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Korovkin type approximation for double sequences via statistical $\mathcal{A}$-summation process on modular spaces

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Abstract. In this work, we introduce the Korovkin type approximation theorems on modular spaces via statistical $\mathcal{A}$-summation process for double sequences of positive linear operators and we construct an example satisfying our new approximation theorem but does not satisfy the classical one.

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1. Introduction and preliminaries

Summability theory is the theory of the assignment of limits in the case of real or complex sequences which are divergent. There are many types of summability methods especially regular summability methods, for example, Abel and Borel methods [6]. Another regular summability method introduced by Fast ([8]) and which is not equivalent to any regular matrix method is called statistical convergence which is also known as $(C, 1)$ statistical convergence. Furthermore, in recent years, various statistical approximation results and theorems have been proved via the concept of statistical convergence ([9, 11, 19]) and the motivation using this type of convergence comes from that the obtained results are more powerful than the classical version of the approximations. One of these frequently used approximation method is the Korovkin-type approximation theorems. As it is known Korovkin theorems allows us to check the convergence with a minimum of computations. In this paper, our main purpose is to study a further generalization of classical Korovkin theorem by considering certain matrix summability process in the frame of statistical convergence in abstract spaces (namely, modular spaces) for double sequences. We also introduce an example satisfying new approximation theorem but does not satisfy the classical one.

Now, let us mention the notion of statistical convergence for double sequences introduced by Moricz [15].

The double sequence $x = \{x_{i,j}\}$ is statistically convergent to L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{mn} \left|\{i \leq m,\ j \leq n : \ |x_{i,j} - L| \geq \varepsilon\}\right| = 0,$$

where $P-$convergent denotes Pringsheim limit ([22]). In that case we write

$$st_2 - \lim_{i,j} x_{i,j} = L.$$

It can be easily seen that a $P-$convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, it is crucial to state that a convergent single sequence needs to be bounded even though this necessity does not exist always for the double sequences. A convergent double sequence does not need to be bounded. For example, take into consideration the double sequence $x = \{x_{i,j}\}$ defined by

$$x_{i,j} = \begin{cases} ij, & i \text{ and } j \text{ are squares} \\ 1, & \text{otherwise.} \end{cases}.$$

Then, clearly $st_2 - \lim_{i,j} x_{i,j} = 1$ but not $P-$convergent and also, it is not bounded. The characterization for the statistical convergence for double sequences is given in [15] as indicated below :

A double sequence $x = \{x_{i,j}\}$ is statistically convergent to L if and only if there exists a set $S \subset \mathbb{N}^2$ such that the natural density of S is 1 and

$$P - \lim_{\substack{i,j \to \infty \\ \text{and } (i,j) \in S}} x_{i,j} = L.$$

In [7] the concepts of *statistical superior limit* and *inferior limit* for double sequences have been introduced by Çakan and Altay. For any real double sequence $x = \{x_{i,j}\}$, the statistical limit superior of x is defined by

$$st_2 - \lim_{i,j} \sup x_{i,j} = \begin{cases} \sup G_x, & \text{if } G_x \neq \varnothing, \\ -\infty, & \text{if } G_x = \varnothing, \end{cases}$$

where $G_x := \{C \in \mathbb{R} : \delta_2 (\{(i,j) : x_{i,j} > C\}) \neq 0\}$ and $\varnothing$ denotes the empty set. Note that, in general, by $\delta_2 (K) \neq 0$ we mean either $\delta_2 (K) > 0$ or K fails to have the double natural density. Similarly, the statistical limit inferior of x is given by

$$st_2 - \lim_{i,j} \inf x_{i,j} = \begin{cases} \inf F_x, & \text{if } F_x \neq \varnothing, \\ \infty, & \text{if } F_x = \varnothing, \end{cases}$$

where $F_x := \{D \in \mathbb{R} : \delta_2 (\{(i,j) : x_{i,j} < D\}) \neq 0\}$. As in the ordinary superior or inferior limit, it was proved that

$$st_2 - \lim_{i,j} \inf x_{i,j} \leq st_2 - \lim_{i,j} \sup x_{i,j}$$

and also that, for any double sequence $x = \{x_{i,j}\}$ satisfying

$$\delta_2 (\{(i,j) : |x_{i,j}| > M\}) = 0$$

for some $M>0$,

$$st_2-\lim_{i,j} x_{i,j}=L \text{ iff } st_2-\liminf_{i,j} x_{i,j}=st_2-\limsup_{i,j} x_{i,j}=L.$$

Let $A=[a_{k,l,i,j}]$, $k,l,i,j\in\mathbb{N}$, be a four-dimensional infinite matrix.
The $A-$transform of $x=\{x_{i,j}\}$, denoted by $Ax:=\{(Ax)_{k,l}\}$, is defined by

$$(Ax)_{k,l}=\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}x_{i,j}, \quad k,l\in\mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$. Then, a double sequence x is $A-$summable to L if the $A-$transform of x exists for all $k,l\in\mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$P-\lim_{p,q}\sum_{i=1}^{p}\sum_{j=1}^{q} a_{k,l,i,j}x_{i,j}=y_{k,l} \text{ and } P-\lim_{k,l} y_{k,l}=L.$$

Now let $\mathcal{A}:=\left\{A^{(m,n)}\right\}=\left\{a^{(m,n)}_{k,l,i,j}\right\}$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x=\{x_{i,j}\}$ is said to be $\mathcal{A}-$summable to L if

$$P-\lim_{k,l}\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j}x_{i,j}=L$$

uniformly in m and n. If $A^{(m,n)}=A$, four-dimensional infinite matrix, then $\mathcal{A}-$summability is the $A-$summability for four-dimensional infinite matrix. Some results concerning matrix summability method for double sequences may be attained in [9], [21], [24].

Now, we recall some definitions and notations on modular spaces.

Let $I=[a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, let $X\left(I^2\right)$ denote the space of all real-valued measurable functions on $I^2=[a,b]\times[a,b]$ provided with equality $a.e.$ As usual, let $C\left(I^2\right)$ denote the space of all continuous real-valued functions, and $C^\infty\left(I^2\right)$ denote the space of all infinitely differentiable functions on I^2. A functional $\rho: X\left(I^2\right)\to[0,+\infty]$ is called a *modular* on $X\left(I^2\right)$ if it satisfies the following conditions:

(i) $\rho(f)=0$ if and only if $f=0$ $a.e.$ in I^2,

(ii) $\rho(-f)=\rho(f)$ for every $f\in X\left(I^2\right)$,

(iii) $\rho(\alpha f+\beta g)\le\rho(f)+\rho(g)$ for every $f,g\in X(I^2)$ and for any $\alpha,\beta\ge 0$ with $\alpha+\beta=1$.

A modular ρ is said to be $N-$*quasi convex* if there exists a constant $N\ge 1$ such that $\rho(\alpha f+\beta g)\le N\alpha\rho(Nf)+N\beta\rho(Ng)$ holds for every $f,g\in X\left(I^2\right)$, $\alpha,\beta\ge 0$ with $\alpha+\beta=1$. In particular, if $N=1$, then ρ is called *convex.*

A modular ρ is said to be $N-$*quasi semiconvex* if there exists a constant $N\ge 1$ such that $\rho(af)\le Na\rho(Nf)$ holds for every $f\in X\left(I^2\right)$ and $a\in(0,1]$.

It is clear that every $N-$quasi convex modular is $N-$quasi semiconvex. Bardaro et. al. introduced and worked through the above two concepts in [3, 5].

We now present some acquired vector subspaces of $X(I^2)$ via a modular ρ as follows:

The modular space $L^{\rho}\left(I^2\right)$ generated by ρ is defined by

$$L^{\rho}\left(I^2\right) := \left\{ f \in X(I^2) : \lim_{\lambda \to 0^+} \rho\left(\lambda f\right) = 0 \right\},$$

and the space of the finite elements of $L^{\rho}\left(I^2\right)$ is given by

$$E^{\rho}\left(I^2\right) := \left\{ f \in L^{\rho}\left(I^2\right) : \rho\left(\lambda f\right) < +\infty \text{ for all } \lambda > 0 \right\}.$$

Observe that if ρ is $N-$quasi semiconvex, then the space

$$\left\{ f \in X\left(I^2\right) : \rho\left(\lambda f\right) < +\infty \text{ for some } \lambda > 0 \right\}$$

coincides with $L^{\rho}\left(I^2\right)$. The notions about modulars are introduced in [16] and widely discussed in [3] (see also [13, 17]).

Bardaro and Mantellini [4] introduced some Korovkin type approximation theorems via the notions of modular convergence and strong convergence. Afterwards Karakuş et al. [11] investigated the modular Korovkin-type approximation theorem via statistical convergence and then, Orhan and Demirci [20] extended these type of approximations to the spaces of double sequences of positive linear operators as follows:

Definition 1.1. [20] A function sequence $\{f_{i,j}\}$ in $L^{\rho}\left(I^2\right)$ is said to be statistical*ly modularly convergent* to a function $f \in L^{\rho}\left(I^2\right)$ iff

$$st_2 - \lim_{i,j} \rho\left(\lambda_0\left(f_{i,j} - f\right)\right) = 0 \text{ for some } \lambda_0 > 0. \tag{1.1}$$

Also, $\{f_{i,j}\}$ is statistically $F-$*norm convergent* (or, statistically *strongly convergent*) to f iff

$$st_2 - \lim_{i,j} \rho\left(\lambda\left(f_{i,j} - f\right)\right) = 0 \text{ for every } \lambda > 0. \tag{1.2}$$

It is known from [16] that (1.1) and (1.2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e.

there exists a constant $M > 0$ such that $\rho\left(2f\right) \leq M\rho\left(f\right)$ for every $f \in X\left(I^2\right)$.

Recently, Orhan and Demirci [19] have introduced the notion of $\mathcal{A}-$summation process on the one dimensional modular space $X\left(I\right)$. Now we introduce the notion of the $\mathcal{A}-$*summation process for double sequences* as follows:

A sequence $\mathbb{T} := \{T_{i,j}\}$ of positive linear operators of D into $X\left(I^2\right)$ is called an $\mathcal{A}-$summation process on D if $\{T_{i,j}\left(f\right)\}$ is $\mathcal{A}-$summable to f (with respect to modular ρ) for every $f \in D$, i.e.,

$$P - \lim_{k,l} \rho\left[\lambda\left(A_{k,l,m,n}^{\mathbb{T}}\left(f\right) - f\right)\right] = 0, \text{ uniformly in } m, n \tag{1.3}$$

for some $\lambda > 0$, where for all $k, l, m, n \in \mathbb{N}$, $f \in D$ the series

$$A_{k,l,m,n}^{\mathbb{T}}\left(f\right) := \sum_{(i,j) \in \mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j} f \tag{1.4}$$

is absolutely convergent almost everywhere with respect to Lebesgue measure and we denote the value of $T_{i,j}f$ at a point $(x,y) \in I^2$ by $T_{i,j}(f(u,v);x,y)$ or briefly, $T_{i,j}(f;x,y)$.

Our goal in the present work is to give the Korovkin theorem for double sequences of positive linear operators using *statistical* $\mathcal{A}-$*summation process* on a modular space. Some results concerning summation processes in the space $L_p[a,b]$ of Lebesgue integrable functions on a compact interval may be found in [18, 23].

It is required to give the following assumptions on a modular ρ :

A modular ρ is *monotone* if $\rho(f) \le \rho(g)$ for $|f| \le |g|$, ρ is said to be *finite* if $\chi_A \in L^\rho\left(I^2\right)$ whenever A is measurable subset of I^2 such that $\mu(A) < \infty$. If ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset I^2$ with $\mu(B) < \delta$, then ρ is *absolutely finite* and if $\chi_{I^2} \in E^\rho\left(I^2\right)$, then ρ is *strongly finite.* A modular ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X\left(I^2\right)$ with $\rho(f) < +\infty$, the following condition holds:

- for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f\chi_B) < \varepsilon$ whenever B is any measurable subset of I^2 with $\mu(B) < \delta$.

Observe now that (see [4, 5]) if a modular ρ is monotone and finite, then we have $C(I^2) \subset L^\rho\left(I^2\right)$. Similarly, if ρ is monotone and strongly finite, then $C(I^2) \subset E^\rho\left(I^2\right)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty\left(I^2\right)} = L^\rho\left(I^2\right)$. (See for more details [2, 3, 14, 17]).

2. Main results

Let ρ be a monotone and finite modular on $X\left(I^2\right)$. Assume that D is a set satisfying $C^\infty\left(I^2\right) \subset D \subset L^\rho\left(I^2\right)$. (Such a subset D can be constructed when ρ is monotone and finite, see [4]). Also, assume that $\mathbb{T} := \{T_{i,j}\}$ is a sequence of positive linear operators from D into $X\left(I^2\right)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ with $C^\infty\left(I^2\right) \subset X_{\mathbb{T}}$ such that

$$st_2 - \limsup_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(h)\right)\right) \le R\rho(\lambda h), \text{ uniformly in } m,n, \tag{2.1}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R.

We will use the test functions f_r $(r = 0,1,2,3)$ defined by $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$ throughout the paper.

We now prove the following Korovkin type theorem.

Theorem 2.1. *Let $\mathcal{A} = \left\{A^{(m,n)}\right\}$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{i,j}\}$ be a sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (2.1) for each $f \in D$. Suppose that*

$$st_2 - \lim_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f_r) - f_r\right)\right) = 0, \quad \textit{uniformly in } m,n \tag{2.2}$$

for every $\lambda > 0$ and $r = 0, 1, 2, 3$. Now let f be any function belonging to $L^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$. Then we have

$$st_2 - \lim_{k,l} \rho\left(\lambda_0\left(A^{\mathbb{T}}_{k,l,m,n}(f) - f\right)\right) = 0, \quad \textit{uniformly in } m, n \tag{2.3}$$

for some $\lambda_0 > 0$.

Proof. We first claim that

$$st_2 - \lim_{k,l} \rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g) - g\right)\right) = 0 \text{ uniformly in } m, n \tag{2.4}$$

for every $g \in C(I^2) \cap D$ and $\eta > 0$ where

$$A^{\mathbb{T}}_{k,l,m,n}(g) = \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j} g.$$

To see this assume that g belongs to $C\left(I^2\right) \cap D$ and η is any positive number. By the continuity of g on I^2 and in consequence of the linearity and positivity of the operators $T_{i,j}$, we can easily see that (see, for instance [20]), for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $(u, v), (x, y) \in I^2$

$$|g(u, v) - g(x, y)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u - x)^2 + (v - y)^2\right\}.$$

where $M := \sup_{(x,y)\in I^2} |g(x, y)|$. Since $T_{i,j}$ is linear and positive, we get

$$\begin{aligned}
&\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(g; x, y) - g(x, y)\right| \\
= &\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(g(.,.) - g(x, y); x, y)\right. \\
&\left.+g(x, y)\left(\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f_0; x, y) - f_0(x, y)\right)\right| \\
\leq &\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(|g(.,.) - g(x, y)|; x, y) \\
&+ |g(x, y)|\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f_0; x, y) - f_0(x, y)\right| \\
\leq &\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(. - x)^2 + (. - y)^2\right\}; x, y\right) \\
&+ |g(x, y)|\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f_0; x, y) - f_0(x, y)\right|
\end{aligned}$$

$$\begin{aligned}
&= \varepsilon + (\varepsilon + M)\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0;x,y) - f_0(x,y)\right| \\
&\quad + \frac{2M}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_3;x,y) - f_3(x,y)\right| \\
&\quad + \frac{4M}{\delta^2}\left(|f_1(x,y)|\left|\sum_{(i,j)\in\mathbb{N}^2}^{\infty} a_{k,l,i,j}^{(m,n)} T_{ij}(f_1;x,y) - f_1(x,y)\right|\right. \\
&\quad \left. + |f_2(x,y)|\left|\sum_{(i,j)\in\mathbb{N}^2}^{\infty} a_{k,l,i,j}^{(m,n)} T_{ij}(f_2;x,y) - f_2(x,y)\right|\right) \\
&\quad + \frac{2M}{\delta^2}|f_3(x,y)|\left|\sum_{(i,j)\in\mathbb{N}^2}^{\infty} a_{k,l,i,j}^{(m,n)} T_{ij}(f_0;x,y) - f_0(x,y)\right|
\end{aligned}$$

for every $x, y \in I$ and $m, n \in \mathbb{N}$. Therefore, from the the last inequality we get

$$\begin{aligned}
&\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g;x,y) - g(x,y)\right| \\
&\le \varepsilon + \left(\varepsilon + M + \frac{4Mc^2}{\delta^2}\right)\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0;x,y) - f_0(x,y)\right| \\
&\quad + \frac{4Mc}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_1;x,y) - f_1(x,y)\right| \\
&\quad + \frac{4Mc}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_2;x,y) - f_2(x,y)\right| \\
&\quad + \frac{2M}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_3;x,y) - f_3(x,y)\right|
\end{aligned}$$

where $c := \max\{|f_1(x,y)|, |f_2(x,y)|\}$.

So, denoting by $K := \max\left\{\varepsilon + M + \frac{4Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$,

$$\begin{aligned}
&\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g;x,y) - g(x,y)\right| \\
&\le \varepsilon + K\left\{\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0;x,y) - f_0(x,y)\right|\right.
\end{aligned}$$

$$+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_1;x,y\right)-f_1\left(x,y\right)\right|$$
$$+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_2;x,y\right)-f_2\left(x,y\right)\right|$$
$$\left.+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_3;x,y\right)-f_3\left(x,y\right)\right|\right\}.$$

Hence, we obtain, for any $\eta>0$, that

$$\begin{aligned}
&\eta\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(g;x,y\right)-g\left(x,y\right)\right| \\
\leq\ &\eta\varepsilon+\eta K\left\{\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_0;x,y\right)-f_0\left(x,y\right)\right|\right. \\
&+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_1;x,y\right)-f_1\left(x,y\right)\right| \\
&+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_2;x,y\right)-f_2\left(x,y\right)\right| \\
&\left.+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_3;x,y\right)-f_3\left(x,y\right)\right|\right\}.
\end{aligned}$$

Now we apply the modular ρ in the both-sides of the above inequality and since ρ is monotone, we get

$$\begin{aligned}
&\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}\left(g\right)-g\right)\right)\leq\rho\left(\eta\varepsilon+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_0\right)-f_0\right)\right. \\
&\left.+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_1\right)-f_1\right)+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_2\right)-f_2\right)+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_3\right)-f_3\right)\right).
\end{aligned}$$

So, we may write that

$$\begin{aligned}
\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}\left(g\right)-g\right)\right) \leq\ &\rho(5\eta\varepsilon)+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_0\right)-f_0\right)\right) \\
&+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_1\right)-f_1\right)\right) \\
&+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_2\right)-f_2\right)\right) \\
&+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_3\right)-f_3\right)\right).
\end{aligned}$$

Since ρ is $N-$quasi semiconvex and strongly finite, we have, assuming $0<\varepsilon\leq 1$

$$\begin{aligned}\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}(g)-g\right)\right) &\leq N\varepsilon\rho(5\eta N)+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_0)-f_0\right)\right)\\ &+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_1)-f_1\right)\right)\\ &+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_2)-f_2\right)\right)\\ &+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_3)-f_3\right)\right).\end{aligned}$$

For a given $\varepsilon^*>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho(5\eta N)<\varepsilon^*$. Now we define the following sets:

$$\begin{aligned}G_\eta &:=\left\{(k,l):\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}(g)-g\right)\right)\geq\varepsilon^*\right\},\\ G_{\eta,r} &:=\left\{(k,l):\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_r)-f_r\right)\right)\geq\frac{\varepsilon^*-N\varepsilon\rho(5\eta N)}{4}\right\},\end{aligned}$$

$r=0,1,2,3$. Then, it is easy to see that $G_\eta\subseteq\bigcup_{r=0}^{3}G_{\eta,r}$. So, we can write that

$$\delta_2(G_\eta)\leq\sum_{r=0}^{3}\delta_2(G_{\eta,r}).$$

Using the hypothesis (2.2), we get

$$\delta_2(G_\eta)=0,$$

which proves our claim (2.4). Obviously, (2.4) also holds for every $g\in C^\infty(I^2)$. Now let $f\in L^\rho\left(I^2\right)$ satisfying $f-g\in X_{\mathbb{T}}$ for every $g\in C^\infty\left(I^2\right)$. Since $\mu\left(I^2\right)<\infty$ and ρ is strongly finite and absolutely continuous, it can be seen that ρ is also absolutely finite on $X(I^2)$ (see [2]). So, it is known from [3, 14] that the space $C^\infty(I^2)$ is modularly dense in $L^\rho\left(I^2\right)$, i.e., there exists a sequence $\{g_{k,l}\}\subset C^\infty\left(I^2\right)$ such that

$$P-\lim_{k,l}\rho\left(3\lambda_0^*\left(g_{k,l}-f\right)\right)=0\quad\text{for some }\lambda_0^*>0,$$

which means, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*\left(g_{k,l}-f\right)\right)<\varepsilon\qquad\text{for every }k,l\geq k_0. \tag{2.5}$$

In addition to that, because the operators $T_{i,j}$ are linear and positive, we can write that

$$\begin{aligned}&\lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(f;x,y)-f(x,y)\right|\\ \leq\ &\lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(f-g_{k_0,k_0};x,y)\right|\\ &+\lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(g_{k_0,k_0};x,y)-g_{k_0,k_0}(x,y)\right|\\ &+\lambda_0^*\left|g_{k_0,k_0}(x,y)-f(x,y)\right|,\end{aligned}$$

holds for every $x,y\in I$ and $m,n\in\mathbb{N}$. Applying the modular ρ and moreover considering the monotonicity of ρ, we have

$$\begin{aligned}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right) &\leq \rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f-g_{k_0,k_0})\right)\right)\\ &\quad+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right)\\ &\quad+\rho\left(3\lambda_0^*\left(g_{k_0,k_0}-f\right)\right). \end{aligned}\tag{2.6}$$

Then, it follows from (2.5) and (2.6) that

$$\begin{aligned}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right) &\leq \varepsilon+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f-g_{k_0,k_0})\right)\right)\\ &\quad+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right). \end{aligned}\tag{2.7}$$

So, taking statistical limit superior as $k,l\to\infty$ in the both-sides of (2.7) and also using the facts that $g_{k_0,k_0}\in C^\infty(I^2)$ and $f-g_{k_0,k_0}\in X_{\mathbb{T}}$, we obtain from (2.1) that

$$\begin{aligned}&st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\\ \leq\ &\varepsilon+R\rho\left(3\lambda_0^*(f-g_{k_0,k_0})\right)\\ &+st_2-\limsup_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right),\end{aligned}$$

which gives

$$\begin{aligned}&st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\\ \leq\ &\varepsilon(R+1)+st_2-\limsup_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right).\end{aligned}\tag{2.8}$$

By (2.4), since

$$st_2-\lim_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right)=0,\ \text{uniformly in } m,n,$$

we get

$$st_2-\limsup_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right)=0,\ \text{uniformly in } m,n. \tag{2.9}$$

From (2.8) and (2.9), we conclude that

$$st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\leq\varepsilon(R+1).$$

Since $\varepsilon>0$ was arbitrary, we find

$$st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)=0 \text{ uniformly in } m,n.$$

Furthermore, since $\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)$ is non-negative for all $k,l,m,n\in\mathbb{N}$, we can easily see that

$$st_2-\lim_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)=0,\ \text{uniformly in } m,n,$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition, then one can get immediately the following result from Theorem 2.1.

Theorem 2.2. *Let $\mathcal{A} = \left\{A^{(m,n)}\right\}$ be a sequence of four dimensional infinite non-negative real matrices. Let ρ and $\mathbb{T} = \{T_{i,j}\}$ be the same as in Theorem 2.1. If ρ satisfies the Δ_2-condition, then the statements (a) and (b) are equivalent:*

(a) $st_2 - \lim_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f_r) - f_r\right)\right) = 0$ *uniformly in m, n, for every $\lambda > 0$ and $r = 0, 1, 2, 3$,*

(b) $st_2 - \lim_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f) - f\right)\right) = 0$ *uniformly in m, n, for every $\lambda > 0$ provided that f is any function belonging to $L^{\rho}(I^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.*

If one replaces the matrices $A^{(m,n)}$ by the identity matrix and taking $P-$limit, then the condition (2.1) reduces to

$$P - \limsup_{i,j} \rho\left(\lambda\left(T_{i,j}h\right)\right) \leq R\rho\left(\lambda h\right) \tag{2.10}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the next results which were obtained by Orhan and Demirci [20] immediately follows from our Theorems 2.1 and 2.2.

Corollary 2.3. ([20]) *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{i,j}\}$ be a sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (2.10). If $\{T_{i,j}f_r\}$ is strongly convergent to f_r for each $r = 0, 1, 2, 3$, then $\{T_{i,j}f\}$ is modularly convergent to f provided that f is any function belonging to $L^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.*

Corollary 2.4. ([20]) *Let $\mathbb{T} = \{T_{i,j}\}$ and ρ be the same as in Corollary 2.3. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

$(a)\{T_{i,j}f_r\}$ is strongly convergent to f_r for each $r = 0, 1, 2, 3$,

$(b)\{T_{i,j}f\}$ is strongly convergent to f provided that f is any function belonging to $L^{\rho}(I^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.

In the following, we construct an example of positive linear operators satisfying the conditions of Theorem 2.1.

Example 2.5. Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.

Hence, let us consider the functional ρ^{φ} on $X(I^2)$ defined by

$$\rho^{\varphi}(f) := \int_0^1\int_0^1 \varphi\left(|f(x, y)|\right) dxdy \quad \text{for} \quad f \in X\left(I^2\right). \tag{2.11}$$

In this case, ρ^{φ} is a convex modular on $X\left(I^2\right)$, which satisfies all assumptions listed in Section 1 (see [4]). Let us consider the Orlicz space generated by φ as follows:

$$L^{\rho}_{\varphi}(I^2) := \left\{f \in X\left(I^2\right) : \rho^{\varphi}\left(\lambda f\right) < +\infty \quad \text{for some } \lambda > 0\right\}.$$

Then let us consider the following bivariate Bernstein-Kantorovich operator

$$\mathbb{U} := \{U_{i,j}\}$$

on the space $L_{\varphi}^{\rho}\left(I^{2}\right)$ which is defined by:

$$U_{i,j}(f;x,y) = \sum_{k=0}^{i}\sum_{l=0}^{j} p_{k,l}^{(i,j)}(x,y)\,(i+1)\,(j+1)\int_{k/(i+1)}^{(k+1)/(i+1)}\int_{l/(j+1)}^{(l+1)/(j+1)} f(t,s)\,dsdt \tag{2.12}$$

for $x, y \in I$, where $p_{k,l}^{(i,j)}(x,y)$ defined by

$$p_{k,l}^{(i,j)}(x,y) = \binom{i}{k}\binom{j}{l} x^{k} y^{l} (1-x)^{i-k} (1-y)^{j-l}.$$

Also, it is clear that,

$$\sum_{k=0}^{i}\sum_{l=0}^{j} p_{k,l}^{(i,j)}(x,y) = 1. \tag{2.13}$$

Observe that the operators $U_{i,j}$ map the Orlicz space $L_{\varphi}^{\rho}\left(I^{2}\right)$ into itself. Because of (2.13), as in the proof of [4] Lemma 5.1 and similar to Example 1[20], we obtain that for every $f \in L_{\varphi}^{\rho}\left(I^{2}\right)$ and $i, j \in \mathbb{N}$ there is an absolute constant $M > 0$ such that

$$\rho^{\varphi}(U_{i,j}f) \leq M\rho^{\varphi}(f).$$

Then, we know that, for any function $f \in L_{\varphi}^{\rho}\left(I^{2}\right)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}\left(I^{2}\right)$, $\{U_{i,j}f\}$ is modularly convergent to f, with the choice of $X_{\mathbf{U}} := L_{\varphi}^{\rho}(I^{2})$. Now define $\{s_{i,j}\}$ by

$$s_{i,j} = \begin{cases} 1, & \text{if } i,\ j \text{ are squares} \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

Now observe that, $st_2 - \lim_{i,j} s_{i,j} = 0$. Also, assume that

$$\mathcal{A} := \left\{A^{(m,n)}\right\} = \left\{a_{k,l,i,j}^{(m,n)}\right\}$$

is a sequence of four dimensional infinite matrices defined by

$$a_{k,l,i,j}^{(m,n)} = \frac{1}{kl} \text{ if } m \leq i \leq m+k-1,\ n \leq j \leq n+l-1,\ (m,n = 1,2,...)$$

and $a_{k,l,i,j}^{(m,n)} = 0$ otherwise. Then, using the operators $U_{i,j}$, we define the sequence of positive linear operators $\mathbb{V} := \{V_{i,j}\}$ *on* $L_{\varphi}^{\rho}(I^{2})$ as follows:

$$V_{i,j}(f;x,y) = (1+s_{i,j})\,U_{i,j}(f;x,y) \quad \text{for } f \in L_{\varphi}^{\rho}(I^{2}),\ x,y \in [0,1] \text{ and } i,j \in \mathbb{N}. \tag{2.15}$$

As in the proof of Lemma 5.1 [4] and using the convexity of φ we get, for every $h \in X_{\mathbb{V}} := L_{\varphi}^{\rho}(I^{2})$, $\lambda > 0$ and for positive constant M, that

$$st_2 - \limsup_{k,l} \rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}(h)\right)\right) \leq M\rho^{\varphi}(\lambda h),\ \text{uniformly in } m,n,$$

where

$$A^{\mathbb{V}}_{k,l,m,n}(h) = \sum_{(i,j)\in\mathbb{N}^2}^{\infty} a^{(m,n)}_{k,l,i,j} V_{i,j} h$$

as in (1.4). Therefore the condition (2.1) works for our operators $V_{i,j}$ given by (2.15) with the choice of $X_{\mathbb{V}} = X_{\mathbf{U}} = L^{\rho}_{\varphi}(I^2)$. We now claim that

$$st_2 - \lim_{k,l} \rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_r) - f_r\right)\right) = 0, \text{ uniformly in } m,n;\ r = 0,1,2,3. \quad (2.16)$$

Observe that

$$U_{i,j}(f_0;x,y) = 1, \quad U_{i,j}(f_1;x,y) = \frac{ix}{i+1} + \frac{1}{2(i+1)},$$

$$U_{i,j}(f_2;x,y) = \frac{jy}{j+1} + \frac{1}{2(j+1)}$$

and

$$U_{i,j}(f_3;x,y) = \frac{i(i-1)x^2}{(i+1)^2} + \frac{2ix}{(i+1)^2} + \frac{1}{3(i+1)^2} + \frac{j(j-1)y^2}{(j+1)^2} + \frac{2jy}{(j+1)^2} + \frac{1}{3(j+1)^2}.$$

So, we can see,

$$\begin{aligned} \rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_0) - f_0\right)\right) &= \rho^{\varphi}\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right) \\ &= \int_0^1\int_0^1 \varphi\left(\left|\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right|\right) dxdy \\ &= \varphi\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right), \end{aligned}$$

because of

$$\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j}) = \begin{cases} 2, & \text{if } i,\ j \text{ are squares} \\ 1 & \text{otherwise.} \end{cases} \quad ,m,n = 1,2,...$$

and using continuity of φ, we get

$$st_2 - \lim_{k,l} \varphi\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right) = 0, \text{ uniformly in } m,n \quad (2.17)$$

and hence

$$st_2 - \lim_{k,l} \rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_0) - f_0\right)\right) = 0, \quad \text{uniformly in } m,n,$$

which guarantees that (2.16) holds true for $r=0$. Also, since

$$
\begin{aligned}
&\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}\left(f_1\right)-f_1\right)\right)\\
=\;&\rho^{\varphi}\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j})\left(\frac{ix}{i+1}+\frac{1}{2(i+1)}\right)-x\right)\right)\\
\leq\;&\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{i}{i+1}-1\right)\right)+\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{2(i+1)}\right)\right)\\
&+\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}s_{i,j}\left(\frac{i}{i+1}+\frac{1}{2(i+1)}\right)\right)\right)\\
=\;&\varphi\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{i}{i+1}-1\right)\right)+\varphi\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{2(i+1)}\right)\right)\\
&+\varphi\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}s_{i,j}\left(\frac{i}{i+1}+\frac{1}{2(i+1)}\right)\right)\right)
\end{aligned}
$$

Since

$$st_2-\lim_{k,l}\left(\sup_{m,n}\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{i}{i+1}-1\right)\right)=0,$$

$$st_2-\lim_{k,l}\left(\sup_{m,n}\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{2(i+1)}\right)=0$$

and

$$st_2-\lim_{k,l}\left(\sup_{m,n}\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}s_{i,j}\left(\frac{i}{i+1}+\frac{1}{2(i+1)}\right)\right)\right)=0$$

we have,

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}\left(f_1\right)-f_1\right)\right)=0,\quad \text{uniformly in } m,n.$$

So (2.16) holds true for $r=1$. Similarly, we have

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}\left(f_2\right)-f_2\right)\right)=0,\quad \text{uniformly in } m,n.$$

Finally, since

$$
\begin{aligned}
&\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}\left(f_3\right)-f_3\right)\right)\\
&=\rho^{\varphi}\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j})\left(\frac{i\left(i-1\right)x^2}{\left(i+1\right)^2}+\frac{2ix}{\left(i+1\right)^2}+\frac{1}{3\left(i+1\right)^2}\right.\right.\right.\\
&\qquad\left.\left.\left.+\frac{j\left(j-1\right)y^2}{\left(j+1\right)^2}+\frac{2jy}{\left(j+1\right)^2}+\frac{1}{3\left(j+1\right)^2}\right)-\left(x^2+y^2\right)\right)\right)
\end{aligned}
$$

$$\leq \rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j})\frac{i\,(i-1)}{(i+1)^2}-1\right)\right)$$
$$+\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j})\frac{j\,(j-1)}{(j+1)^2}-1\right)\right)$$
$$+\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j})\left(\frac{2i}{(i+1)^2}+\frac{1}{3\,(i+1)^2}\quad+\frac{2j}{(j+1)^2}+\frac{1}{3\,(j+1)^2}\right)\right)\right)$$

Hence we can easily see that

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}\left(f_3\right)-f_3\right)\right)=0,\quad \text{uniformly in } m,n.$$

So, our claim (2.16) holds true for each $r=0,1,2,3$. $\{V_{i,j}\}$ satisfies all hypothesis of Theorem 2.1 and we immediately see that,

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}\left(f\right)-f\right)\right)=0,\quad \text{uniformly in } m,n,$$

on $I^2=[0,1]\times[0,1]$ for all $f\in L^{\rho}_{\varphi}(I^2)$. Also, since $\{s_{i,j}\}$ does not converge modularly, $\{V_{i,j}\}$ does not satisfy Corollary 2.3.

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