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Fekete-Szegö problem for a class of analytic functions defined by Carlson-Shaffer operator

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Abstract. In the present paper, authors study a Fekete-Szegö problem for a class of analytic functions defined by Carlson-Shaffer operator. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and S denote the subclass of A that are univalent in U. Fekete and Szegö [10] proved a interesting result that the estimate

$$\left|a_3 - \lambda a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right) \tag{1.2}$$

holds for any normalized univalent function $f(z)$ of the form (1.1) in the open unit disk U for $0 \leq \lambda \leq 1$. This inequality is sharp for each λ.

The coefficient functional

$$\phi_\lambda(f) = a_3 - \lambda a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\lambda}{2}\left[f''(0)\right]^2\right), \tag{1.3}$$

on normalized analytic functions f in the unit disk represents various geometric quantities, for example, when $\lambda = 1$, $\phi_\lambda(f) = a_3 - a_2^2$, becomes $\frac{S_f(0)}{6}$, where S_f denote the Schwarzian derivative $(f'''/f')' - (f''/f')^2/2$ of locally univalent functions f in

U. The problem of maximising the absolute value of the functional $\phi_\lambda(f)$ is called the Fekete-Szegö problem.

The Fekete-Szegö problem is one of the interesting problems in Geometric Function Theory. This attracts many researchers (see the work of [1]-[5], [7]-[9], [12], [13], [16], [17], [20] and [3]) to study the Fekete-Szegö problem for the various classes of analytic univalent functions. Very recently, Bansal [4] introduced the class $R_\gamma^\tau(\phi)$ of functions in $f \in S$ for which

$$1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) \prec \phi(z), \ z \in U$$

where $0 \leq \gamma < 1$, $\tau \in C \setminus \{0\}$, $\phi(z)$ is an analytic function with positive real part on U with $\phi(0) = 0$, $\phi'(0) > 0$ which maps the unit like disk U onto a starlike region with respect to 1 which is symmetric with respect to the real axis and $\prec$ denotes the subordination between analytic functions and studied the Fekete-Szegö problem for this class.

Now, by using the Carlson-Shaffer operator we introduce a new subclass $R_\gamma^\tau(\phi, a, c)$ for functions $f \in A$ and $0 \leq \gamma < 1$, $\tau \in C\setminus\{0\}$, $a, c \in C$, $\{c \neq 0, -1, -2, ...\}$ satisfying the condition

$$1 + \frac{1}{\tau}\left((L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1\right) \prec \phi(z) \ \ (z \in U) \tag{1.4}$$

where $\phi(z)$ is defined the same as above and $L(a, c)$ denotes the Carlson-Shaffer operator introduced in [6] and defined in the following way:

$$L(a,c)f(z) = \quad f(z) * zh(a,c;z),$$

where

$$h(a,c;z) = \quad 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^n$$

$L(a, c)$ maps A into itself. $L(c, c)$is the identity and if $a \neq 0, -1, -2...$, then $L(a, c)$ has a continuous inverse $L(c, a)$ and is an one-to-one mapping of A onto itself. $L(a, c)$ provides a convenient representation of differentiation and integration. If $g(z) = zf'(z)$, then $g = L(2, 1)f$ and $f = L(1, 2)g$. If we set

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \qquad (-1 \leq B < A \leq 1; z \in U),$$

in (1.4), we obtain

$$R_\gamma^\tau\left(\frac{1 + Az}{1 + Bz}, a, c\right) = R_\gamma^\tau(A, B, a, c)$$

$$= \left\{ f \in A : \left| \frac{(L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1}{\tau(A - B) - B\left((L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1\right)} \right| < 1 \right\}$$

which is again a new class.

By specializing parameters in the subclass $R_\gamma^\tau(A, B, a, c)$ we obtain the following known subclasses studied earlier by various authors.

1. $R_\gamma^\tau(A, B, a, a) \equiv R_\gamma^\tau(A, B)$ studied by Bansal [4].
2. $R_\gamma^\tau(1 - 2\beta, -1, a, a) \equiv R_\gamma^\tau(\beta)$ for $0 \leq \beta < 1$, studied by Swaminathan [21].

3. $R_\gamma^\tau(1-2\beta,-1,a,a) \equiv R_\gamma^\tau(\beta)$ for $\tau = e^{i\eta}cos\eta, 0 \le \beta < 1$, where $-\pi/2 < \eta < \pi/2$ introduced by Ponnusamy and Rønning [19], (see also [18]).
4. $R_1^\tau(0,-1,a,a) \equiv R^\tau(\beta)$ for $\tau = e^{i\eta}cos\eta$ was considered in [14].

To prove our main result, we shall require the following lemma.

Lemma 1.1. *(see [11], [15]). If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ $(z \in U)$ is a function with positive real part, then for any complex number μ ,*

$$\left|c_2 - \mu c_1^2\right| \le \ 2 \ \max\{1, \ |2\mu - 1|\} \tag{1.5}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z} \quad (z \in U). \tag{1.6}$$

2. Main results

Our main result is contained in the following theorem.

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, where $\phi(z) \in A$ with $\phi'(0) > 0$. If $f(z)$ given by (1.1) belongs to $R_\gamma^\tau(\phi,a,c)$ $(0 \le \gamma \le 1,\ \tau \in C \setminus \{0\},\ a,c \in C,$ $\{c \ne 0,-1,-2,....\},\ z \in U)$, then for any complex number μ*

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1|\tau|c(c+1)}{3a(a+1)(1+2\gamma)} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3\mu\tau B_1 c(a+1)(1+2\gamma)}{4a(c+1)(1+\gamma)^2}\right|\right\}. \tag{2.1}$$

This result is sharp.

Proof. If $f(z) \in R_\gamma^\tau(\phi,a,c)$, then there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$1 + \frac{1}{\tau}\left((L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1\right) = \phi(w(z)), \quad (z \in U). \tag{2.2}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + ... \tag{2.3}$$

Since $w(z)$ is a Schwarz function, we see that $Re\{p_1(z)\} > 0$ and $p_1(0) = 1$.

Define the function $p(z)$ by,

$$p(z) = 1 + \frac{1}{\tau}\left((L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1\right) = 1 + b_1 z + b_2 z^2 + ... \tag{2.4}$$

In view of (2.2), (2.3), (2.4), we have

$$\begin{aligned} p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) &= \phi\left(\frac{c_1 z + c_2 z^2 + ...}{2 + c_1 z + c_2 z^2 + ...}\right) \\ &= \phi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + ...\right) \qquad (2.5) \\ = 1 + B_1\frac{1}{2}c_1 z &+ B_1\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + B_2\frac{1}{4}c_1^2 z^2 + ... \end{aligned}$$

Thus,

$$b_1 = \frac{1}{2} B_1 c_1; \quad b_2 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2. \tag{2.6}$$

From (2.4), we obtain

$$a_2 = \frac{\tau\, B_1 c_1 c}{4a\,(1+\gamma)}; \quad a_3 = \frac{\tau\, c(c+1)}{6a(a+1)\,(1+2\gamma)} \left[B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} B_2 c_1^2 \right]. \tag{2.7}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{B_1 \tau\, c(c+1)}{6a(a+1)\,(1+2\gamma)} \left(c_2 - \nu c_1^2 \right) \tag{2.8}$$

where

$$\nu = \frac{1}{2} \left(1 - \frac{B_2}{B_1} + \frac{3\tau\mu B_1 c(a+1)\,(1+2\gamma)}{4a(c+1)(1+\gamma)^2} \right). \tag{2.9}$$

Our result now is followed by an application of Lemma 1.1. Also, by the application of Lemma 1.1 equality in (2.1) is obtained when

$$p_1(z) = \frac{1+z^2}{1-z^2} \text{ or } p_1(z) = \frac{1+z}{1-z} \tag{2.10}$$

but

$$p(z) = 1 + \frac{1}{\tau} \left((L(a,c)f(z))' + \gamma z (L(a,c)f(z))' - 1 \right) = \phi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right). \tag{2.11}$$

Putting value of $p_1(z)$ we get the desired results. Thus the proof of Theorem 2.1 is established. □

For the class $R_\gamma^\tau(A,\ B, a, c)$,

$$\phi(z) = \frac{1+Az}{1+Bz} = (1+Az)\,(1+Bz)^{-1} = 1 + (A-B)\,z - \left(AB - B^2\right) z^2 + \ldots \tag{2.12}$$

Thus, putting $B_1 = A - B$ and $B_2 = -B\,(A-B)$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *If $f(z)$ given by (1.1) belongs to $R_\gamma^\tau(A,\ B, a, c)$, then*

$$\left| a_3 - \mu a_2^2 \right| \le \frac{(A-B)\ \ |\tau|\, c(c+1)}{3a(a+1)\,(1+2\gamma)} \max \left\{ 1,\ \left| B + \frac{3\tau\mu c(a+1)\,(A-B)\,(1+2\gamma)}{4a(c+1)(1+\gamma)^2} \right| \right\}. \tag{2.13}$$

If we put $a = c$ in Theorem 2.1, then we obtain the following result of Bansal [4].

Corollary 2.3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, where $\phi(z) \in A$ with $\phi'(0) > 0$. If $f(z)$ given by (1.1) belongs to $R_\gamma^\tau(\phi)(0 \le \gamma \le 1, \tau \in C\backslash\{0\}, z \in U)$ then for any complex number μ*

$$\left| a_3 - \mu a_2^2 \right| \le \frac{B_1\,|\tau|}{3(1+2\gamma)} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{3\mu\tau B_1 (1+2\gamma)}{4(1+\gamma)^2} \right| \right\}.$$

This result is sharp.

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