Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 3–15

New fractional estimates of Hermite-Hadamard inequalities and applications to means

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Abstract. The main objective of this paper is to obtain some new fractional estimates of Hermite-Hadamard type inequalities via h-convex functions. A new fractional integral identity for three times differentiable function is established. This result plays an important role in the development of new results. Several new special cases are also discussed. Some applications to means of real numbers are also discussed.

Mathematics Subject Classification (2010): 26A33, 26D15, 26A51.

Keywords: Convex functions, h-convex, fractional integrals, Hermite-Hadamard inequality.

1. Introduction and preliminaries

Throughout the sequel of the paper, let set of real numbers be denoted by $\mathbb{R}$, $I = [a, b] \subset \mathbb{R}$ be the real interval and I° be the interior of I unless otherwise specified.

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be classical convex function, if*

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in I, t \in [0, 1]. \tag{1.1}$$

In recent years numerous generalizations of classical convex functions have been proposed, see [1, 2, 3, 4, 5, 6, 11, 19]. Varosanec [19] investigated a new class of convex functions which she named as h-convex functions. This class is unifying one and it includes some other classes of convex functions, such as, s-Breckner convex functions [1], s-Godunova-Levin-Dragomir convex functions [3], Godunova-Levin functions [6] and P-functions [5].
The h-convexity is defined as:

Definition 1.2. [19] *Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function. A non-negative function $f : I \to \mathbb{R}$ is said to be h-convex function, if*

$$f((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y), \quad \forall x, y \in I, t \in [0, 1]. \tag{1.2}$$

For different suitable choices of function $h(.)$ one can have other classes of convex functions.
Every one is familiar with the fact that theory of convex functions has a close relation with theory of inequalities. In fact many classical inequalities are derived using convexity property. Thus these facts inspired a number of researchers to investigate both theories. Consequently several new generalizations of classical inequalities have been obtained via different generalizations of convex functions, see [3, 4, 5, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24].
Nowadays fractional calculus is a vibrant area of research in mathematics. The history of fractional calculus started with the letter of L'Hospital to Leibniz on 30th September 1695 in which he enquired Leibniz about the notation he used in his publications for n-th order derivative of the linear function $f(x) = x$, $\frac{\mathrm{D}^n x}{\mathrm{D}x^n}$. L'Hospital asked a question to Leibniz that what would happen if $n = \frac{1}{2}$. Leibniz's replied: "An apparent paradox, from which one day useful consequences will be drawn." With this the study of fractional calculus had begun. Several applications of fractional calculus have been found till now. For some useful information on fractional calculus and its applications, see [7, 8, 9]. A recent approach of obtaining fractional version of classical integral inequalities has also attracted researchers. For example, see [11, 12, 15, 19, 22]. The motivation of this article is to establish some new fractional estimates of Hermite-Hadamard type inequalities via h-convex functions. Some special cases which can be derived from our main results are also discussed. In the end some application to special means of real numbers are also discussed.
We now recall some preliminary concepts which are widely used throughout the paper.

Definition 1.3. [9] *Let $f \in L_1[a, b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a^+} f$ and $J^{\alpha}_{b^-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a,$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1}\mathrm{d}x,$$

is the Gamma function.

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x, y; c; z) = \frac{1}{\mathrm{B}(y, c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\mathrm{d}t$$

for $|z| < 1, c > y > 0$.

Recall that

1. For arbitrary $a, b \in \mathbb{R} \setminus \{0\}$ and $a \neq b$, $L(b,a) = \dfrac{b-a}{\log b - \log a}$, is the logarithmic mean.
2. For arbitrary $a, b \in \mathbb{R}$ and $a \neq b$, $A(a,b) = \dfrac{a+b}{2}$, is the arithmetic mean.
3. The extended logarithmic mean L_p of two positive numbers a, b is given for $a = b$ by $L_p(a,a) = a$ and for $a \neq b$ by

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0, \\ \frac{b-a}{\log b - \log a}, & p = -1, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0. \end{cases}$$

2. Main results

To prove our main results, we need following auxiliary result.

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$, then*

$$L_f(a,b;n;\alpha) = \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}$$

$$\times \int_0^1 (1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + f'''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right] \mathrm{d}t,$$

where

$$\begin{aligned} L_f(a,b;n;\alpha) &= \frac{(n+1)^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-} f(a) + J^\alpha_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+} f(b)\right] \\ &- \frac{(b-a)^2}{(n+1)^3(\alpha+1)(\alpha+2)}\left[f''\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)\right. \\ &\left.+ f''\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)\right] + \frac{b-a}{(n+1)^2(\alpha+1)}\left[f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)\right. \\ &\left.+ f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)\right] - \frac{1}{n+1}\left[f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)\right. \\ &\left.+ f\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)\right]. \end{aligned}$$

Proof. Let

$$I \triangleq \int_0^1 (1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t$$
$$= -\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t+\int_0^1 (1-t)^{\alpha} f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t$$
$$= -I_1 + I_2. \tag{2.1}$$

Integrating I_1 on $[0,1]$ yields

$$\begin{aligned} I_1 &\triangleq \int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t \\ &= \frac{n+1}{b-a}f''\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)-\frac{(n+1)^2(\alpha+2)}{(b-a)^2}f'\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right) \\ &+\frac{(n+1)^3(\alpha+1)(\alpha+2)}{(b-a)^3}f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right) \\ &-\frac{(n+1)^{\alpha+3}\Gamma(\alpha+3)}{(b-a)^{\alpha+3}}J^{\alpha}_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-}f(a). \end{aligned} \tag{2.2}$$

Similarly, integrating I_2 on $[0,1]$, we have

$$\begin{aligned} I_2 &\triangleq \int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t \\ &= -\frac{n+1}{b-a}f''\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)-\frac{(n+1)^2(\alpha+2)}{(b-a)^2}f'\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right) \\ &-\frac{(n+1)^3(\alpha+1)(\alpha+2)}{(b-a)^3}f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right) \\ &+\frac{(n+1)^{\alpha+3}\Gamma(\alpha+3)}{(b-a)^{\alpha+3}}J^{\alpha}_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+}f(b). \end{aligned} \tag{2.3}$$

Summation of (2.2), (2.3) and (2.1) and then multiplying both sides by

$$\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}$$

completes the proof. □

Note that for $n=1$ and $\alpha=1$ in Lemma 2.1, we have previously Lemma [24].
If $n=1$ in Lemma 2.1, then, we have Lemma 3.1 [14].

Theorem 2.2. *Let $f: I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$ and $|f'''|$ is h-convex function, then*

$$|L_f(a,b;n;\alpha)| \le \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\Psi(n;h;t)\left[|f'''(a)|+|f'''(b)|\right],$$

where

$$\Psi(h;n;t)=\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{n+t}{n+1}\right)+h\left(\frac{1-t}{n+1}\right)\right]\mathrm{d}t.$$

Proof. Using Lemma 2.1 and the given hypothesis, we have

$$\begin{aligned}
&|L_f(a,b;n;\alpha)|\\
&=\Bigg|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\\
&\times\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t\Bigg|\\
&\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\\
&\times\left\{\left|\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t\right|\right.\\
&\left.+\left|\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t\right|\right\}\\
&\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|\mathrm{d}t\\
&+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|\mathrm{d}t\\
&\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{n+t}{n+1}\right)|f'''(a)|+h\left(\frac{1-t}{n+1}\right)|f'''(b)|\right]\mathrm{d}t\\
&+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{1-t}{n+1}\right)|f'''(a)|+h\left(\frac{n+t}{n+1}\right)|f'''(b)|\right]\mathrm{d}t\\
&=\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\\
&\times\left(\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{n+t}{n+1}\right)+h\left(\frac{1-t}{n+1}\right)\right]\mathrm{d}t\right)\left[|f'''(a)|+|f'''(b)|\right].
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $f: I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$ and $|f'''|^q$ is h-convex function where $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].$$

Proof. Using given hypothesis, Lemma 2.1 and the Hölder's inequality, we have

$$|L_f(a,b;n;\alpha)|$$
$$=\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\right.$$
$$\left.\times\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t\right|$$
$$\leq\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t\right|$$
$$+\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t\right|$$
$$\leq\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|\mathrm{d}t$$
$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|\mathrm{d}t$$
$$\leq\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{p(\alpha+2)}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{p(\alpha+2)}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Theorem 2.4. *Let $f: I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$ and $|f'''|^q$ is h-convex function where $q > 1$, then*

$$|L_f(a,b;n;\alpha)|$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{n+t}{n+1}\right)|f'''(a)|^q+h\left(\frac{1-t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{1-t}{n+1}\right)|f'''(a)|^q+h\left(\frac{n+t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

Proof. Using given hypothesis, Lemma 2.1 and power mean inequality, we have

$$|L_f(a,b;n;\alpha)|$$

$$=\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\right.$$

$$\left.\times\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t\right|$$

$$\leq\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t\right|$$

$$+\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t\right|$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1 (1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 (1-t)^{\alpha+2}\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1 (1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 (1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1 (1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 (1-t)^{\alpha+2}\left\{h\left(\frac{n+t}{n+1}\right)|f'''(a)|^q+h\left(\frac{1-t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1 (1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 (1-t)^{\alpha+2}\left\{h\left(\frac{1-t}{n+1}\right)|f'''(a)|^q+h\left(\frac{n+t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 (1-t)^{\alpha+2}\left\{h\left(\frac{n+t}{n+1}\right)|f'''(a)|^q+h\left(\frac{1-t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 (1-t)^{\alpha+2}\left\{h\left(\frac{1-t}{n+1}\right)|f'''(a)|^q+h\left(\frac{n+t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

We now discuss some special cases of the results proved in previous section.
I. If $h(t)=t^s$ in Theorem 2.2, then, we have result for s-Breckner convex function.

Corollary 2.5. *Under the assumptions of Theorem 2.2, if $|f'''|$ is s-Breckner convex function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{(b-a)^3}{(n+1)^{s+4}(\alpha+1)(\alpha+2)}\Psi(n;s;t)\left[|f'''(a)|+|f'''(b)|\right],$$

where

$$\begin{aligned}\Psi(n;s;t) &= \int_0^1 (1-t)^{\alpha+2}\left[(n+t)^s+(1-t)^s\right]\mathrm{d}t \\ &= \frac{n^s}{\alpha+3}\,{}_2F_1\left[1,-s;\alpha+4;-\frac{1}{n}\right]+\frac{1}{\alpha+s+3}.\end{aligned}$$

II. If $h(t)=t^{-s}$ in Theorem 2.2, then, we have result for s-Godunova-Levin-Dragomir function.

Corollary 2.6. *Under the assumptions of Theorem 2.2, if $|f'''|$ is s-Godunova-Levin-Dragomir function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{(b-a)^3}{(n+1)^{4-s}(\alpha+1)(\alpha+2)}\Psi(n;-s;t)\left[|f'''(a)|+|f'''(b)|\right],$$

where

$$\begin{aligned}\Psi(n;-s;t) &= \int_0^1 (1-t)^{\alpha+2}\left[(n+t)^{-s}+(1-t)^{-s}\right]\mathrm{d}t \\ &= \frac{1}{n^s(\alpha+3)}\,{}_2F_1\left[1,s;\alpha+4;-\frac{1}{n}\right]+\frac{1}{\alpha-s+3}.\end{aligned}$$

III. If $h(t)=1$ in Theorem 2.2, then, we have result for P-function.

Corollary 2.7. *Under the assumptions of Theorem 2.2, if $|f'''|$ is P-function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{2(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)(\alpha+3)}\left[|f'''(a)|+|f'''(b)|\right].$$

IV. If $h(t)=t^s$ in Theorem 2.3, then, we have result for s-Breckner convex function.

Corollary 2.8. *Under the assumptions of Theorem 2.3, if $|f'''|^q$ is s-Breckner convex function, then*

$$\begin{aligned}&|L_f(a,b;n;\alpha)| \\ &\leq \frac{(b-a)^3}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}} \\ &\times\left[\left(\left\{\frac{(1+n)^{1+s}-n^{1+s}}{1+s}\right\}|f'''(a)|^q+\left\{\frac{1}{1+s}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\left\{\frac{1}{1+s}\right\}|f'''(a)|^q+\left\{\frac{(1+n)^{1+s}-n^{1+s}}{1+s}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

V. If $h(t) = t^{-s}$ in Theorem 2.3, then, we have result for s-Godunova-Levin-Dragomir convex function.

Corollary 2.9. *Under the assumptions of Theorem 2.3, if $|f'''|^q$ is s-Godunova-Levin-Dragomir convex function, then*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{(b-a)^3}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\left\{\frac{(n(1+n))^{-s}(-n^s(1+n)+n(1+n)^s)}{s-1}\right\}|f'''(a)|^q+\left\{\frac{1}{1-s}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left\{\frac{1}{1-s}\right\}|f'''(a)|^q+\left\{\frac{(n(1+n))^{-s}(-n^s(1+n)+n(1+n)^s)}{s-1}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

VI. If $h(t) = 1$ in Theorem 2.3, then, we have result for P-function.

Corollary 2.10. *Under the assumptions of Theorem 2.3, if $|f'''|^q$ is P-function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{2(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}\left[|f'''(a)|^q+|f'''(b)|^q\right]^{\frac{1}{q}}.$$

VII. If $h(t) = t^s$ in Theorem 2.4, then, we have result for s-Breckner convex function.

Corollary 2.11. *Under the assumptions of Theorem 2.4, if $h(t) = t^s$, then, we have result for s-Breckner convex function.*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{(b-a)^3}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\left\{\left(\frac{n^s\,{}_2F_1\left[1,-s;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)|f'''(a)|^q+\left(\frac{1}{\alpha+s+3}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left\{\left(\frac{1}{\alpha+s+3}\right)|f'''(a)|^q+\left(\frac{n^s\,{}_2F_1\left[1,-s;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

VIII. If $h(t) = t^{-s}$ in Theorem 2.4, then, we have result for s-Godunova-Levin-Dragomir convex function.

Corollary 2.12. *Under the assumptions of Theorem 2.4, if $h(t) = t^s$, then, we have result for s-Godunova-Levin-Dragomir convex function.*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{(b-a)^3}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\left\{\left(\frac{{}_2F_1\left[1,2;\alpha+4;\frac{1}{n}\right]}{n^s(\alpha+3)}\right)|f'''(a)|^q+\left(\frac{1}{\alpha-s+3}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left\{\left(\frac{1}{\alpha-s+3}\right)|f'''(a)|^q+\left(\frac{{}_2F_1\left[1,2;\alpha+4;\frac{1}{n}\right]}{n^s(\alpha+3)}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

IX. If $h(t) = 1$ in Theorem 2.4, then, we have result for P-function.

Corollary 2.13. *Under the assumptions of Theorem 2.4, if $h(t) = t^s$, then, we have result for p-function.*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{2(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)(\alpha+3)}\left[|f'''(a)|^q+|f'''(b)|^q\right].
\end{aligned}$$

3. Applications

In this section, we present some applications to means of real numbers.

Proposition 3.1. *For some $s \in (0,1)$, $0 \leq a < b$, then*

$$\begin{aligned}
&\left|L(a,b)-\frac{s(s-1)(b-a)^3}{24}A^{s-2}(a,b)-A^s(a,b)\right| \\
&\leq \frac{s(s-1)(s-2)(b-a)^3}{192}\left[|a|^{s-3}+|b|^{s-3}\right].
\end{aligned}$$

Proof. The assertion directly follows from Theorem 2.2 applying for $h(t) = t^s$, $f : [0,1] \to [0,1]$, $f(x) = x^s$ and $\alpha = 1, n = 1$. □

Proposition 3.2. *For some $s \in (0,1)$, $0 \leq a < b$ and $\frac{1}{p}+\frac{1}{q} = 1$, $1 < q < \infty$, then*

$$\begin{aligned}
&\left|L(a,b)-\frac{s(s-1)(b-a)^3}{24}A^{s-2}(a,b)-A^s(a,b)\right| \\
&\leq \frac{s(s-1)(s-2)(b-a)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{3}{4}|a|^{q(s-3)}+\frac{1}{2}|b|^{q(s-3)}\right)^{\frac{1}{q}}+\left(\frac{1}{2}|a|^{q(s-3)}+\frac{3}{4}|b|^{q(s-3)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The assertion directly follows from Theorem 2.3 applying for $h(t) = t^s$, $f : [0,1] \to [0,1]$, $f(x) = x^s$ and $\alpha = 1, n = 1$. □

Proposition 3.3. *For some $s \in (0,1)$, $0 \leq a < b$ and $q > 1$, then*

$$\left|L(a,b) - \frac{s(s-1)(b-a)^3}{24} A^{s-2}(a,b) - A^s(a,b)\right|$$
$$\leq \frac{s(s-1)(s-2)(b-a)^3}{384}\left(\frac{4}{5}\right)^{\frac{1}{q}}$$
$$\times \left[\left(\frac{3}{2}|a|^{q(s-3)} + |b|^{q(s-3)}\right)^{\frac{1}{q}} + \left(|a|^{q(s-3)} + \frac{3}{2}|b|^{q(s-3)}\right)^{\frac{1}{q}}\right].$$

Proof. The assertion directly follows from Theorem 2.4 applying for $h(t) = t^s$, $f : [0,1] \to [0,1]$, $f(x) = x^s$ and $\alpha = 1, n = 1$. □

Acknowledgements. The authors would like to thank anonymous referees for their comments and suggestions. The authors are grateful to Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research and academic environment.

References

[1] Breckner, W.W., *Stetigkeitsaussagen für eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen*, Publ. Inst. Math., **23**(1978), 13-20.

[2] Cristescu, G., *Improved integral inequalities for products of convex functions*, J. Ineq. Pure Appl. Math., **6**(2)(2005).

[3] Dragomir, S.S., *Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, RGMIA Research Report Collection, **16**(2013), Article 72.

[4] Dragomir, S.S., Pearce, C.E.M., *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, Australia, 2000.

[5] Dragomir, S.S., Pečarić, J., Persson, L.E., *Some inequalities of Hadamard type*, Soochow J. Math., **21**(1995), 335-341.

[6] Godunova, E.K., Levin, V.I., *Neravenstva dlja funkcii sirokogo klassa soderzascego vypuklye monotonnye i nekotorye drugie vidy funkii*, Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. MGPI Moskva, (1985), 138142 (in Russian).

[7] Katugampola, U.N., *A new approach to generalized fractional derivatives*, Bulletin of Math. Anal. Appl., **6**(4)(2014), 1-15.

[8] Katugampola, U.N., *Mellin transforms of generalized fractional integrals and derivatives*, Appl. Math. Comput., **257**(2015), 566-580.

[9] Kilbas, A., Srivastava, H.M., Trujillo, J.J., *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, Netherlands, 2006.

[10] Khattri, S.K., *Three proofs of the inequality $e < \left(1+\frac{1}{n}\right)^{n+0.5}$*, Amer. Math. Monthly, **117**(3)(2010), 273-277.

[11] Noor, M.A., Awan, M.U., *Some integral inequalities for two kinds of convexities via fractional integrals*, Trans. J. Math. Mech. **5**(2)(2013), 129-136.

[12] Noor, M.A., Cristescu, G., Awan, M.U., *Generalized fractional Hermite-Hadamard inequalities for twice differentiable s-convex functions*, Filomat, **29**(4)(2015), 807-815.

[13] Noor, M.A., Noor, K.I., Awan, M.U., *Hermite-Hadamard inequalities for relative semi-convex functions and applications*, Filomat, **28**(2)(2014), 221-230.

[14] Noor, M.A., Noor, K.I., Awan, M.U., *Fractional Hermite-Hadmard inequalities for convex functions and applications*, Tbilisi J. Math., **8**(2)(2015), 103-113.

[15] Ozdemir, M.E., Kavurmaci, H., Yildiz, C., *Fractional integral inequalities via s-convex functions*, available online at: arXiv:1201.4915v1, (2012).

[16] Park, J., *Some inequalities of Hermite-Hadamard type via differentiable are (s,m)-convex mappings*, Far East J. Math. Sci., **52**(2)(2011), 209-221.

[17] Park, J., *On the left side inequality of Hermite-Hadamard inequality for differentiable (α,m)-convex mappings*, Far East J. Math. Sci., **58**(2011)(2), 179-191.

[18] Park, J., *Hermite-Hadamard-like type inequalities for n-times differentiable functions which are m-convex and s-convex in the second sense*, Int. J. Math. Anal., **8**(25)(2014), 1187-1200.

[19] Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N., *HermiteHadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comp. Modelling, **57**(2013), 2403-2407.

[20] Shuang, Y., Yin, H.P., Qi, F., *HermiteHadamard type integral inequalities for geometric-arithmetically s-convex functions*, Analysis, 33(2013), 197208.

[21] Varosanec, S., *On h-convexity*, J. Math. Anal. Appl., 326(2007), 303-311.

[22] Wang, J., Li, X., Feckan, M., Zhou, Y., *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal., 2012, http://dxdoi.org/10.1080/00036811.2012.727986.

[23] Xi, B.Y., Qi, F., *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Func. Spaces Appl., 2012(2012), Article ID 980438.

[24] Xi, B.Y., Wang, S.H., Qi, F., *Some inequalities of Hermite-Hadamard type for functions whose third derivatives are P-convex*, Appl. Math., **3**(2012), 1898-1902.

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