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Generalized g-fractional calculus and iterative methods

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Abstract. We approximated solutions of some iterative methods on a generalized Banach space setting in [5]. Earlier studies such as [7-12] the operator involved is Fréchet-differentiable. In [5] we assumed that the operator is only continuous. This way we extended the applicability of these methods to include generalized fractional calculus and problems from other areas. In the present study applications include generalized g-fractional calculus. Fractional calculus is very important for its applications in many applied sciences.

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1. Introduction

Many problems in Computational sciences can be formulated as an operator equation using Mathematical Modelling [8, 10, 13, 14, 15]. The fixed points of these operators can rarely be found in closed form. That is why most solution methods are usually iterative.

The semilocal convergence is, based on the information around an initial point, to give conditions ensuring the convergence of the method.

We presented a semilocal convergence analysis for some iterative methods on a generalized Banach space setting in [5] to approximate fixed point or a zero of an operator. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in section 2). Earlier studies such as [7-12] for Newton's method have shown that a more precise convergence analysis is obtained when compared to the real norm theory. However, the main assumption is that the operator involved is Fréchet-differentiable. This hypothesis limits the applicability of Newton's method. In [5] study we only assumed the continuity of the operator. This may be expanded the applicability of these methods.

The rest of the paper is organized as follows: section 2 contains the basic concepts on generalized Banach spaces and the semilocal convergence analysis of these methods. Finally, in the concluding section 3, we present special cases and applications in generalized g-fractional calculus.

2. Generalized Banach spaces

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [5-12], and the references there in.

Definition 2.1. *A generalized Banach space is a triplet $(X, E, /\cdot/)$ such that*

(i) X is a linear space over $\mathbb{R}(\mathbb{C})$.
(ii) $E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e.
(ii_1) $(E, \|\cdot\|)$ is a real Banach space,
(ii_2) E is partially ordered by a closed convex cone K,
(iii_3) The norm $\|\cdot\|$ is monotone on K.
(iii) The operator $/\cdot/ : X \to K$ satisfies
$/x/ = 0 \Leftrightarrow x = 0$, $/\theta x/ = |\theta| /x/$,
$/x + y/ \leq /x/ + /y/$ for each $x, y \in X$, $\theta \in \mathbb{R}(\mathbb{C})$.
(iv) X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|\cdot\| \cdot /\cdot/$.

Remark 2.2. The operator $/\cdot/$ is called a generalized norm. In view of (iii) and (ii_3) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm.

Let $L\left(X^j, Y\right)$ stand for the space of j-linear symmetric and bounded operators from X^j to Y, where X and Y are Banach spaces. For X, Y partially ordered $L_+\left(X^j, Y\right)$ stands for the subset of monotone operators P such that

$$0 \leq a_i \leq b_i \Rightarrow P(a_1, ..., a_j) \leq P(b_1, ..., b_j).$$

Definition 2.3. *The set of bounds for an operator $Q \in L(X, X)$ on a generalized Banach space $(X, E, /\cdot/)$ is defined to be:*

$$B(Q) := \{P \in L_+(E, E),\ /Qx/ \leq P/x/ \text{ for each } x \in X\}.$$

Let $D \subset X$ and $T : D \to D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0)$$

is well defined. We write in case of convergence

$$T^\infty(x_0) := \lim(T^n(x_0)) = \lim_{n\to\infty} x_n.$$

Let $(X, (E, K, \|\cdot\|), /\cdot/)$ and Y be generalized Banach spaces, $D \subset X$ an open subset, $G : D \to Y$ a continuous operator and $A(\cdot) : D \to L(X, Y)$. A zero of operator G is to be determined by a method starting at a point $x_0 \in D$. The results are presented for an operator $F = JG$, where $J \in L(Y, X)$. The iterates are determined through a fixed point problem:

$$x_{n+1} = x_n + y_n, \;\; A(x_n) y_n + F(x_n) = 0 \qquad (2.1)$$
$$\Leftrightarrow y_n = T(y_n) := (I - A(x_n)) y_n - F(x_n).$$

Let $U(x_0, r)$ stand for the ball defined by

$$U(x_0, r) := \{x \in X : /x - x_0/ \leq r\}$$

for some $r \in K$.

Next, we state the semilocal convergence analysis of method (2.1) using the preceding notation.

Theorem 2.4. [5] *Let $F : D \subset X$, $A(\cdot) : D \to L(X, Y)$ and $x_0 \in D$ be as defined previously. Suppose:*

(H_1) There exists an operator $M \in B(I - A(x))$ for each $x \in D$.

(H_2) There exists an operator $N \in L_+(E, E)$ satisfying for each $x, y \in D$

$$/F(y) - F(x) - A(x)(y - x)/ \leq N /y - x/.$$

(H_3) There exists a solution $r \in K$ of

$$R_0(t) := (M + N) t + /F(x_0)/ \leq t.$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \to 0$ as $k \to \infty$.

Then, the following hold:

(C_1) The sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n + T_n^{\infty}(0), \;\; T_n(y) := (I - A(x_n)) y - F(x_n) \qquad (2.2)$$

is well defined, remains in $U(x_0, r)$ for each $n = 0, 1, 2, ...$ and converges to the unique zero of operator F in $U(x_0, r)$.

(C_2) An apriori bound is given by the null-sequence $\{r_n\}$ defined by $r_0 := r$ and for each $n = 1, 2, ...$

$$r_n = P_n^{\infty}(0), \;\; P_n(t) = Mt + Nr_{n-1}.$$

(C_3) An aposteriori bound is given by the sequence $\{s_n\}$ defined by

$$s_n := R_n^{\infty}(0), \;\; R_n(t) = (M + N) t + N a_{n-1},$$

$$b_n := /x_n - x_0/ \leq r - r_n \leq r,$$

where

$$a_{n-1} := /x_n - x_{n-1}/ \;\; \text{for each } n = 1, 2, ...$$

Remark 2.5. The results obtained in earlier studies such as [7-12] require that operator F (i.e. G) is Fréchet-differentiable. This assumption limits the applicability of the earlier results. In the present study we only require that F is a continuous operator. Hence, we have extended the applicability of these methods to include classes of operators that are only continuous.

Example 2.6. The j-dimensional space $\mathbb{R}^j$ is a classical example of a generalized Banach space. The generalized norm is defined by componentwise absolute values. Then, as ordered Banach space we set $E = \mathbb{R}^j$ with componentwise ordering with e.g. the maximum norm. A bound for a linear operator (a matrix) is given by the corresponding matrix with absolute values. Similarly, we can define the "N" operators. Let $E = \mathbb{R}$. That is we consider the case of a real normed space with norm denoted by $\|\cdot\|$. Let us see how the conditions of Theorem 2.4 look like.

Theorem 2.7. *(H_1)* $\|I - A(x)\| \leq M$ *for some* $M \geq 0$.

(H_2) $\|F(y) - F(x) - A(x)(y - x)\| \leq N \|y - x\|$ *for some* $N \geq 0$.

(H_3) $M + N < 1$,

$$r = \frac{\|F(x_0)\|}{1 - (M + N)}. \tag{2.3}$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \to 0$ *as* $k \to \infty$, *where* r *is given by* (2.3).

Then, the conclusions of Theorem 2.4 *hold.*

3. Applications to g-fractional calculus

We apply Theorem 2.7 in this section. Here basic concepts and facts come from [4]. We need

Definition 3.1. *Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = m$, $\lceil \cdot \rceil$ *the ceiling of the number. Here* $g \in AC([a, b])$ *(absolutely continuous functions) and* g *is strictly increasing.*

Let $G : [a, b] \to \mathbb{R}$ *such that* $\left(G \circ g^{-1}\right)^{(m)} \circ g \in L_\infty([a, b])$.

We define the left generalized g*-fractional derivative of* G *of order* α *as follows:*

$$\left(D^\alpha_{a+;g} G\right)(x) := \frac{1}{\Gamma(m - \alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t) \left(G \circ g^{-1}\right)^{(m)} (g(t))\, dt, \tag{3.1}$$

$a \leq x \leq b$, *where* Γ *is the gamma function.*

We also define the right generalized g*-fractional derivative of* G *of order* α *as follows:*

$$\left(D^\alpha_{b-;g} G\right)(x) := \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left(G \circ g^{-1}\right)^{(m)} (g(t))\, dt, \tag{3.2}$$

$a \leq x \leq b$.

Both $\left(D^\alpha_{a+;g} G\right), \left(D^\alpha_{b-;g} G\right) \in C([a, b])$.

(I) Let $a < a^* < b$. In particular we have that $\left(D^\alpha_{a+;g} G\right) \in C([a^*, b])$. We notice that

$$\left|\left(D^\alpha_{a+;g} G\right)(x)\right| \leq \frac{\left\|\left(G \circ g^{-1}\right)^{(m)} \circ g\right\|_{\infty,[a,b]}}{\Gamma(m - \alpha)} \left(\int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t)\, dt\right) \tag{3.3}$$

$$= \frac{\left\| \left(G \circ g^{-1} \right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha)} \frac{(g(x)-g(a))^{m-\alpha}}{(m-\alpha)}$$

$$= \frac{\left\| \left(G \circ g^{-1} \right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)} (g(x)-g(a))^{m-\alpha}, \quad \forall\, x \in [a,b].$$

We have proved that

$$\left| \left(D^{\alpha}_{a+;g} G \right)(x) \right| \leq \frac{\left\| \left(G \circ g^{-1} \right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)} (g(x)-g(a))^{m-\alpha}$$

$$\leq \frac{\left\| \left(G \circ g^{-1} \right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)} (g(b)-g(a))^{m-\alpha} < \infty, \quad \forall\, x \in [a,b], \tag{3.4}$$

in particular true $\forall\, x \in [a^*, b]$.
We obtain that

$$\left(D^{\alpha}_{a+;g} G \right)(a) = 0. \tag{3.5}$$

Therefore there exist $x_1, x_2 \in [a^*, b]$ such that $D^{\alpha}_{a+;g} G(x_1) = \min D^{\alpha}_{a+;g} G(x)$, and $D^{\alpha}_{a+;g} G(x_2) = \max D^{\alpha}_{a+;g} G(x)$, for $x \in [a^*, b]$.
We assume that

$$D^{\alpha}_{a+;g} G(x_1) > 0. \tag{3.6}$$

(i.e. $D^{\alpha}_{a+;g} G(x) > 0$, $\forall\, x \in [a^*, b]$).
Furthermore

$$\left\| D^{\alpha}_{a+;g} G \right\|_{\infty,[a^*,b]} = D^{\alpha}_{a+;g} G(x_2). \tag{3.7}$$

Here it is

$$J(x) = mx,\ m \neq 0. \tag{3.8}$$

The equation

$$JG(x) = 0, \quad x \in [a^*, b], \tag{3.9}$$

has the same set of solutions as the equation

$$F(x) := \frac{JG(x)}{2D^{\alpha}_{a+;g} G(x_2)} = 0, \quad x \in [a^*, b]. \tag{3.10}$$

Notice that

$$D^{\alpha}_{a+;g} \left(\frac{G(x)}{2D^{\alpha}_{a+;g} G(x_2)} \right) = \frac{D^{\alpha}_{a+;g} G(x)}{2D^{\alpha}_{a+;g} G(x_2)} \leq \frac{1}{2} < 1, \quad \forall\, x \in [a^*, b]. \tag{3.11}$$

We call

$$A(x) := \frac{D^{\alpha}_{a+;g} G(x)}{2D^{\alpha}_{a+;g} G(x_2)}, \quad \forall\, x \in [a^*, b]. \tag{3.12}$$

We notice that

$$0 < \frac{D^{\alpha}_{a+;g} G(x_1)}{2D^{\alpha}_{a+;g} G(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{3.13}$$

Hence it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D^{\alpha}_{a+;g} G(x_1)}{2D^{\alpha}_{a+;g} G(x_2)} =: \gamma_0, \quad \forall\, x \in [a^*, b]. \tag{3.14}$$

Clearly $\gamma_0 \in (0,1)$.
We have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0,1), \quad \forall\, x \in [a^*, b]. \tag{3.15}$$

Next we assume that $F(x)$ is a contraction over $[a^*, b]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall\, x, y \in [a^*, b], \tag{3.16}$$

and $0 < \lambda < \frac{1}{2}$.
Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda \left(D^{\alpha}_{a+;g} G(x_2)\right) |x - y|, \quad \forall\, x, y \in [a^*, b]. \tag{3.17}$$

We observe that

$$|F(y) - F(x) - A(x)(y - x)| \leq |F(y) - F(x)| + |A(x)| |y - x|$$

$$\leq \lambda |y - x| + |A(x)| |y - x| = (\lambda + |A(x)|) |y - x| =: (\xi_1), \quad \forall\, x, y \in [a^*, b]. \tag{3.18}$$

Hence by (3.4), $\forall\, x \in [a^*, b]$ we get that

$$|A(x)| = \frac{\left|D^{\alpha}_{a+;g} G(x)\right|}{2 D^{\alpha}_{a+;g} G(x_2)} \leq \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} < \infty. \tag{3.19}$$

Consequently we observe

$$(\xi_1) \leq \left(\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} \right) |y - x|, \tag{3.20}$$

$\forall\, x, y \in [a^*, b]$.
Call

$$0 < \gamma_1 := \lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)}, \tag{3.21}$$

choosing $(g(b) - g(a))$ small enough we can make $\gamma_1 \in (0,1)$.
We proved that

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad \text{where } \gamma_1 \in (0,1), \ \forall\, x, y \in [a^*, b]. \tag{3.22}$$

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{D^{\alpha}_{a+;g} G(x_1)}{2 D^{\alpha}_{a+;g} G(x_2)} + \lambda$$

$$+ \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} < 1, \tag{3.23}$$

equivalently we find,

$$\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} < \frac{D^{\alpha}_{a+;g} G(x_1)}{2 D^{\alpha}_{a+;g} G(x_2)}, \tag{3.24}$$

equivalently,

$$2\lambda D^{\alpha}_{a+;g}G\left(x_2\right)+\frac{\left(g\left(b\right)-g\left(a\right)\right)^{m-\alpha}}{\Gamma\left(m-\alpha+1\right)}\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}<D^{\alpha}_{a+;g}G\left(x_1\right), \quad (3.25)$$

which is possible for small λ, $\left(g\left(b\right)-g\left(a\right)\right)$.
That is $\gamma\in(0,1)$. Hence equation (3.9) can be solved with our presented iterative algorithms.

Conclusion 3.2. (for (I))

Our presented earlier semilocal results, see Theorem 2.7, can apply in the above generalized fractional setting for $g\left(x\right)=x$ for each $x\in[a,b]$ since the following inequalities have been fulfilled:

$$\left\|1-A\right\|_{\infty}\leq\gamma_0, \quad (3.26)$$

and

$$\left|F\left(y\right)-F\left(x\right)-A\left(x\right)\left(y-x\right)\right|\leq\gamma_1\left|y-x\right|, \quad (3.27)$$

where $\gamma_0,\gamma_1\in(0,1)$, furthermore it holds

$$\gamma=\gamma_0+\gamma_1\in\left(0,1\right), \quad (3.28)$$

for all $x,y\in[a^*,b]$, where $a<a^*<b$.

The specific functions $A\left(x\right)$, $F\left(x\right)$ have been described above, see (3.12) and (3.10), respectively.

(II) Let $a<b^*<b$. In particular we have that $\left(D^{\alpha}_{b-;g}G\right)\in C\left([a,b^*]\right)$. We notice that

$$\left|\left(D^{\alpha}_{b-;g}G\right)\left(x\right)\right|\leq\frac{\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}}{\Gamma\left(m-\alpha\right)}\left(\int_x^b\left(g\left(t\right)-g\left(x\right)\right)^{m-\alpha-1}g'\left(t\right)dt\right) \quad (3.29)$$

$$=\frac{\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}}{\Gamma\left(m-\alpha+1\right)}\left(g\left(b\right)-g\left(x\right)\right)^{m-\alpha}$$

$$\leq\frac{\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}}{\Gamma\left(m-\alpha+1\right)}\left(g\left(b\right)-g\left(a\right)\right)^{m-\alpha}<\infty,\quad\forall\ x\in\left[a,b\right], \quad (3.30)$$

in particular true $\forall\ x\in\left[a,b^*\right]$.
We obtain that

$$\left(D^{\alpha}_{b-;g}G\right)\left(b\right)=0. \quad (3.31)$$

Therefore there exist $x_1,x_2\in[a,b^*]$ such that $D^{\alpha}_{b-;g}G\left(x_1\right)=\min D^{\alpha}_{b-;g}G\left(x\right)$, and $D^{\alpha}_{b-;g}G\left(x_2\right)=\max D^{\alpha}_{b-;g}G\left(x\right)$, for $x\in[a,b^*]$.
We assume that

$$D^{\alpha}_{b-;g}G\left(x_1\right)>0. \quad (3.32)$$

(i.e. $D^{\alpha}_{b-;g}G\left(x\right)>0$, $\forall\ x\in[a,b^*]$).
Furthermore

$$\left\|D^{\alpha}_{b-;g}G\right\|_{\infty,[a,b^*]}=D^{\alpha}_{b-;g}G\left(x_2\right). \quad (3.33)$$

Here it is

$$J(x) = mx,\ m \neq 0. \tag{3.34}$$

The equation

$$JG(x) = 0, \quad x \in [a, b^*], \tag{3.35}$$

has the same set of solutions as the equation

$$F(x) := \frac{JG(x)}{2D^{\alpha}_{b-;g}G(x_2)} = 0, \quad x \in [a, b^*]. \tag{3.36}$$

Notice that

$$D^{\alpha}_{b-;g}\left(\frac{G(x)}{2D^{\alpha}_{b-;g}G(x_2)}\right) = \frac{D^{\alpha}_{b-;g}G(x)}{2D^{\alpha}_{b-;g}G(x_2)} \leq \frac{1}{2} < 1, \quad \forall\ x \in [a, b^*]. \tag{3.37}$$

We call

$$A(x) := \frac{D^{\alpha}_{b-;g}G(x)}{2D^{\alpha}_{b-;g}G(x_2)}, \quad \forall\ x \in [a, b^*]. \tag{3.38}$$

We notice that

$$0 < \frac{D^{\alpha}_{b-;g}G(x_1)}{2D^{\alpha}_{b-;g}G(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{3.39}$$

Hence it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D^{\alpha}_{b-;g}G(x_1)}{2D^{\alpha}_{b-;g}G(x_2)} =: \gamma_0, \quad \forall\ x \in [a, b^*]. \tag{3.40}$$

Clearly $\gamma_0 \in (0, 1)$.
We have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0, 1), \quad \forall\ x \in [a, b^*]. \tag{3.41}$$

Next we assume that $F(x)$ is a contraction over $[a, b^*]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall\ x, y \in [a, b^*], \tag{3.42}$$

and $0 < \lambda < \frac{1}{2}$.
Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda \left(D^{\alpha}_{b-;g}G(x_2)\right) |x - y|, \quad \forall\ x, y \in [a, b^*]. \tag{3.43}$$

We observe that

$$|F(y) - F(x) - A(x)(y - x)| \leq |F(y) - F(x)| + |A(x)| |y - x| \leq$$

$$\lambda |y - x| + |A(x)| |y - x| = (\lambda + |A(x)|) |y - x| =: (\xi_2), \quad \forall\ x, y \in [a, b^*]. \tag{3.44}$$

Hence by (3.30), $\forall\ x \in [a, b^*]$ we get that

$$|A(x)| = \frac{\left|D^{\alpha}_{b-;g}G(x)\right|}{2D^{\alpha}_{b-;g}G(x_2)} \leq \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\|\left(G \circ g^{-1}\right)^{(m)} \circ g\right\|_{\infty,[a,b]}}{D^{\alpha}_{b-;g}G(x_2)} < \infty. \tag{3.45}$$

Consequently we observe

$$(\xi_2) \leq \left(\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D_{b-;g}^{\alpha} G(x_2)} \right) |y - x|, \tag{3.46}$$

$\forall\ x, y \in [a, b^*]$.
Call

$$0 < \gamma_1 := \lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D_{b-;g}^{\alpha} G(x_2)}, \tag{3.47}$$

choosing $(g(b) - g(a))$ small enough we can make $\gamma_1 \in (0, 1)$.
We proved that

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad \text{where } \gamma_1 \in (0,1), \ \forall\ x, y \in [a, b^*]. \tag{3.48}$$

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{D_{b-;g}^{\alpha} G(x_1)}{2D_{b-;g}^{\alpha} G(x_2)} + \lambda$$

$$+ \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D_{b-;g}^{\alpha} G(x_2)} < 1, \tag{3.49}$$

equivalently we find,

$$\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D_{b-;g}^{\alpha} G(x_2)} < \frac{D_{b-;g}^{\alpha} G(x_1)}{2D_{b-;g}^{\alpha} G(x_2)}, \tag{3.50}$$

equivalently,

$$2\lambda D_{b-;g}^{\alpha} G(x_2) + \frac{(g(b) - g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)} \left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]} < D_{b-;g}^{\alpha} G(x_1), \tag{3.51}$$

which is possible for small λ, $(g(b) - g(a))$.
That is $\gamma \in (0, 1)$. Hence equation (3.35) can be solved with our presented iterative algorithms.

Conclusion 3.3. (for (II))

Our presented earlier semilocal iterative methods, see Theorem 2.7, can apply in the above generalized fractional setting for $g(x) = x$ for each $x \in [a, b]$ since the following inequalities have been fulfilled:

$$\|1 - A\|_{\infty} \leq \gamma_0, \tag{3.52}$$

and

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \tag{3.53}$$

where $\gamma_0, \gamma_1 \in (0, 1)$, furthermore it holds

$$\gamma = \gamma_0 + \gamma_1 \in (0, 1), \tag{3.54}$$

for all $x, y \in [a, b^*]$, where $a < b^* < b$.

The specific functions $A(x)$, $F(x)$ have been described above, see (3.38) and (3.36), respectively.

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