0}^{s} \frac{||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||}{k(\tau)^{d_2}} d\tau \leq$$

$$\leq N(s)k(t_0)^{d_2}||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||.$$

For $t \geq s$, $s \in [t_0, t_0 + 1)$, $(x_0, v_0) \in Y$ we have

$$k(s)^{d_2}||\Psi(t, t_0, x_0)Q(t, \varphi(t, t_0, x_0))v_0|| \leq$$

$$\leq k(s)^{d_2}M(s) \left(\frac{k(s)}{k(t_0)}\right)^{\omega_2} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| \leq$$

$$\leq M(s)k(t_0)^{d_2} \left(\frac{k(s)}{k(t_0)}\right)^{\omega_2+d_2} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| \leq$$

$$\leq N(s)k(t_0)^{d_2}||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||.$$

We deduce that (ks_1'') is verified, for all $(t, s, t_0) \in T$, $(x_0, v_0) \in Y$.

Using Proposition 2.16, it follows that (C, P) admits a (h, k)-splitting. □

In particular, we emphasize the following consequences:

Corollary 3.8. *Let (C, P) be a pair with (h, k)-growth,where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) is (h, k)-dichotomic if and only if there exist two constants $d_1 < 0 < d_2$*

and a nondecreasing map $D : \mathbb{R}_+ \to [1, +\infty)$ with:

$$(Dhd_1) \quad \int_s^{+\infty} \frac{||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0||}{h(\tau)^{d_1}} d\tau \le \frac{D(s)}{h(s)^{d_1}} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Dkd_1') \quad \int_{t_0}^{s} \frac{||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||}{k(\tau)^{d_2}} d\tau \le$$

$$\le \frac{D(s)}{k(s)^{d_2}} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Corollary 3.9. *Let (C, P) be with nonuniform exponential growth. Then (C, P) has a nonuniform exponential splitting if and only if exist two real constants $d_1 < d_2$ and a nondecreasing function $D : \mathbb{R}_+ \to [1, +\infty)$ such that:*

$$(Dnes_1) \quad \int_s^{+\infty} e^{-\tau d_1} ||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0|| d\tau \le$$

$$\le D(s)e^{-sd_1} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Dnes_2') \quad \int_{t_0}^{s} e^{-\tau d_2} ||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| d\tau \le$$

$$\le D(s)e^{-sd_2} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Corollary 3.10. *If (C, P) has an exponential growth, then it admits an exponential splitting if and only if there exist $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$, $D \ge 1$ and $\delta \ge 0$ such that:*

$$(Des_1) \quad \int_s^{+\infty} e^{-\tau d_1} ||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0|| d\tau \le$$

$$\le De^{(\delta - d_1)s} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Des_2') \quad \int_{t_0}^{s} e^{-\tau d_2} ||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| d\tau \le$$

$$\le De^{(\delta - d_2)s} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Remark 3.11. Theorem 3.7, Corollary 3.8, Corollary 3.9 and Corollary 3.10 are characterizations of Barbashin-type for the splitting concepts considered in this paper.

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