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Coefficient inequality for subclass of analytic univalent functions related to simple logistic activation functions

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Abstract. The author investigates the relationship between unified subclasses of analytic univalent functions and simple logistic activation function to determine the initial Taylor series coefficients alongside classical Fekete-Szegő problem.

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1. Introduction and preliminaries

The theory of special functions are significantly important to scientist and engineers with mathematical calculations. Though not with any specific definition but its applications extends to physics, computer etc. In the recent time, theory of special function has been overshadowed by other fields such as real analysis, functional analysis, differential equation, algebra and topology.

There are various special functions but we shall concern ourselves with one of the activation function popularly known as sigmoid function or simple logistic function. By activation function, we meant an information process inspired by the same way biological nervous system (such as brain) process information. This composed of large number of highly interconnected processing element, that is neurons, working as a unit to solve or process a specific task. It also learns by examples, can not be programmed to solve a specific task. Sigmoid function (simple logistic activation function) has a gradient descendent learning algorithm, its evaluation could be done in several ways (even by truncated series expansion).

The simple logistic activation function is given as

$$L(z) = \frac{1}{1+e^{-z}} \tag{1.1}$$

which is differentiable, it outputs real number between 0 and 1, it maps a very large input domain to a small range of outputs, it never loses information because it is one-to-one function and it increases monotonically. It is evidently clear from the aforementioned that sigmoid function is a great tool in geometric function theory.
As usual we denote by A the class of function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ (z \in U) \tag{1.2}$$

which are analytic in the open unit disk $U = \{z : |z| < 1, z \in C\}$ with normalization $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of univalent functions. For two functions f and φ analytic in the open unit disk, we say that f is subordinate to φ written as $f \prec \varphi$ in U or $f(z) \prec \varphi(z)$ if there exist Schwarz function $\omega(z)$ analytic in U with $w(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = \varphi(\omega(z)), z \in U$. It is clear from the Schwarz lemma that $f(z) \prec \varphi(z), (z \in U)$ which implies that $f(0) = \varphi(0)$ and $f(U) \subset \varphi(U)$. Suppose that φ is univalent in U then $f(z) \prec \varphi(z)$ if and only if $f(0) = \varphi(0)$ and $f(U) \subset \varphi(U)$.
Lemma A. [7] *If a function $p \in P$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \ (z \in U)$$

then $|p_k| \leq 2, k \in N$ where P is the family of all functions analytic in U for which $p(0) = 1$ and $Re(p(z)) > 0, \ (z \in U)$.
Let $\phi(z)$ be an analytic univalent function with positive real part in U and $\phi(U)$ be symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$ and $\Phi'(0) > 0$. Ma and Minda [6] gave unified representation of various subclasses of starlike and convex functions using the classes $S^*(\phi)$ and $C(\phi)$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$ respectively, which includes several well-known classes as special case.
Take for example, if

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \ (-1 \leq B < A \leq 1)$$

the class $S^*(\phi)$ reduces to the class $S^*[A, B]$ introduces by Janowski in [4].
In 1933, Fekete and Szegő [3] proved that

$$|a_2^2 - \mu a_3| \leq \begin{cases} 4\mu - 3, & \mu \geq 1 \\ 1 + \exp^{-\frac{2\mu}{1-\mu}}, & 0 \leq \mu \leq 1 \\ 3 - 4\mu & \mu \leq 0 \end{cases}$$

holds for function $f \in S$ and the result is sharp. The problem of finding the sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ of any compact family of functions is popularly known as the Fekete-Szegő problem. Several known authors at different time have applied the classical Fekete-Szegő to various classes to obtain various sharp bounds the likes of Keogh and Merkes in 1969 [5] obtained the sharp upper bound of the Fekete-Szegő functional $|a_2^2 - \mu a_3|$ for some subclasses of univalent function S

(see also [1,11,12,14,15]). The Hadamard product (or convolution) of $f(z)$ given by (1.2) and

$$\varphi(z) = z + \sum_{k=2}^{\infty} \varphi_k z^k$$

is defined by

$$(f * \varphi)(z) = z + \sum_{k=2}^{\infty} a_k \varphi_k z^k = (\varphi * f)(z)$$

Therefore, $D^n(f*\varphi)(z) = D(D^{n-1}(f*\varphi)(z)) = z + \sum_{k=2}^{\infty} k^n a_k \varphi_k z^k$ where D^n is the well known Sălăgean derivative operator[13] defined as

$$D^0 f(z) = f(z), \;\; D^1 f(z) = D(f(z)) = zf'(z), \ldots,$$

$$D^n f(z) = D(D^{n-1} f(z)) = z + \sum_{k=2}^{\infty} k^n a_k z^k$$

Recently, Murugusundaramoorthy et al [8] also applied the Hadamard product to discuss a new class of functions denoted by $M_{g,h}(\phi)$ see for detail in [8].
Our major focus in this work is to investigate the simple logistic sigmoid activation function as related to the unified subclass of starlike and convex functions $M_{n,g}^{\alpha,h}(b, \Phi_{k,m})$ to determine the initial Taylor series coefficients and discuss its Fekete-Szegő functional.
For the purpose of our intention we recall the following:
Lemma B. [2] *Let L be a Sigmoid function defined in (1.1) and*

$$\Phi_{k,m} = 2L(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m \right]^k$$

then $\Phi_{k,m} \in P, |z| < 1$ where $\Phi_{k,m}$ is a modified sigmoid function.
Lemma C. [2] *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m \right]^k$$

then

$$|\Phi_{k,m}| < 2.$$

Lemma D. [2] *If $\Phi_{k,m} \in P$ is starlike then f is a normalized univalent function of the form (1.2).*
Taking $k = 1$, Joseph-Fadipe et al [2] proved that
Remark A. Let

$$\Phi(z) = 1 + \sum_{m=1}^{\infty} C_m z^m$$

where $C_m = \frac{(-1)(-1)^m}{2m!}$ $|C_m| \le 2, m = 1, 2, 3, \ldots$ this result is sharp for each m (see also [10]).

Definition 1. For $b \in C$. Let the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ denote the subclass of A consisting of functions f of the form (1.2), and

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k, \quad h(z) = z + \sum_{k=2}^{\infty} h_k z^k,$$

$$g_k > 0,\ h_k > 0,\ g_k - h_k > 0$$

satisfying the following subordination condition

$$1 + \frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'} - 1\right] \prec \Phi_{k,m}(z)$$

where $\alpha \geq 0, n \in N_0$, $\Phi_{k,m}$ is a simple logistic sigmoid activation function and D^n is the Sălăgean derivative operator [13].
We state here that we are not assuming $\Phi_{k,m}(U)$ in Definition 1 to be symmetric with respect to the real axis and starlike with respect to $\Phi_{k,m}(0) = 1$. To show that class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ is non empty, let us consider the function $f(z) = \frac{z}{1-z}$. We assume

$$\gamma(z) = 1 + \frac{1}{b}\left[(1-\alpha))\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{[D^n(f*g)(z)]'}{[D^n(f*h)(z)]'} - 1\right],$$

we have

$$\gamma(z) = 1 + \frac{2^n}{b}(1-\alpha)(g_2 - h_2)z + \dots.$$

Clearly $\gamma(0) = 1$ and

$$\gamma'(0) = \frac{2^n}{b}(1-\alpha)(g_2 - h_2) > 0,$$

hence

$$f(z) = \frac{z}{1+z} \in M_{n,h}^{\alpha,g}(b, \Phi_{k,m}).$$

Remark B. With various special choices of functions $g, h, \Phi_{k,m}, b$ and the real number α, the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ reduces to several known classes and lead to other new classes.
Examples. 1. Suppose $\Phi_{k,m}(z) = \phi(z)$, then the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ reduces to the class $M_{n,h}^{\alpha,g}(b, \phi)$
2. If $\Phi_{k,m} = \phi, n = 0, \alpha = 0$, the class $M_{0,h}^{0,g}(b, \Phi_{k,m}) = M_{g,h}(b, \phi)$ and if $b = 1$ in Example 2 the class reduces to class $M_{g,h}(\phi)$ studied in [8].
3. Furthermore, if we put $g(z) = \frac{z}{(1-z)^2}, h(z) = \frac{z}{1-z}$ then the class

$$M_{n,h}^{\alpha,g}(b, \Phi_{k,m}) = M_{n,\frac{z}{1-z}}^{\alpha,\frac{z}{(1-z)^2}}(b, \Phi_{k,m}),$$

we can continue to generate many classes with various special choices of the functions and parameters involved.
Suppose we let

$$\Phi_{k,m}(z) = \frac{\sqrt{1 \pm z^2} + z}{\sqrt{1 \pm z^2}},$$

then the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ becomes $M_{n,h}^{\alpha,g}(b, \frac{\sqrt{1\pm z^2}+z}{\sqrt{1\pm z^2}})$.

2. Main result

Theorem 2.1. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} [\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m]^k$$

where $\Phi_{k,m} \in A$ *is a modified logistic sigmoid activation function and* $\Phi'_{k,m}(0) > 0$. *If* $f(z)$ *given by (1.2) belongs to the class* $M_{g,h}^{\alpha,n}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$, $k \geq 2$ *then*

$$|a_2| \leq \frac{b}{2^{n+1}(1+\alpha)(g_2-h_2)}$$

$$|a_3| \leq \frac{b^2(1+3\alpha)h_2}{2^2 \times 3^n(1+2\alpha)(1+\alpha)^2(g_2-h_2)(g_3-h_3)}$$

$$|a_4| \leq \frac{6^n b^3(h_2g_3+h_3g_2-2h_2h_3)}{2^{3n+3}\cdot 3^n(g_4-h_4)}$$

$$\times \left[\frac{(1+5\alpha)h_2}{(1+2\alpha)(1+\alpha)^3(g_3-h_3)(g_2-h_2)^2} - \frac{2^n \times 3^{n-1}}{6^n b^2(1+3\alpha)(h_2g_3+h_3g_2-2h_2h_3)}\right].$$

Proof. If $f \in M_{g,h}^{\alpha,n}(b, \Phi_{k,m})$, then

$$1 + \frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{[D^n(f*g)(z)]'}{[D^n(f*h)(z)]'} - 1\right] = \Phi_{k,m}(z) \tag{2.1}$$

A computation shows that

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)} = 1 + 2^n a_2(g_2-h_2)z + [2^{2n}(h_2^2 - g_2h_2) + 3^n a_3(g_3-h_3)]z^2 + \tag{2.2}$$

$$[4^n a_4(g_4-h_4) + 6^n a_2a_3(2h_2h_3 - h_3g_2 - h_2g_3)]z^3 + \dots$$

$$\frac{[D^n(f*g)(z)]'}{[D^n(f*h)(z)]'} = 1 + 2^{n+1}a_2(g_2-h_2)z + [2^{2n+2}(h_2^2 - g_2h_2) + 3^{n+1}a_3(g_3-h_3)]z^2 + \tag{2.3}$$

$$[4^{n+1}a_4(g_4-h_4) + 6^{n+1}a_2a_3(2h_2h_3 - h_3g_2 - h_2g_3)]z^3 + \dots$$

and Taylor series expansion of $\Phi_{k,m}$ is given as

$$\Phi_{k,m}(z) = 1 + \frac{1}{2}z - \frac{1}{24z^3} + \frac{1}{240z^5} - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \dots \tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) we have

$$2^{n+1}(1+\alpha)(g_2-h_2)a_2 = b \tag{2.5}$$

$$3^n(1+2\alpha)(g_3-h_3)a_3 = \frac{b^2(1+3\alpha)h_2}{4(1+\alpha)^2(g_2-h_2)} \tag{2.6}$$

and

$$4^n(1+3\alpha)(g_4-h_4)a_4 = 6^n(1+5\alpha)(h_3g_2+h_2g_3-2h_2h_3)a_2a_3 - \frac{b}{24} \tag{2.7}$$

Equations (2.5), (2.6) and (2.7) give the desired results of Theorem 2.1.

Theorem 2.2. *Let $\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$ where $\Phi_{k,m} \in A$ is a modified logistic sigmoid activation function and $\Phi'_{k,m}(0) > 0$. If $f(z)$ given by (1.2) belongs to the class $M_{g,h}^{\alpha,n}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$ and $\mu \in R$, $k \geq 2$ then*

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{2^2(1+\alpha)^2(g_2-h_2)(g_3-h_3)} \left[\frac{(1+3\alpha)h_2}{3^n(1+2\alpha)} - \frac{\mu(g_3-h_3)}{2^n(g_2-h_2)}\right]. \tag{2.8}$$

Proof. A simple computation from (2.5) and (2.6) gives the desire result of Theorem 2.2.

Corollary 2.3. *Let $\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$ where $\Phi_{k,m} \in A$ is a modified logistic sigmoid activation function and $\Phi'_{k,m}(0) > 0$. If $f(z)$ given by (1.2) belongs to the class $M_{0,h}^{\alpha,g}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$, $k \geq 2$ then*

$$|a_2| \leq \frac{b}{2(1+\alpha)(g_2-h_2)}$$

$$|a_3| \leq \frac{b^2(1+3\alpha)h_2}{4(1+2\alpha)(1+\alpha)^2(g_2-h_2)(g_3-h_3)}$$

$$|a_4| \leq \frac{b^3(h_2g_3+h_3g_2-2h_2h_3)}{8(g_4-h_4)} \left[\frac{(1+5\alpha)h_2}{(1+2\alpha)(1+\alpha)^3(g_3-h_3)(g_2-h_2)^2} - \frac{1}{3b^2(1+3\alpha)(h_2g_3+h_3g_2-2h_2h_3)}\right].$$

Corollary 2.4. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$$

where $\Phi_{k,m} \in A$ is a modified logistic sigmoid activation function and $\Phi'_{k,m}(0) > 0$. If $f(z)$ given by (1.2) belongs to the class $M_{0,h}^{\alpha,0}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$ and $\mu \in R$, $k \geq 2$ then

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(1+\alpha)^2(g_2-h_2)(g_3-h_3)} \left[\frac{(1+3\alpha)h_2}{(1+2\alpha)} - \frac{\mu(g_3-h_3)}{2^n(g_2-h_2)}\right].$$

Furthermore, suppose we put $\alpha = 0$ in Corollaries 2.3 and 2.4 we have respectively the following

Corollary 2.5. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$$

where $\Phi_{k,m} \in A$ is a modified logistic sigmoid activation function and $\Phi'_{k,m}(0) > 0$. If $f(z)$ given by (1.2) belongs to the class $M_{0,h}^{0,g}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$, $k \geq 2$ then

$$|a_2| \leq \frac{b}{2(g_2-h_2)}$$

$$|a_3| \leq \frac{b^2h_2}{4(g_2-h_2)(g_3-h_3)}$$

$$|a_4| \leq \frac{b^3(h_2g_3 + h_3g_2 - 2h_2h_3)}{8(g_4 - h_4)}\left[\frac{h_2}{(g_3 - h_3)(g_2 - h_2)^2} - \frac{1}{3b^2(h_2g_3 + h_3g_2 - 2h_2h_3)}\right].$$

Corollary 2.6. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m)^k$$

where $\Phi_{k,m} \in A$ *is a modified logistic sigmoid activation function and* $\Phi'_{k,m}(0) > 0$. *If* $f(z)$ *given by (1.2) belongs to the class* $M_{0,h}^{0;g}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$ *and* $\mu \in R$, $k \geq 2$ *then*

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(g_2 - h_2)(g_3 - h_3)}\left[h_2 - \frac{\mu(g_3 - h_3)}{2^n(g_2 - h_2)}\right].$$

Concluding, with various special choices of α, n, b and other parameters involved, many interesting coefficient bounds and Fekete-Szegő inequalities could be obtained.

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