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Convergence of the Neumann series for a Helmholtz-type equation

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Abstract. We pursue a constructive solution to the Robin problem of a Helmholtz-type equation in the form of a single layer potential. This representation method leads to a boundary integral equation. We study the problem on a bounded planar domain of class C^2. We prove the convergence of the Neumann series of iterations of the layer potential operators to the solution of the boundary integral equation. This study is inspired by several recent papers which cover the iteration techniques. In [7], [8], [9], D. Medkova obtained results regarding the successive approximation method for Neumann, Robin and transmission problems.

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1. Introduction

The study of the Helmholtz equation has received broad attention in [1], [2], [3]. The equation is connected to several physical phenomena. The general form of the equation is

$$\Delta u + \lambda^2 u = 0$$

with $Im\ \lambda > 0$. In this paper we study the Robin problem for the Helmholtz equation in a bounded planar domain $D \subset \mathbb{R}^2$ of class C^2. We present an iteration technique which is suited to be used for a numerical computation of the solution of the Robin problem. The technique is based on the Neumann series of iterations of the layer potential operators. In the past the technique was studied by W.L. Wendland (see [11], [12]). More recently the Neumann series were used by D. Medkova for several problems associated with the Stokes system, including Robin and transmission problems, in the papers [7], [8], [9].

In general, the boundary value problems associated with the Helmholtz equation are not uniquely solvable when coupled with the general condition $Im\ \lambda > 0$.

The values of λ for which the Helmholtz equation is not uniquely solvable are called irregular frequencies (see also [2], section 2.1). In this paper we restrict the study of the equation to the case $Re\ \lambda = 0$. In this particular case the equation is

$$\Delta u - k^2 u = 0, \tag{1.1}$$

with $k > 0$. This equation is also known as the Klein-Gordon equation. It is connected to quantum mechanics. We consider the Robin boundary condition

$$\frac{\partial u}{\partial \nu} + \alpha u = g, \tag{1.2}$$

where ν is the outward unit normal vector of D, $\alpha > 0$ is a constant and $g \in C(\partial D, \mathbb{R}^2)$.

We pursue the solution $u \in C^2(D, \mathbb{R}^2) \cap C^1(\overline{D}, \mathbb{R}^2)$ of the boundary value problem (1.1),(1.2) in the form of a single layer potential

$$u(x) = \int_{\partial D} E(x,y)h(y)\, dy,$$

where $E(x,y)$ is the fundamental solution of the equation (1.1) and $h \in C(\partial D, \mathbb{R}^2)$ is a boundary function called density. The function u defined above as a single layer potential solves equation (1.1).

The fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ is given by

$$E(x,y) = \frac{1}{2\pi} K_0(k|x-y|) = \frac{i}{4} H_0^{(1)}(k|x-y|),$$

where K is the modified Bessel function of the second kind and $H^{(1)}$ is the Hankel function of the first kind.

We will require to assume that the domains D have smooth boundaries because several proofs in this paper will use Green's formula and the compactness of the layer potential operators which are true for smooth domains. There are several established properties regarding the layer potentials on smooth domains (see [1], [2]). We simply state the following well known facts. In the sequel we will assume that the bounded domain $D \subset \mathbb{R}^2$ is of class C^2.

Definition 1.1. *For $h \in C(\partial D, \mathbb{C}^2)$ define the single layer potential S with density h by*

$$Sh(x) = \int_{\partial D} E(x,y)h(y)\, dy,\ x \in \mathbb{R}^2 \setminus \partial D,$$

and the double layer potential D with density h by

$$Dh(x) = \int_{\partial D} \frac{\partial E(x,y)}{\partial \nu} h(y)\, dy,\ x \in \mathbb{R}^2 \setminus \partial D.$$

Lemma 1.2. *The single layer potential operator $S : C(\partial D, \mathbb{C}^2) \to C(\partial D, \mathbb{C}^2)$ is given by*

$$Sh(x) = \int_{\partial D} E(x,y)h(y)\, dy = \lim_{z \to x} \int_{\partial D} E(z,y)h(y)\, dy,\ x \in \partial D.$$

The double layer potential operator $K : C(\partial D, \mathbb{C}^2) \to C(\partial D, \mathbb{C}^2)$ *is given by*

$$Kh(x) = \int_{\partial D} \frac{\partial E(x,y)}{\partial \nu} h(y)\, dy = \lim_{z \to x,\, z \in D} Dh(z) + \frac{1}{2} h(x),\ x \in \partial D.$$

The single layer potential operator satisfies

$$\frac{\partial Sh(x)}{\partial \nu} = \frac{1}{2} h(x) + K' h(x),\ x \in \partial D,$$

where K' *is the adjoint operator of* K.

The equalities above are called limiting relations. They relate the values of the layer potentials in the domain with the boundary values.

Lemma 1.3. *The operators* S, K *and* K' *are compact.*

Furthermore there are several other properties of the layer potential operators. If we define the operators $S,\ K,\ K'$ on the space $L^2(\partial D)$, then $S,\ K,\ K'$ are bounded.

2. Convergence of the Neumann series

Consider a solution of the problem (1.1),(1.2) in the form of a single layer potential $u = Sh$ with $h \in C(\partial D, \mathbb{R}^2)$. Since the single layer potential satisfies the limiting relations in Lemma 1.2, the Robin boundary condition (1.2) becomes

$$\frac{1}{2} h(x) + K' h(x) + \alpha Sh(x) = g(x),\ x \in \partial D. \tag{2.1}$$

This is a boundary integral equation. The invertibility of the operator

$$\frac{1}{2} I + K' + \alpha S$$

and the solvability of the equation were proved in [4]. The proof of the invertibility uses the Fredholm theory. Since the operators S and K' are compact, the operator $I/2 + K' + \alpha S$ has index 0. One can use Green's formula to prove the injectivity of the operator, from which it follows that the operator $I/2 + K' + \alpha S$ is invertible.

We will prove the convergence of a series of iterations of the layer potential operators to the solution of the boundary integral equation (2.1). The series is called Neumann series. In this way we give a constructive solution to the boundary value problem (1.1),(1.2).

In the proof of the convergence we will use the following lemmas which were proved for the Stokes system by D.Medkova in [7], [8], [10]. We prove the lemmas corresponding to the layer potential operators associated with the Klein-Gordon equation. They are instrumental in finding a range for the spectrum of the operator $I/2 + K' + \alpha S$.

Lemma 2.1. *Denote* $\|S\|_{L^2(\partial D, \mathbb{C}^2)} = M$. *Let* $h \in C(\partial D, \mathbb{C}^2)$. *Then*

$$\int_{\partial D} |Sh|^2\, dy \le M \int_{\partial D} h \cdot S\overline{h}\, dy.$$

Proof. For $f, g \in L^2(\partial D, \mathbb{C}^2)$ define

$$\langle f, g\rangle = \int_{\partial D} f \cdot S\overline{g}\, dy.$$

The integral operator S has a symmetric kernel $E(x,y)$. Therefore the product defined before is conjugate symmetric. Since the kernel $E(x,y)$ is positive, we deduce that the product $\langle\cdot,\cdot\rangle$ is positive definite. Then $\langle\cdot,\cdot\rangle$ is an inner product on $L^2(\partial D, \mathbb{C}^2)$. Holder's inequality gives

$$\int_{\partial D} |Sh|^2\, dy = \left(\sup_{f\in L^2, \|f\|=1} |\langle f, h\rangle|\right)^2.$$

From the Schwartz inequality we deduce

$$\int_{\partial D} |Sh|^2\, dy \le \langle h, h\rangle \sup_{f\in L^2, \|f\|=1} \langle f, f\rangle,$$

from which we get

$$\int_{\partial D} |Sh|^2\, dy \le \langle h, h\rangle \sup_{f\in L^2, \|f\|=1} \left(\int_{\partial D} |Sf|^2\, dy\right)^{1/2}.$$

This means

$$\int_{\partial D} |Sh|^2\, dy \le \|S\|_{L^2(\partial D, \mathbb{C}^2)} \int_{\partial D} h \cdot S\overline{h}\, dy.$$

The lemma is proved. □

Lemma 2.2. *Let $h \in C(\partial D, \mathbb{C}^2)$. Then*

$$\int_{\partial D} \overline{Sh} \cdot \left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right) dy = \int_D \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy + \int_{\partial D} \alpha |Sh|^2\, dy.$$

Proof. If we apply Green's formula

$$\int_G (\psi\Delta\varphi + \nabla\varphi\cdot\nabla\psi)\, dy = \int_{\partial G} \psi \frac{\partial\varphi}{\partial\nu} dy$$

for the vector components of $\overline{Sh}$ and Sh on the domain D, we obtain

$$\int_D \left(\overline{Sh}\cdot\Delta Sh + \nabla\overline{Sh}\cdot\nabla Sh\right) dy = \int_{\partial D} \overline{Sh}\cdot\frac{\partial Sh}{\partial\nu} dy.$$

The equality implies

$$\int_D \left(\overline{Sh}\cdot k^2 Sh + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh}\cdot\frac{\partial Sh}{\partial\nu} dy,$$

and therefore

$$\int_{\partial D} \overline{Sh} \cdot \left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right) dy = \int_D \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy + \int_{\partial D} \alpha |Sh|^2\, dy. \qquad \square$$

Lemma 2.3. *Let $h \in C(\partial D, \mathbb{C}^2)$. Then*

$$\int_{\mathbb{R}^2\setminus\partial D} \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} h\cdot\overline{Sh}\, dy.$$

Proof. From Green's formula we have

$$\int_D \left(\overline{Sh}\cdot k^2 Sh + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh}\cdot \frac{\partial Sh}{\partial \nu} dy,$$

Using the limiting relations in Lemma 1.2, we get

$$\int_D \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh}\cdot (h/2 + K'h) dy. \tag{2.2}$$

If we apply Green's formula on the expanding domains $D^c \cap B(0,r)$ that converge to D^c and we use the Sommerfeld condition (see also [2], [10])

$$\frac{\partial u}{\partial |x|}(x) + ku(x) = o\left(|x|^{-1/2}\right),$$

then we deduce

$$\int_{D^c} \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh}\cdot (h/2 - K'h) dy. \tag{2.3}$$

From (2.2) and (2.3) we get

$$\int_{\mathbb{R}^2\setminus\partial D} \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} h\cdot \overline{Sh}\, dy. \qquad \square$$

The following theorem gives a range for the spectrum of the operator $I/2 + K' + \alpha S$. It will be used to find a suitable norm on $C(\partial D, \mathbb{C}^2)$, in order to prove the convergence of the Neumann series.

Theorem 2.4. *The spectrum σ of the operator*

$$\frac{1}{2}I + K' + \alpha S \;:\; C(\partial D, \mathbb{C}^2) \to\; C(\partial D, \mathbb{C}^2)$$

satisfies

$$\sigma(I/2 + K' + \alpha S) \subset (0, 1 + M\alpha].$$

Proof. Suppose λ is a complex eigenvalue of the operator $I/2 + K' + \alpha S$ with the corresponding eigenvector $h \in C(\partial D, \mathbb{C}^2)$. Then

$$\lambda \int_{\partial D} h\cdot S\overline{h}\, dy = \int_{\partial D} S\overline{h}\cdot (I/2 + K' + \alpha S)\, h\, dy,$$

from which it follows

$$\lambda \int_{\partial D} h\cdot S\overline{h}\, dy = \int_{\partial D} S\overline{h}\left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right) dy.$$

We showed in Lemma 2.3 that

$$\int_{\partial D} h\cdot S\overline{h}\, dy = \int_{\mathbb{R}^2\setminus\partial D} \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy \geq 0.$$

Assume that $\int_{\partial D} h\cdot S\overline{h}\, dy = 0$. Then $Sh \equiv 0$ and therefore

$$(I/2 + K' + \alpha S)h = \frac{\partial Sh}{\partial \nu} + \alpha Sh = 0.$$

From the invertibility of the operator $I/2 + K' + \alpha S$ we deduce $h = 0$, which is a contradiction. Therefore

$$\int_{\partial D} h \cdot S\overline{h}\, dy > 0.$$

In Lemma 2.2 we proved that

$$\int_{\partial D} S\overline{h}\left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right)\, dy \geq 0.$$

It follows that $\lambda \geq 0$ and, since the operator $I/2 + K' + \alpha S$ is invertible, we obtain $\lambda > 0$, which proves the first part of the estimate of the range of

$$\sigma\left(\frac{1}{2}I + K' + \alpha S\right).$$

If we use Lemmas 2.1, 2.2 and 2.3, then we successively deduce

$$\lambda = \frac{\int_{\partial D} S\overline{h}\left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right)\, dy}{\int_{\partial D} h \cdot S\overline{h}\, dy},$$

$$\lambda = \frac{\int_D \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy + \int_{\partial D} \alpha |Sh|^2\, dy}{\int_{\mathbb{R}^2 \setminus \partial D} \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy},$$

$$\lambda \leq 1 + \frac{\int_{\partial D} \alpha |Sh|^2\, dy}{\int_{\mathbb{R}^2 \setminus \partial D} \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy},$$

$$\lambda \leq 1 + \frac{\int_{\partial D} \alpha |Sh|^2\, dy}{\int_{\partial D} h \cdot S\overline{h}\, dy} \leq 1 + M\alpha.$$

The theorem is proved. □

Theorem 2.5. *Let $g \in C(\partial D, \mathbb{R}^2)$ and $0 < c < 2/(1 + M\alpha)$. Define the operator $T = I - c(I/2 + K' + \alpha S)$. Then the series*

$$\sum_{j=0}^{\infty} c\, T^j g \tag{2.4}$$

converges in $C(\partial D, \mathbb{R}^2)$ to the solution of the boundary integral equation

$$\frac{1}{2}h + K'h + \alpha Sh = g.$$

Remark 2.6. The series (2.4) is called Neumann series. We will use the spectrum of the operator $I/2 + K' + \alpha S$ to prove the convergence. It is well known (see [8]) that if $\|T\| < 1$, then

$$\sum_{j=0}^{\infty} T^j = (I - T)^{-1}.$$

We need the following lemma about the relation between the eigenvalues and the norms in a complex Banach space. We state the lemma without proof. The lemma can be found in [8].

Lemma 2.7. *Let X be a complex Banach space and B the set of the norms on X that are equivalent to the original norm. Suppose A is a bounded linear operator in X and $r(A)$ is the spectral radius of A. Then*

$$r(A) = \inf_{\|\cdot\| \in B} \|A\|.$$

Proof. (proof of theorem 2.5) From Theorem 2.4 we have

$$\sigma(I/2 + K' + \alpha S) \subset (0, 1 + M\alpha].$$

Using the definitions that we made, $T = I - c(I/2 + K' + \alpha S)$ and

$$c \in \left(0, \frac{2}{1 + M\alpha}\right),$$

we obtain $\sigma(T) \subset (-1, 1)$ and therefore $r(T) < 1$.

From Lemma 2.7 we deduce that there is an equivalent norm $\|\cdot\|_*$ on $C(\partial D, \mathbb{C}^2)$, such that $\|T\|_* < 1$. It follows that the Neumann series

$$\sum_{j=0}^{\infty} c\, T^j g$$

converges to

$$c(I - T)^{-1} g = \left(\frac{1}{2} I + K' + \alpha S\right)^{-1} g = h,$$

which ends the proof. □

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